Normally Stable Hamiltonian Systems

Ken Meyer, Jesús Palacián, Patricia Yanguas

University of Cincinnati, Universidad Pública de Navarra

Consider a Hamiltonian system with an equilibrium at the origin

$$\mathcal{H}(z) = \mathbb{H}(z) + \mathcal{K}(z), \qquad z \in \mathbb{R}^{2n}$$

were the quadratic system is

$$\mathbb{H}(z) = \frac{1}{2} z^T S z,$$

and the linear Hamiltonian system is

$$\dot{z} = Az, \qquad A = JS$$

and $\mathcal{K}(z)$ is the higher order terms.

Assume $\mathcal{H}$ is smooth and $\mathbb{H}$ is positive definite then:

The origin is stable. (Dirichlet, 1846)

There are n periodic solutions on each small level set $\mathcal{H} = \epsilon > 0$.
(Weinstein, 1973)

Assume $\mathcal{H}$ is smooth and $\mathbb{H}$ satisfies the MWC, (Moser-Weinstein Condition).

There are n periodic solutions near the origin on the level sets $\mathcal{H} = \pm\epsilon$. (Moser, 1976)

What is the MWC? How can we use it?

Linear Hamiltonian Systems:

$$\textbf{Ln}: \qquad 2\mathbb{H}(z) = z^T S z, \qquad \dot{z} = Az, \qquad A = JS$$

System **Ln** is *stable* if all solutions are bounded for all $t \in \mathbb{R}$.

System **Ln** satisfies the PIDC if all eigenvalues of A are pure imaginary and A is diagonalizable.

Theorem: *System* **Ln** *is stable iff it satisfies PIDC.*

System **Ln** is *parametrically stable* if it and all sufficiently small linear Hamiltonian perturbations of it are stable.

Define $\eta(\lambda) = \text{kernel } (A - \lambda I)$

Let A have distinct eigenvalues $\pm\beta_1 i, \dots, \pm\beta_s i$, $1 \le s \le n$.
$\mathbb{W}_j = \eta(+\beta_j i) \oplus \eta(-\beta_j i)$ is the complexification of a real space $\mathbb{V}_j$.
Let $\mathbb{H}_j$ be the restriction of $\mathbb{H}$ to $\mathbb{V}_j$.

System **Ln** satisfies the *Krein–Gel'fand-Lidskii condition*, KGLC, if A is nonsingular, A is stable, and the Hamiltonian $\mathbb{H}_j$ is positive or negative definite for each $j = 1, \dots s$.

Theorem: *System* **Ln** *is parametrically stable iff KGLC holds.*

Group the eigenvalues of A as follows:

$$\begin{array}{l} \pm ik_{11}\omega_1, \pm ik_{12}\omega_1, \ldots, \pm ik_{1s_1}\omega_1 \\ \ldots \\ \pm ik_{r1}\omega_r, \pm ik_{r2}\omega_r, \ldots, \pm ik_{rs_r}\omega_r \end{array}$$

$\omega_1, \ldots, \omega_r$ are rationally independent and $k_{11} \ldots k_{rs_r} \in \mathbb{Z} \backslash \{0\}$.

$\mathbb{W}_j = [\eta(ik_{j1}\omega_j) \oplus \eta(-ik_{j1}\omega_j)] \oplus \cdots \oplus [\eta(ik_{js_j}\omega_j) \oplus \eta(-ik_{js_j}\omega_j)]$.
$\mathbb{W}_j$ is the complexification of a real $\mathbb{V}_j$.

$$\mathbb{R}^{2n} = \mathbb{V}_1 \oplus \mathbb{V}_2 \oplus \cdots \oplus \mathbb{V}_r$$

Let $\mathbb{H}_j$ be the restriction of $\mathbb{H}$ to $\mathbb{V}_j$.

System **Ln** satisfies the *Moser-Weinstein condition*, MWC, if each $\mathbb{H}_j$ is either positive or negative definite.

$\mathbb{H}$ satisfies PIDC iff there are symplectic coordinates such that

$$\mathbb{H} = \frac{\omega_1}{2}(x_1^2 + y_1^2) + \cdots + \frac{\omega_n}{2}(x_n^2 + y_n^2) = \omega_1 I_1 + \cdots + \omega_n I_n$$

eigenvalues of A: $\pm\omega_1 i, \ldots, \pm\omega_n i$ and
$I_j = \frac{1}{2}(x_j^2 + y_j^2), \phi_j = \tan^{-1} y_j/x_j$ are action-angle variables.

Examples:

$\mathbb{H} = I_1 + I_2$ satisfies KGLC and MWC — positive definite.
$\mathbb{H} = I_1 - I_2$ does not satisfies KGLC or MWC.
$\mathbb{H} = I_1 - 2I_2$ satisfies KGLC but not MWC.
$\mathbb{H} = I_1 + 2I_2 - \sqrt{2}(I_3 + 3I_4)$ satisfies KGLC and MWC.

A formal system $\mathcal{H}_n$ is in *normal form* if

$$\mathcal{H}_n(z) = \mathbb{H}(z) + \bar{\mathcal{H}}(z)$$

with $\bar{\mathcal{H}}(e^{At}z) \equiv \bar{\mathcal{H}}(z)$ or $\{\mathbb{H}, \bar{\mathcal{H}}\} = 0$.

$\mathcal{H}_n$ is *normally stable* if for every $\bar{\mathcal{H}}$ there exists a formal integral

$$\mathcal{L}_n(z) = \mathbb{L}(z) + \mathcal{L}_*(z)$$

where $\mathbb{L}$ is a positive definite quadratic form in z.

Theorem: *$\mathcal{H}_n$ is normally stable iff $\mathbb{H}$ satisfies MWC.*

Consider a real analytic Hamiltonian

$$\mathcal{H}_a(w) = \mathbb{H}(w) + \mathcal{H}^*(w)$$

such that the origin is an equilibrium point.

$\mathcal{H}_a$ is *formally stable* if there exists a formal positive definite integral $\mathcal{L}_f(w)$, i.e., $\{\mathcal{H}_a, \mathcal{L}_f\} = 0$.

Remark: This implies some asymptotic estimates – see Seigel, Moser, Bruno et al.

Corollary: *If $\mathbb{H}$ satisfies MWC then the analytic Hamiltonian system $\mathcal{H}_a$ is formally stable.*

Proof:

MWC implies

$$\mathbb{H} = \omega_1(k_{11}I_{11} + \cdots + k_{1s_1}I_{1s_1}) + \cdots + \omega_r(k_{r1}I_{r1} + \cdots + k_{rs_r}I_{rs_r})$$

were all the ks are positive. Define the positive definite

$$\mathbb{L} = |\omega_1|(k_{11}I_{11} + \cdots + k_{1s_1}I_{1s_1}) + \cdots + |\omega_r|(k_{r1}I_{r1} + \cdots + k_{rs_r}I_{rs_r}).$$

The rational independence of the ωs imply
Lemma : Let

$$\mathbb{T} = cI_{11}^{\alpha_{11}/2} \cdots I_{r\sigma}^{\alpha_{r\sigma}/2} \cos(\Sigma_{j=1}^{r}[\beta_{j1}\phi_{j1} + \cdots + \beta_{j\sigma}\phi_{j\sigma}])$$

be a typical term in the Poisson series for $\bar{\mathcal{H}}$ then

$$k_{j1}\beta_{j1} + k_{j2}\beta_{j2} + \cdots + k_{j\sigma}\beta_{j\sigma} = 0.$$

Using the lemma a direct computation yields

$$\{\mathbb{L}, \mathbb{T}\} = 0.$$

QED

Three Degrees of Freedom: $\mathbb{H} = \omega_1 I_1 + \omega_2 I_2 + \omega_3 I_3$

Case: (i) ω_i's all have the same sign: $\omega_i > 0$
Case (ii) two of one sign and one of other sign:
$\omega_1 < 0, \omega_2 > 0, \omega_3 > 0$

In case (ii) MWC holds iff ω_2/ω_1 and ω_3/ω_1 are irrational.

Application: L_4 in the spatial restricted three body problem.

$\omega_2/\omega_1 = r \in \mathbb{Q}$, $r \in (0,1)$ and $\omega_3/\omega_1 = 1/s \in \mathbb{Q}$, $s \in (1/\sqrt{2}, 1)$

Excluded Values:

$$\mu_r = \frac{1}{2} - \frac{\sqrt{27r^4 + 38r^2 + 27}}{6\sqrt{3}(r^2+1)}, \quad \mu_s = \frac{1}{2} - \frac{\sqrt{48s^4 - 48s^2 + 81}}{18}.$$