Normal forms at relative equilibria

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Mechanical Systems

(P, Ω) = a finite dimensional symplectic manifold

$H : P \to \mathbb{R}$ determines the dynamics:

$$d\Omega_z\,(X_H(z), v) = dH(z)\cdot v \qquad \text{for all } v \in T_zP$$

Alternatively, one can use the Poisson bracket

$$\{F, H\} := \Omega\,(X_F, X_H) \qquad \text{for all } \; F, H \in \mathcal{C}^\infty(P)$$

and then X_H is determined by $\dot{F} = \{F, H\}$ for all $F, H \in \mathcal{C}^\infty(P)$.

Interested in $(T^*Q, \Omega_{can}) \equiv (T^*Q, \{\cdot,\cdot\}_{can})$

$$\dot{q} = \frac{\partial H}{\partial p}\,, \qquad \dot{p} = -\frac{\partial H}{\partial q}$$

(P, Ω_{can}) Dynamics near Equilibria. Poincaré-Birkhoff normalization

Let z_0 be an equilibrium ($DH(z_0) = 0$). Wlog $z_0 = 0$.

We apply iteratively changes of coordinates $H \to \hat{H}$ such that $\hat{H}$, the k-jet of $\hat{H}$, becomes

$$j^k \hat{H} = \hat{H}^{(2)} + \hat{H}^{(3)} + \ldots + \hat{H}^{(k)}$$

so that

$$\{H^{(2)}, \hat{H}^{(i)}\} = 0 \quad \forall\ i = 2, 3, \ldots k$$

For the term "$H^{(i)}$" of degree i we look for a homogeneous polynomial F of degree i so that

$$H^{(i)} + \{H_2, F\} = 0 \quad \text{(as much as possible)}$$

Having F, we apply a time-1 flow X_F^1 change of coordinates and obtain the new H.

Symmetries

G = a (compact) Lie group acting freely and properly on Q
Denote $\mathfrak{g}$ and $\mathfrak{g}^*$ its Lie algebra and co-algebra, respectively.

The *momentum map* is $J : P \to \mathfrak{g}^*$ such that for any $\omega \in \mathfrak{g}$ the Hamiltonian vector field of J_ω where $J_\omega(z) := \langle J(z), \omega \rangle$ satisfies

$$X_{J_\omega}(z) = \omega_P(z) = \frac{d}{dt}\Big|_{t=0} \exp(t\omega) \cdot (z)$$

(i.e., X_{J_ω} points along group orbits)

E.g. $N-$body problems in $\mathbb{R}^3$: $G = SO(3)$, $\mathfrak{g} \simeq \mathbb{R}^3$, $\mathfrak{g}^* \simeq \mathbb{R}^3$

$$J_\omega(q,p) = \sum \langle p_i, \omega \times q_i \rangle \quad \text{and} \quad J(q \times p) = \sum q_i \times p_i$$

Theorem (Noether) If $H : P \to \mathbb{R}$ is G-invariant, then J is conserved along the motion.

Special solutions

Definition: a *relative equilibrium* is a solution of the dynamics that is also a group orbit; that is, there exist $\omega \in \mathfrak{g}$ and $z_0 \in T^*Q$ such that

$$z(t) = \exp(t\omega) z_0$$

is a solution.

E.g. For N-body problems,

$$(q(t), p(t)) = R(t) \cdot (q_0, p_0) \quad \text{where} \quad R(t) = \exp(t\omega)$$

for some fixed angular velocity $\omega \in \mathbb{R}^3 \simeq so(3)^*$.

Co-tangent Bundle Reduction (with $G = SO(3)$)

$(T^*Q, \Omega_{can}, SO(3))$. The momentum map is $J : T^*Q \to so(3)^*$.

N-body systems: $J(q,p) = \sum q_i \times p_i$.

H invariant $\Rightarrow$ for each momentum $\mu \in so(3)^*$

$$J^{-1}(\mu) := \{(q,p) \mid J(q,p) = \mu\}$$ are invariant submanifolds.

Fix $\mu_0 = J(q \times p) \in so(3)^*$ (e.g. a rotation about Oz).

$$J(q \times p) = \mu_0 = R_z\mu_0 = J(R_z q, R_z p) \quad \forall\, R_z = \text{Rot. about } Oz$$

$\Longrightarrow$ $J^{-1}(\mu_0)$ quotients by the subgroup of vertical rotations

$$SO(3)_{\mu_0} := \{R \in SO(3) \mid R\mu_0 = \mu_0\}$$ isotropy group of μ_0

Let $\mu_0 \in so(3)^*$ and $SO(3)_{\mu_0}$ its isotropy group. Then $J^{-1}(\mu_0)$ quotients by $SO(3)_{\mu_0}$ and, provided the action is free, the *reduced* space

$$(T^*Q)_{\mu_0} := J^{-1}(\mu_0)/SO(3)_{\mu_0}$$

is a smooth manifold.

Theorem (Meyer; Marsden-Weinstein)

There is a unique symplectic structure Ω_{μ_0} on $(T^*Q)_{\mu_0}$ such that for every G-invariant Hamiltonian H, dynamical solutions of $(T^*Q, \Omega_{can}, H, G)$ project into dynamical solutions of $((T^*Q)_{\mu_0}, \Omega_{\mu_0}, h)$ where $h \circ \pi = H$.

In general, we want to know: dynamics in the reduced space, its reconstruction to the un-reduced space, understand the mechanism of symmetry-breaking perturbations, etc.

Good starting point $\Rightarrow$ relative equilibria $=$ equilibria in the reduced space.

For non-symmetric systems, dynamics near equilibria are studied using Poincaré-Birkhoff normalization.

For symmetric "$(q, p) \in T^*Q$" co-tangent bundle systems, we have (local) Darboux coordinates, both for the unreduced and the symplectically reduced spaces, but how does one "sit" inside the other?

$(T^*Q, \Omega_{can}, SO(3), H) \longrightarrow$ the reduced space$((T^*Q)_{\mu_0}, \Omega_{\mu_0})$.

$$[1] \qquad (T^*Q)_{\mu_0} = J^{-1}(\mu_0)/\left(SO(3)\right)_{\mu_0} \longrightarrow T^*\left(Q/\left(SO(3)\right)_{\mu_0}\right)$$

where one uses a *shift* map $(q,p) \to (q,p) - \mathcal{A}_{\mu_0}(q)$.

Then $\Omega_{\mu_0} = \omega_{can} - \beta_{\mu_0}$. *Non-canonical*, unless $\mu = 0$.

$$[2] \qquad (T^*Q)_{\mu_0} = J^{-1}(\mu_0)/\left(SO(3)\right)_{\mu_0} \simeq T^*\left(Q/SO(3)\right) \times \mathcal{O}_{\mu_0}$$

where $Q/SO(3) :=$ the shape space and

$$\mathcal{O}_{\mu_0} := \{R\mu_0 \,|\, R \in SO(3)\} = \text{a 2-sphere of radius } |\mu_0|$$

$(T^*\left(Q/G\right), \Omega_{can})$ and $(\mathcal{O}_{\mu_0}, \Omega_{can})$

Adopt [2], since it comes with a canonical symplectic form.

$$(T^*Q)_{\mu_0} = J^{-1}(\mu_0)/\left(SO(3)\right)_{\mu_0} \simeq T^*\left(Q/SO(3)\right) \times \mathcal{O}_{\mu_0}$$

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$$Q/SO(3) := \text{the shape space}$$

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Easiest case: $Q = SO(3)$. In plain words, the *rigid body*.

$$(T^*SO(3))_{\mu_0} = J^{-1}(\mu_0)/\left(SO(3)\right)_{\mu_0} \simeq \mathcal{O}_{\mu_0}$$

The free rigid body with a fixed point.

$$(T^*SO(3))_{\mu_0} = J^{-1}(\mu_0)/(SO(3))_{\mu_0} \simeq \mathcal{O}_{\mu_0}$$

1) Reduced dynamics: we have canonical coordinates on the 2-sphere $\mathcal{O}_{\mu_0}$ (obviously, one needs two charts).

2) If we are interested in the dynamics in the full phase space, we can use Serret-Andoyer-Deprit coordinates.

However, if we want to study the dynamics of systems with more general Lie symmetries (e.g., $SO(4)$), we need a systematic approach.

Free rigid body with a fixed point

$$(T^*SO(3))_{\mu_0} = J^{-1}(\mu_0)/(SO(3))_{\mu_0} \simeq \mathcal{O}_{\mu_0}$$

$$\begin{array}{ccccc} T^*SO(3) & \to & SO(3) \times so(3)^* & \simeq & SO(3) \times \mathbb{R}^3 \\ (A, P) & \to & (A, A^{-1}P) & = & (A, \mu) \end{array}$$

body coordinates

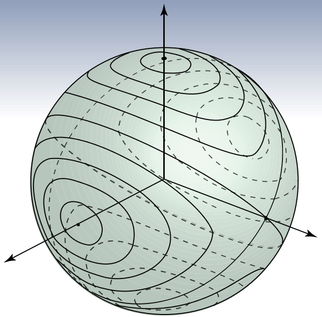
Let $\mathbb{I}_1$, $\mathbb{I}_2$, $\mathbb{I}_3$ be the principal moments of inertia of the body.

$$H(A, \mu) = H(\mu) = \frac{1}{2}\left(\frac{\mu_1^2}{\mathbb{I}_1} + \frac{\mu_2^2}{\mathbb{I}_2} + \frac{\mu_3^2}{\mathbb{I}_3}\right)$$

Spatial angular momentum is conserved $\Longrightarrow \dfrac{d}{dt}(A\mu) = 0 \Longleftrightarrow$

$\dot{\mu} = \mu \times (\mathbb{I}^{-1}\mu)$ Euler's equations and $|\mu| = const. =: \mu_0$

Symplectically reduced spaces: 2-spheres = symplectic leafs of the Poisson reduced space $(SO(3) \times so(3)^*)\,/SO(3) = so(3)^*$.



$$\mathcal{O}_{\mu_0} = \{R\mu_0 \,|\, R \in SO(3)\} = \text{ sphere of radius } |\mu_0|$$

$$H(A,\mu) = H(\mu) = \frac{1}{2}\left(\frac{\mu_1^2}{\mathbb{I}_1} + \frac{\mu_2^2}{\mathbb{I}_2} + \frac{\mu_3^2}{\mathbb{I}_3}\right) = h = const.$$

Given μ_0 fixed, how the dynamics changes when μ_0 is increased to a $\mu_0 + \delta$? ($\delta \not\parallel \mu_0$; say δ small).

In the reduced space, we just move to the sphere $\mathcal{O}_{\mu_0+\delta}$ of radius $|\mu_0 + \delta|$. And the energy increases:

$$H_{\mu_0+\nu} = \frac{1}{2}\left(\frac{(\mu_{10}+\delta_1)^2}{\mathbb{I}_1} + \frac{(\mu_{20}+\delta_2)^2}{\mathbb{I}_2} + \frac{(\mu_{30}+\delta_3)^2}{\mathbb{I}_3}\right) = const.$$

Coordinates for the perturbed rigid-body:

(a) $|\mu_0 + \delta| =$ the radius of the sphere of the new momentum level $\rightarrow$ (modified) Serret-Deprit-Andoyer coordinates

(b) perhaps a *Slice Theorem* would provide a new parametrization.

Slice Theorems → a symmetry-adapted framework

Theorem (Symplectic Slice Theorem - free action)

Consider $\mathcal{P}$ *be a symplectic manifold,* $z_0 \in \mathcal{P}$ *a RE with momentum* μ_0*, and let* $\mathcal{N}$ *a normal space transverse to* $G \cdot z_0$ *and* z_0*, i.e.*

$$T_{z_0} P \overset{loc.}{=} T_{z_0}(G\, z_0) \oplus \mathcal{N}$$

There is a choice of $\mathcal{N}$ *and coordinates such that near* $G\, z_0$ *we have* $\mathcal{N} = \mathcal{N}_0 \oplus \mathcal{N}_1 \simeq \mathfrak{g}^*_{\mu_0} \oplus (kerDJ(\mu_0) \cap \mathcal{N})$ *s. t.* $z_0 \simeq (e, 0, 0)$*,*

$$z \overset{loc.}{\simeq} (g, \nu, w)\,, \qquad g \in G\,,\ \nu \in \mathfrak{g}^*_{\mu_0}\,,\ w \in \mathcal{N}_1\,.$$

$$\dot{g} = g\, D\nu h(\nu, w) \qquad \dot{A} = A\, D\nu h(\nu, w) \ (\textit{attitude dynamics})$$

$$\dot{\nu} = ad^*_{D_\nu h(\nu, w)} \nu \qquad \dot{\nu} = \nu \times D_\nu h(\nu, w) \ (\textit{where } \nu \parallel \mu_0)$$

$$\dot{w} = \mathbb{J}_{\mathcal{N}_1} D_w h(\nu, w) \qquad \dot{w} = \mathbb{J}\, D_w h(\nu, w) \ (\textit{dynamics on } \mathcal{O}_{\mu_0})$$

In this framework there is quite a body of work - long bibliography - . Some relevant papers (free case):

M. Roberts and de M.E.R. Sousa Dias: *Bifurcations from relative equilibria of Hamiltonian systems*, Nonlinearity, 10, 1997

J.P. Ortega and T. Ratiu: *Stability of Hamiltonian relative equilibria*, Nonlinearity, 12, 1999

C. Wulff, A. Schebesch: *Numerical continuation of Hamiltonian relative periodic orbits*, J. Nonl. Science 18, 2008.

C. Wulff and F. Schilder: *Numerical bifurcation of Hamiltonian relative periodic orbits*, SIAM J. Appl. Dyn. Syst., 8, 2009.

For co-tangent bundles, the slice framework is great at the theoretical level, but there are no *constructive* slice theorems (even for free actions) except for

- abelian groups -

- for compact groups at zero momentum $\rightarrow$ T. Schmah: *A cotangent bundle slice theorem* Diff. Geom. Appl. 25, 2007

- for $SO(3)$ $\rightarrow$ T. Schmah & C.S.: *Normal forms for Lie symmetric cotangent bundle systems with free and proper actions*, to appear in Fields Institute Communications series, Vol. "Geometry, Mechanics and Dynamics: the Legacy of Jerry Marsden"

$$T^*SO(3) \to SO(3) \times so(3)^*$$
$$(A, P) \to \qquad (A, \mu)$$

$$so(3)^* = \left(so(3)^*\right)_{\mu_0} \times so(3)^{\perp}_{\mu_0} \simeq \left(so(3)^*\right)_{\mu_0} \times T_{\mu_0}\mathcal{O}_{\mu_0}$$
$$\mu \quad \leftrightarrow \quad (\nu\,, \qquad (\eta_x,\, \eta_y))$$

Look for a $SO(3)$-equivariant symplectic diffeomorphism

$$\left(SO(3) \times so(3)^*_{\mu_0} \times T_{\mu_0}\mathcal{O}_{\mu_0}\,, \Omega_Y\right) \longrightarrow \left(SO(3) \times so(3)^*, \Omega_{can}\right)\,,$$
$$\text{such that } \ (\text{Id}, 0, 0) \longrightarrow (\text{Id}, \mu_0),$$

Theorem (A symplectic tube for $SO(3)$, Schmah 2007/2013)

The following is an $SO(3)$-equivariant symplectic local diffeomorphism with respect to the symplectic form

$$\Omega_Y(R,\nu,\eta)\left((\xi_1,\dot\nu_1,\eta_1),(\xi_2,\dot\nu_2,\eta_2)\right) := \langle \mu_0+\nu,[\xi_1,\xi_2]\rangle + \langle \dot\nu_2,\xi_1\rangle - \langle \dot\nu_1,\xi_2\rangle - \langle \mu_0,[\eta_1,\eta_2]\rangle$$

in a neighbourhood of $(Id,0,0)$*:*

$$\phi : SO(3)\times so(3)^*_{\mu_0}\times so(3)^{\perp}_{\mu_0} \longrightarrow SO(3)\times so(3)^*,$$
$$(R,\nu,\eta) \longrightarrow \left(RF(\nu,\eta)^{-1}, F(\nu,\eta)(\mu_0+\nu)\right) = (A,\mu)$$

where

$$F(\nu,\eta) = \exp\left(\theta\frac{\hat\eta}{\|\eta\|}\right), \qquad \sin\left(\frac{\theta}{2}\right) = \frac{\|\eta\|}{2}\sqrt{\frac{\|\mu_0\|}{\|\mu_0+\nu\|}}.$$

$(R, \nu, \eta) \in SO(3) \times so(3)^*_{\mu_0} \times T_{\mu_0}\mathcal{O}_{\mu_0}$ symplectic form

$$\Omega_Y(R,\nu,\eta) = \begin{bmatrix} 0 & (\mu_0+\nu) & 0 & 0 & 0 & 0 \\ -(\mu_0+\nu) & 0 & 0 & 0 & 0 & 0 \\ & & & & & \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ & & & & & \\ 0 & 0 & 0 & 0 & 0 & -\mu_0 \\ 0 & 0 & 0 & 0 & \mu_0 & 0 \end{bmatrix}$$

Equations of motion

$$
\begin{bmatrix} \xi_x \\ \xi_y \\ \xi_z \\ \dot{\nu} \\ \dot{\eta}_x \\ \dot{\eta}_y \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{\mu_0+\nu} & 0 & 0 & 0 & 0 \\ -\frac{1}{\mu_0+\nu} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{\mu_0} \\ 0 & 0 & 0 & 0 & \frac{1}{\mu_0} & 0 \end{bmatrix} \begin{bmatrix} \left(R^{-1}\partial_R H\right)_x \\ \left(R^{-1}\partial_R H\right)_y \\ \left(R^{-1}\partial_R H\right)_z \\ \partial_\nu H \\ \partial_{\eta_x} H \\ \partial_{\eta_y} H \end{bmatrix}
$$

where $\xi = R^{-1}\dot{R} \in so(3)$.

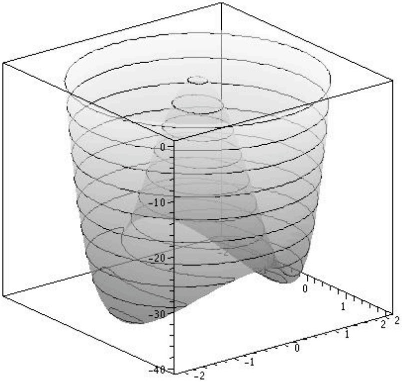
$SO(3)$ symmetric systems defined on $T^*SO(3)$

Let $H : T^*SO(3) \to \mathbb{R}$ be $SO(3)$-invariant.

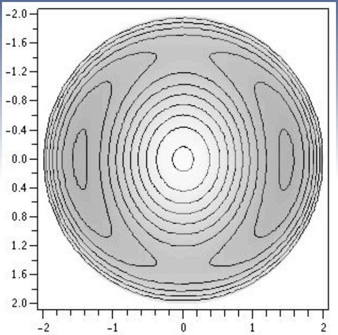
$$h : so(3)^*_{\mu_0} \times T_{\mu_0}\mathcal{O}_{\mu_0} \simeq so(3)^* \to \mathbb{R}\,, \qquad h = h(\nu, \eta) = h(\mu)$$

$$\xi_z = \partial_\nu h\Big|_{(\nu=\nu_0\,,\,\eta(t))}$$
$$\dot{\nu} = 0 \implies \nu = const. = \nu_0 \implies h = h(\eta\,;\nu_0)$$
$$\dot{\eta} = -\frac{1}{\mu_0}\mathbb{J}\,\partial_\eta h$$

On $\mathcal{O}_{\mu_0}$ is just a one degree of freedom system.



(a) 3-d view



(b) top view

$$h(\eta_x, \eta_y; \nu_0) = \frac{1}{2}\mu_0(\mu_0+\nu_0)\left(1 - \frac{\mu_0}{4(\mu_0+\nu_0)}\right)\left(\eta_x^2+\eta_y^2\right)\left(\frac{\eta_y^2}{\mathbb{I}_1}+\frac{\eta_x^2}{\mathbb{I}_2}\right) + \frac{(\mu_0+\nu_0)^2}{2\mathbb{I}_3}\left(\eta_x^2+\eta_y^2\right)$$

Coupled systems (e.g. N-body problems)

Applying a slice theorem $\implies S \overset{loc.}{\simeq} Q/SO(3)$ shape space (or internal space)

$$SO(3) \times S \overset{loc.}{\simeq} Q$$

$$SO(3) \times S \overset{loc.}{\simeq} Q \Rightarrow \ \dots \ \Rightarrow T^*SO(3) \times T^*S \overset{loc.}{\simeq} T^*Q$$

"Body" coordinates

$$(A, \mu, (s, \sigma)) \in SO(3) \times so^*(3) \times T^*S \simeq T^*SO(3) \times T^*S$$

Symplectic slice coordinates: $(A, \mu, (s, \sigma)) \to (R, \nu, \eta, (s, \sigma))$

$$(R,\nu,\eta,(s,\sigma)) \in \left(SO(3)\times so(3)^*_{\mu_0}\times T_{\mu_0}\mathcal{O}_{\mu_0}\right)_{\Omega_Y}\times T^*S_{\Omega_{can}} \overset{loc.}{\simeq} T^*Q_{\Omega_{can}}$$

$$\dot{R} = R\left[\begin{pmatrix} 0 & -(\mu_0+\nu) & 0 \\ (\mu_0+\nu) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\left(R^{-1}\frac{\partial H}{\partial R}\right) + \begin{pmatrix} 0 \\ 0 \\ \frac{\partial H}{\partial \nu} \end{pmatrix}\right]$$

$$\dot{\nu} = -\left(R^{-1}\frac{\partial H}{\partial R}\right)_z, \qquad \dot{\eta} = -\frac{1}{\mu_0}\,\mathbb{J}\,\partial_\eta H, \qquad \begin{pmatrix} \dot{s} \\ \dot{\sigma} \end{pmatrix} = \mathbb{J}\begin{pmatrix} \frac{\partial H}{\partial s} \\ \frac{\partial H}{\partial \sigma} \end{pmatrix}$$

If $H(R,\nu,\eta,s,\sigma) \equiv h(\nu,\eta,s,\sigma) \implies$

$$\nu(t) = \nu_0 \text{ and } h = h(\eta,(s,\sigma);\nu_0).$$

Simple mechanical systems

$$H(q,p) = \frac{1}{2} p^t \, \mathbb{K}^{-1}(q) \, p + V(q) \,, \quad (q,p) \in T^*Q$$

$$(A, \mu, s, \sigma) \in SO(3) \times so(3)^* \times T^*S \overset{loc.}{\simeq} T^*Q$$

H invariant $\Rightarrow \mathbb{K}(q) \equiv \mathbb{K}(s)$ and $V(q) \equiv V(s)$

$$\mathbb{K}(s) = \begin{bmatrix} \mathbb{I}(s) & \mathbb{C}(s) \\ \mathbb{C}^T(s) & m(s) \end{bmatrix}$$

Define $\mathbb{A} := \mathbb{I}^{-1}\mathbb{C}$ and $\mathbb{M} := m - \mathbb{C}^T \mathbb{I}^{-1} \mathbb{C}$.

$$h(\mu, s, \sigma) = \frac{1}{2} [\mu \,, s] \begin{bmatrix} \mathbb{I}^{-1} + \mathbb{A}\mathbb{M}^{-1}\mathbb{A}^t & -\mathbb{A}\mathbb{M}^{-1} \\ -\mathbb{M}^{-1}\mathbb{A}^t & \mathbb{A}^t \end{bmatrix} \begin{bmatrix} \mu \\ s \end{bmatrix} + V(s)$$

The rigid body in the full phase space

Option (a): the Serret-Andoyer-Deprit canonical coordinates

Option (b): the parametrization given by the Slice Theorem

Option (a): there is a singularity at $\mu = \mu_3$. Not a problem, it is removable.

G. Benettin & F. Fasso: *Long Term Stability of Proper Rotations of the Perturbed Euler Rigid Body*, Commun. Math Phys. 250, 2004

M.L. Lidov & A.I.Neishtadt: *The method of canonical transformations in problems of the rotation of celestial bodies and Cassini Laws*, Determination of the motion of a spacecraft (in Russian), P.E. Eliasberg, Ed., Moscow: Nauka, 1975

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But a calculation shows that the Serret-Andoyer-Deprit $\equiv$ the slice coordinates :(

Coming back to:

Many general bifurcation and persistence results are done in the framework of the symplectic slice theorem, but we don't have constructive methods for finding these coordinates.

"General" $\rightarrow$ most of the dynamical results are for free and proper actions.

Can we say anything constructive in this more general case?

Compact G which acts freely and properly on a T^*Q

We want a constructive method to find a G-equivariant symplectic diffeomorphism

$$\phi : \left(G \times \mathfrak{g}^*_{\mu_0} \times \mathfrak{g}^{\perp}_{\mu_0}, \Omega_Y \right) \longrightarrow \left(G \times \mathfrak{g}^*, \Omega_{can} \right) ,$$
$$\text{such that } (e, 0, 0) \to (e, \mu_0)$$

Lucky to find the *tube* ϕ in general. $SO(3)$ is quite special and no wonder the calculations lead to the regularized Serret-Andoyer-Deprit coordinates.

The key relation for finding the "tube" ϕ

$$\phi : \left(G \times \mathfrak{g}^*_{\mu_0} \times \mathfrak{g}^{\perp}_{\mu_0}, \Omega_Y \right) \longrightarrow (G \times \mathfrak{g}^*, \Omega_{can}) \ ,$$

such that $(e, 0, 0) \to (e, \mu_0)$

$$\phi^* \Omega_{can} = \Omega_Y \Rightarrow \ldots \Rightarrow \phi(g, \nu, \eta) = \left(gF(\nu, \eta)^{-1}, \ \mathrm{Ad}^*{}_{F(\nu,\eta)}(\mu_0 + \nu) \right)$$

for some $F : \mathfrak{g}^*_{\mu_0} \times \mathfrak{g}^{\perp}_{\mu_0} \to G$. Moreover, F must be of the form

$$F(\nu, \eta) = \exp \left(h(\nu, \eta) \frac{\eta}{\|\eta\|} \right)$$

for some $h : \mathfrak{g}^*_{\mu_0} \times \mathfrak{g}^{\perp}_{\mu_0} \to \mathbb{R}$.

Note: if we know F, then we know the "tube" ϕ

$$F(\nu,\eta)=\exp\left(h(\nu,\eta)\,\frac{\eta}{\|\eta\|}\right),\ \ \nu\in\mathfrak{g}^*_{\mu_0}\simeq\mathcal{M}(\mathbb{R}^?)\,,\ \eta\in\mathfrak{g}^\perp_{\mu_0}\simeq\mathcal{M}(\mathbb{R}^?)$$

must satisfy

$$\Big\langle\mu_0+\nu,\Big[F(\nu,\eta)^{-1}\left(DF(\nu,\eta)\cdot(\dot\nu_1,\zeta_1)\right),F(\nu,\eta)^{-1}\left(DF(\nu,\eta)\cdot(\dot\nu_2,\zeta_2)\right)\Big]\Big\rangle$$
$$+\Big\langle\dot\nu_2,F(\nu,\eta)^{-1}\left(DF(\nu,\eta)\cdot(\dot\nu_1,\zeta_1)\right)\Big\rangle$$
$$-\Big\langle\dot\nu_1,F(\nu,\eta)^{-1}\left(DF(\nu,\eta)\cdot(\dot\nu_2,\zeta_2)\right)\Big\rangle=\langle\mu_0,[\zeta_1,\zeta_2]\rangle\,.$$

One may compute: $DF(\nu,\eta)\Big|_{(0,0)}$. Then take the derivative of the above and compute $D^2F(\nu,\eta)\Big|_{(0,0)}$, and so forth...

Note: if we know F, then we know the “tube” ϕ

Unlikely to find F globally, but one can calculate the its derivatives at $(0,0)$.

$$\phi : \left(G \times \mathfrak{g}^*_{\mu_0} \times \mathfrak{g}^{\perp}_{\mu_0}, \Omega_Y \right) \longrightarrow (G \times \mathfrak{g}^*, \Omega_{can}) ,$$

$$(e,0,0) \to \phi(e,0,0) = (e, \mu_0)$$

$$\phi(g,\nu,\eta) = \left(gF(\nu,\eta)^{-1}, \mathrm{Ad}^*{}_{F(\nu,\eta)}(\mu_0 + \nu) \right)$$

...and so we know the derivatives of ϕ at the base point (i.e., at the relative equilibrium).

The Poincaré-Birkhoff normal forms

is a method based on canonical changes of coordinates which are applied to term of a *truncated* Taylor expansion at the equilibrium of the Hamiltonian.

At each step $H \to \hat{H}$ the k-jet of $\hat{H}$ at the equilibrium becomes

$$j^k \hat{H} = \hat{H}^{(2)} + \hat{H}^{(3)} + \ldots + \hat{H}^{(k)}$$

so that $\{H^{(2)}, \hat{H}^{(i)}\} = 0 \quad \forall\ i = 2, 3, \ldots k$.

$$H_{\text{tube}}(R, \nu, \eta) = (H \circ \phi)(A, \mu)$$

Knowing the derivatives at $(e, 0, 0)$ of the tube ϕ (*and these can be computed for any group !*) is sufficient for calculating the normal form near a relative equilibrium.

Conclusions

We “re-discovered the wheel” when about spatial rotations. Oh, well...However,

- for $SO(3)$-symmetric systems we understand how nice and useful the Serret-Andoyer-Deprit coordinates are. In particular, they allow the studying of perturbations of the spatial N-body problem in coordinates which are

a) canonical in the full phase-space, and

b) “split” the reduced dynamics into rigid-body-like and internal (vibrational) parts at a chosen (non-isotropic) point;

Conclusions

We “re-discovered the wheel” when about spatial rotations. Oh, well...However,

- for $SO(3)$-symmetric systems we understand how nice and useful the Serret-Andoyer-Deprit coordinates are. In particular, they allow the studying of perturbations of the spatial N-body problem in coordinates which are

a) canonical in the full phase-space, and

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- for the case of free and proper symmetries (the group does not have to be compact) we do provide an iterative methodology to compute a normal form of the Hamiltonian near a relative equilibrium.

Thank you for your attention!

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