Remarks on Lagrangian singularities, caustics, minimum distance lines

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June 2014
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OUTLINE

- Lagrangian singularities, caustics.
- Wavefront sets and singularities.
- Counting singularities. Indices.
- Minimum distance lines.
- Minimization and holonomic constraints, a simpler picture
- Reduction, and minimization
- dynamics and rotation of Lagrange planes

An example

- Consider the problem of minimizing (Euclidean) distance from an elliptical boundary to a point interior to the ellipse.

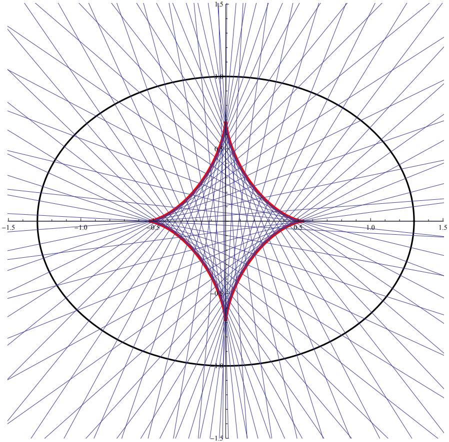


Figure: Minimum distance, caustic of the family of rays

- We use Euclidean geodesic segments to measure this distance function. The distance function is not smooth. Ridge line (of the graph of distance function) along the line segment joining the focii.

- The family of such geodesic segments develops caustic singularities along certain curves (caustics).
- The straight lines, normal to the ellipse, parameterized proportional to arclength, form a (1D) Lagrangian family $y(x, \theta)$. . The caustic curve comes from the condtion $\frac{\partial y}{\partial \theta} = 0$.
- In addition to the distance function, we consider the time function (contstant speed).

- The set of points which are equidistant (isochrones) from the boundary (called wavefront sets) are level sets of (local) solutions to the Hamilton-Jacobi equation $\|\nabla S\|^2 = 1$. $(H(q, \frac{\partial S}{\partial q}) = h)$.

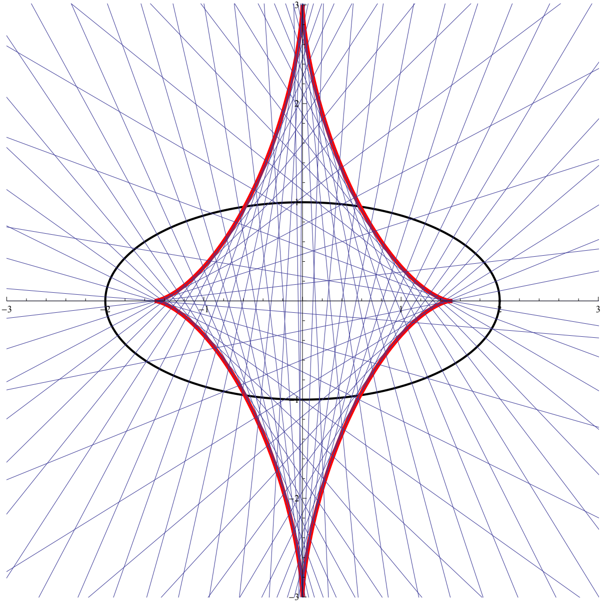


Figure: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. (ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}} = \left(a^2 - b^2\right)^{\frac{2}{3}}$.

Hamiltonian dynamics

$$\dot{z} = X_H(z) = J\nabla H(z), \quad J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$$

$$H : T^*\mathcal{X} \to \mathbb{R}, \quad \frac{\partial^2 H}{\partial p^2} > 0.$$

Hamiltonian flow $\phi_t : T^*\mathcal{X} \to T^*\mathcal{X}$,

periodicity $\phi_T z = z$

Maslov class of Lagrangian submanifolds

An n-dimensional subspace λ of a 2n-dimensional symplectic space $\mathbb{R}^{2n}$ is *Lagrangian* if

$$\omega|_\lambda = 0, \text{ where } \omega = \sum dq_i \wedge dp_i$$

An n-dimensional submanifold $i : \mathcal{L} \to T^*\mathcal{X}$ is *Lagrangian* if $i^*\omega = 0$.
Lagrangian singularities of the projection

$$\pi : T^*\mathcal{X} \to \mathcal{X}, \quad \pi|_{\mathcal{L}} : \mathcal{L} \to \mathcal{X}.$$

the linearization of the projection restricted to $\mathcal{L}$, $d_z\pi|_{\mathcal{L}}$ is not surjective at $z \in \mathcal{L}$.

$$\det|\xi_1, \ldots, \xi_n| = 0, \quad \xi_i = d\pi\zeta_i, \quad \langle \zeta_1, \ldots, \zeta_n \rangle = T_z\mathcal{L}.$$

The *Maslov cycle* is the locus $\Gamma \subset \mathcal{L}$ of such singular points.
If $\{z(t)\} \subset \mathcal{L}$ is closed curve, we count the number (algebraic) of intersections with the singular cycle Γ in $[0, T]$

$$\begin{aligned} [T_{z(t)}\mathcal{L}; \Gamma] &= \sum_{0<t\leq T} \pm \mathrm{codim}\ \left(d\pi T_{z(t)}\mathcal{L}\right) \\ &= \sum_{0<t\leq T} \pm \dim\ \left(T_{z(t)}\mathcal{L} \cap \mathcal{V}\right) \end{aligned}$$

$$\mathcal{V}|_z = \ker\ d\pi|_z$$

jacobi metric for fixed energy

- $H(q,p) = \frac{1}{2}K(p,p) + V(q)$, K is dual metric for Riemannian $\|\cdot\|_R^2$.
- Hamilton Jacobi eq $\frac{1}{2}K\left(\frac{\partial S}{\partial q}\right) + V(q) = h$, or

$$\frac{\frac{1}{2}K\left(\frac{\partial S}{\partial q}\right)}{2(h - V(q))} = \frac{1}{2}$$

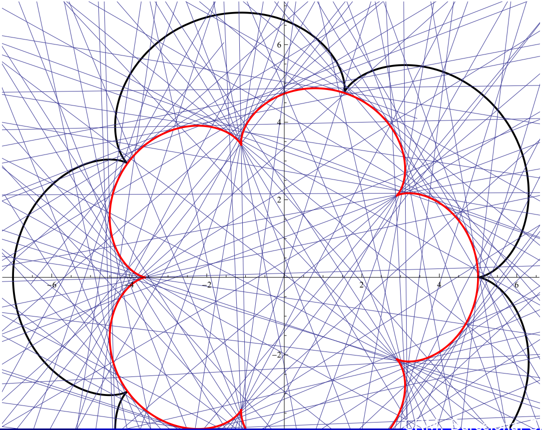
- This is HJ equation for "new Hamiltonian" $\tilde{H}(q,p) = \frac{\frac{1}{2}K(p,p)}{2(h-V(q)}$
- The corresponding Lagrangian gives action of Jacobi metric $L(q,q') = (h - V(q))\|q'\|_R^2$. Cogeodesic equations (Euclidean case)

$$q' = \frac{p}{2(h - V(q))}, \quad p' = \frac{-L^2 DV(q)}{2(h - V(q))}$$

caustics and wavefront sets

- Giiven convex hamiltonian H(q,p), we construct the Jacobi metric action measuring minimum distance to the boundary

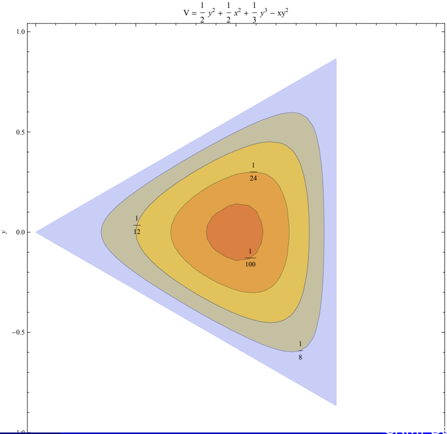
$$\gamma \to \int_0^1 (h - V(\gamma))\|\gamma'\|_R^2 ds$$



Wavefront sets in jacobi metric

$$s = \int_0^{g(s,z)} 2\left(h - V\left(\pi\phi_t z\right)\right) dt,$$

$$M_i(s) = \pi\phi_{g(s,z)}(z), \quad z = (x, 0) \in M_i \subset \partial V^h$$



minimum distance lines

- Wavefront sets key to understanding minimization and stability/instability.

- Normal bundle of wavefront sets
$\mathcal{M}(s) = \{(p, q) \in H^{-1}(h) \mid p(T_q M(s)) = 0.\}$

$$\text{index } \pi z = \sum_{0<s\leq 1} \text{codim } \left(d\pi T_{z(s)}\mathcal{L}\right) + \text{index } \omega(T\mathcal{M}_0(1), T\mathcal{M}_1(0))$$

- **Theorem**: minimium distance lines for $H = \frac{1}{2}K + V$ are always unstable and generically hyperbolic.
- Not true without restrictions, for example closed minimizing geodesics are not always unstable.

hip hop symmetries

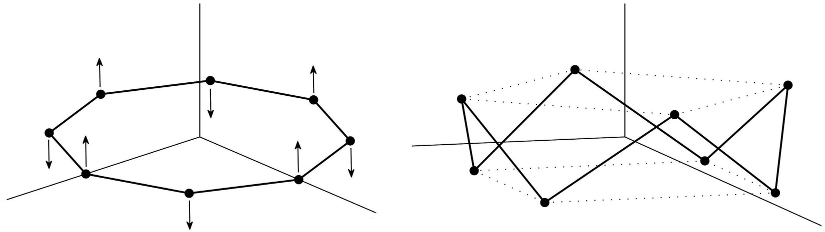


Figure: Qualitative diagram of the hip-hop configuration for eight masses.

Chenciner-Venturelli, Terracini- Venturelli (2001, 2007) existence

$$\mathcal{A}_{4T}(\widehat{q}) = \inf_{q \in \Lambda_{\mathbb{Z}_{2N}}} \mathcal{A}_{4T}(q), \tag{1}$$

$$\Lambda_{\mathbb{Z}_{2N}} = \left\{ q \in H^1(\mathbb{R}/4T\mathbb{Z}, \mathcal{C}) \;\middle|\; q(t+2T) = -q(t) \right\}.$$

Lewis, O. Buono (2013) show that this family of orbits are minimum distance lines on reduced energy surface.

- Given four masses, $m_1 = m_2, \quad m_3 = m_4, \quad \sum_{i=1,4} m_i q_i = 0.$

$$\mathcal{M} = \{q = (q_1, \dots, q_4) \| q_3 - q_1 = q_2 - q_4, q_4 - q_1 = q_2 - q_3\}$$

- $T\mathcal{M}$ is invariant under the flow of Newtonian four body problem.
- Study homographic rhombus solutions (diagonals perpendicular)

$$\mathcal{M}' = \{q \in \mathcal{M} | q_3 - q_1 \perp q_4 - q_2\}$$

- N-body Newtonian systems, collisions have finite action, so collision orbits must be considered when minimizing the action.
- In this setting, we impose holonomic constraints which force a rhomboid configuration.
- On the constraint set minimization for parameterized curves of period T, the action can be shown to be minimizing along the entire family of rhombus homographic solutions.

$$\mathcal{M}' = \{q \in \mathcal{M} | q_3 - q_1 \perp q_4 - q_2\}, \quad \tilde{L} = L|_{T\mathcal{M}'}$$

$$\int^{T}$$

reduction $e > 0$

- As mentioned, the dynamics on the 3D.F. constraint $T\mathcal{M}'$ are determined by restriction $\tilde{L} = L|_{T\mathcal{M}'}$. For the following, we set eccentricity $e > 0$.
- To study stability, we restrict to angular momentum level set, and remove the angle from rotation symmetry (symplectic reduction).

  This brings us to a 2D.F. subsystem on $T^*(\mathcal{M}'/S^1) = J^{-1}(\Theta)/S^1$ where $\Theta = 2(m_1\alpha^2 + m_3\beta^2)\mu$.
- Important to recognize at this juncture, that the minimizing property of the rhombus homographic solution may disappear upon reduction. We do not use the reduced variational principle to study the reduced solutions.

Lagrange planes

- On the reduced constrained system, along the reduced rhomboid solutions $\gamma(t) = \phi_t(x, p)$, we study the linearized equations, especially the movement of Lagrangian isoenergetic subspaces.

  These are 2D planes of tangent variations in the reduced phase space, maximally isotropic with respect to the symplectic form.

  These planes contain the (reduced) flow direction X_H together with one additional isoenergetic direction of variations.

$$(\xi(t), \eta(t)) \in T_{\gamma(t)} T^*\mathcal{M}'/S^1, \qquad (\xi(t), \eta(t)) = d_{(x,p)}\phi_t(\xi_0, \eta_0).$$

- We are interested in the time evolution of these planes as measured by their rotation around the vertical distribution $V = \ker d\pi, \;\; \pi : T^*(\mathcal{M}') \to \mathcal{M}'$.

focal points

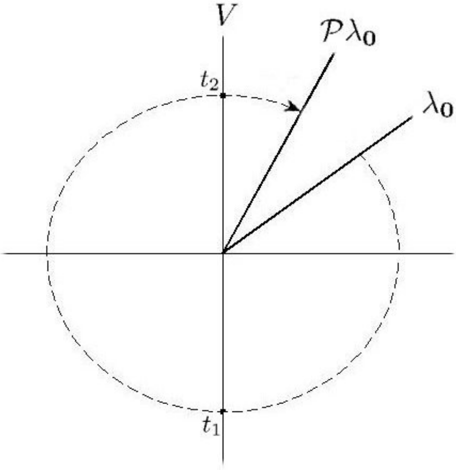


Figure: Focal points of λ_0 occur at $t = t_i$, $\quad d\phi_t\lambda_0 \bigcap V \neq 0$.

Jacobi fields

In order to study the stability properties of the orbit, we must consider the space $\mathcal{J}$ of *Jacobi fields* along $\gamma(t)$ which are given by

$$(\xi(t), \eta(t)) \in T_{\gamma(t)} T^*\mathcal{M}'/S^1, \qquad (\xi(t), \eta(t)) = d_{(x,p)}\phi_t(\xi_0, \eta_0).$$

The mapping $\mathcal{P}$ acts on $\mathcal{J}$ by advancing the initial conditions of a Jacobi field through the period T,

$$\mathcal{P}(\xi(0), \eta(0)) = (\xi(T), \eta(T)). \tag{2}$$

The periodic orbit $\gamma(t)$ is *non-degenerate* if $\ker(\mathcal{P} - Id) = X_H(x, p)$.

isoenergetic Jacobi fields

Let the energy level be denoted $\Sigma = H^{-1}(h)$.

Proposition

Assume that the rhombus orbit $\gamma(t)$ is non-degenerate. The Jacobi fields along $\gamma(t)$ are a four dimensional linear symplectic space, which may be decomposed as $\mathcal{J} = \lambda_1 \oplus \lambda_2$, where λ_1 is a symplectic subspace which contains the flow direction X_H , and λ_2 is a complimentary symplectic subspace of transverse Jacobi fields which are everywhere parallel to the energy surface Σ. λ_1 and λ_2 are both invariant under the Poincaré map. Moreover, $\mathcal{J}$ contains a three dimensional subspace $\mathcal{W}$, consisting of Jacobi fields which satisfy the periodic boundary conditions $\xi(0) = \xi(T)$. $\mathcal{W}$ restricted to the energy surface is two dimensional Lagrange plane Λ containing the flow direction X_H.

- $\mathcal{W}$ satisfies the inclusion relation $(\mathcal{P} - Id)\mathcal{W} \subseteq V$. Therefore dim $\mathcal{W} \leq 3$ with equality when $(\mathcal{P} - Id)|_{\mathcal{W}}$ is onto V.
- Choose the time $t = 0$ to coincide with perihelion along the elliptical orbits, then $d\pi X_H(\gamma(0)) = 0 \;\; X_H(\gamma(0)) \in V$.
  It follows that $\mathcal{P}$ is of the form

$$\mathcal{P} = \left(\begin{array}{cc|c} 1 & 1 & \multirow{2}{*}{0} \\ 0 & 1 & \\ \hline \multicolumn{2}{c|}{0} & \hat{\mathcal{P}} \end{array} \right),$$

  where $\hat{\mathcal{P}} = d\hat{\phi}_T$ is the Poincaré map restricted to $T\Sigma / X_H$, and has no $+1$ eigenvalues.
- V/X_H is one dimensional. $(\hat{\mathcal{P}} - Id)$ is invertible, therefore $\mathcal{W}/X_H = (\hat{\mathcal{P}} - Id)^{-1} V/X_H = [\lambda] \neq 0$. Hence $\mathcal{W} = \mathrm{span}\{X_H, \nu, \lambda\}$ and $\Lambda = \mathcal{W}|_{T\Sigma} = \mathrm{span}\{X_H, \lambda\}$.

more reduction

We are going to study the invariant Lagrangian curve described in Proposition 1: $\Lambda = \mathcal{W}|_{T\Sigma} = \mathrm{span}\{X_H, \lambda\}$.

$$\lambda_t = \left\{ (\xi(t), \eta(t)) \;\middle|\; \xi(0) = \xi(T),\ dH(\xi(t), \eta(t)) = 0 \right\}.$$

Final reduction to consider, removing the flow direction $X_H(\gamma(t))$ from the tangent spaces $T_{\gamma(t)} H^{-1}(h)$.
Consider the corresponding sub-bundle $(T\Sigma / X_H)$, along the orbit $\gamma(t)$.

Lemma

The quotient space $T\Sigma / X_H(\gamma((t))$ is symplectic, with symplectic form given by $\hat{\omega}([u],[v]) = \omega(u,v)$. The linearized Hamiltonian flow $d\phi_t$ projects to a Hamiltonian flow on the sub-bundle $(T\Sigma / X_H)$, defined by $\hat{\phi}_t[v] = [d\phi_t v]$.

Second variation and $\omega(\lambda_0, \mathcal{P}\lambda_0)$

- The minimizing rhombus curve denoted $\hat{q}(t)$. The variational problem is called *non-degenerate* if $\ker(\delta^2\mathcal{A}(\hat{q})) = d\pi X_H$.
- The second variation along $\hat{q}(t)$ evaluated in the direction of the field λ_t:

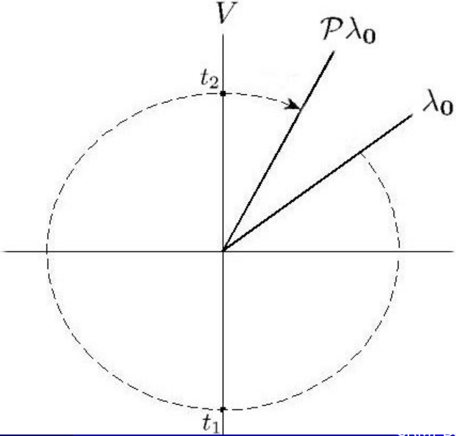
$$\begin{aligned} \delta^2\mathcal{A}(\hat{q})\cdot\xi &= \langle\eta(t),\xi(t)\rangle|_0^T \\ &= \langle\eta(T),\xi(T)\rangle - \langle\eta(0),\xi(0)\rangle \\ &= \langle\eta(T)-\eta(0),\xi(0)\rangle \ > \ 0, \end{aligned}$$

  where the last equality follows since the Jacobi fields $(\xi(t),\eta(t))$ in $\mathcal{W}$ satisfy the periodic boundary conditions $\xi(T)=\xi(0)$.
- We have shown that $\omega(\lambda_0,\mathcal{P}\lambda_0) > 0$, due to minimization in the *nonreduced setting*.

no focal point condition for λ_t

Proposition

Assume that the variational problem is non-degenerate, then the curve λ_t *is focal point free on the interval* $[0, T]$.



hyperbolicty of the reduced rhombus orbit on Σ

Proposition

The Lagrange planes $\mathcal{P}^n\lambda_0$ have no focal points in the interval $[0, T]$.

Proposition

The Lagrange plane λ_0 of isoenergetic variations transverse to the flow is focal point free on the interval $0 \leq t < \infty$.

Theorem

The reduced rhombus orbit is hyperbolic in the reduced energy manifold when it is not degenerate.

Instability of the orbits in the parallegram 4 body problem

- We have shown above, that when we minimize in the holonomically constrained system (rhomboid loops), the resulting orbits $\gamma(t)$ are unstable. The resulting invariant Lagrangian submanifolds contain asymptotic orbits.

- These same orbits $\gamma(t)$ are also orbits of the unconstrained parallelogram four body problem.

- The relation of the constrained dynamics to the unconstrained dynamics is one of projection (at least locally), where the unconstrained dynamics are projected along coordinate directions normal to the constraint set.

- It follows that the local stable and unstable manifolds in the constraint set are the projections of asymptotic orbits in the unconstrained system.

Thank you !