Exchange orbits in the general planar five-body problem

Symmetric exchange orbits in the planar N-body problem

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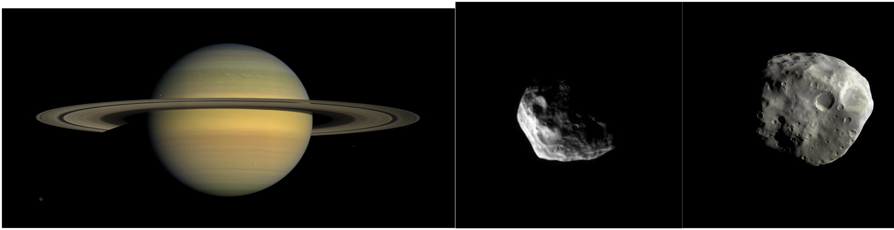
on the 70th birthday of Prof. Clark Robinson

Two co-orbital satellites of Saturn

- Janus and Epimetheus are satellites of Saturn with coplanar orbits that are very close to each other.
- $m_S = 5.69 \times 10^{26} kg$, $R_S = 60268 Km$,
  $m_J = 1.98 \times 10^{18} Kg$, $196 \times 192 \times 150$ and
  $m_E = 5.50 \times 10^{17} Kg$, $144 \times 108 \times 98$ Km .

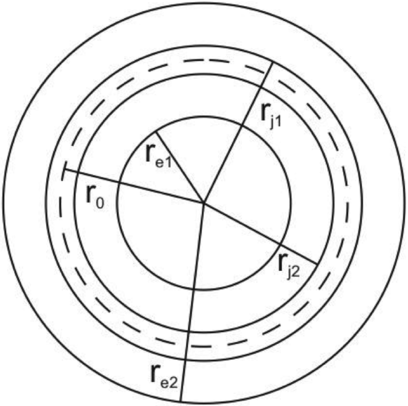


The orbits

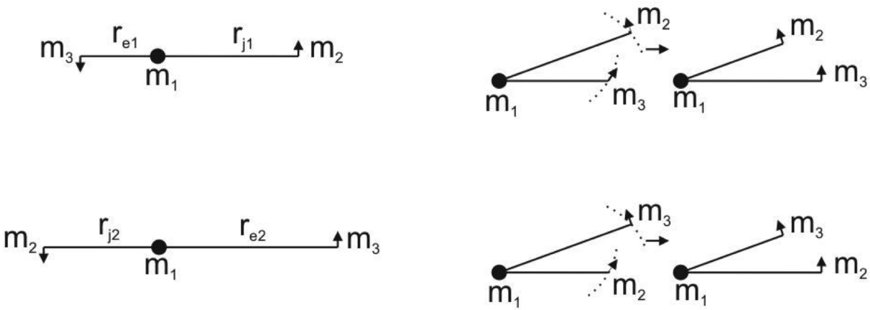
- The motion of both satellites occurs in the same plane.
- Most of the time the satellites do not feel each other (two body Kepler solution).
- According to Kepler's laws the inner satellite goes faster than the outer one and eventually after some full revolutions will catch it (encounter).
- Only when they are close to each other they feel the mutual gravitational attraction (encounter).

The orbits

- $$r = \frac{a(1-e^2)}{1 - e\cos\theta}$$
- $a_J = 151460, e_J = 0.0068$
- $a_E = 151410, e_E = 0.0098$

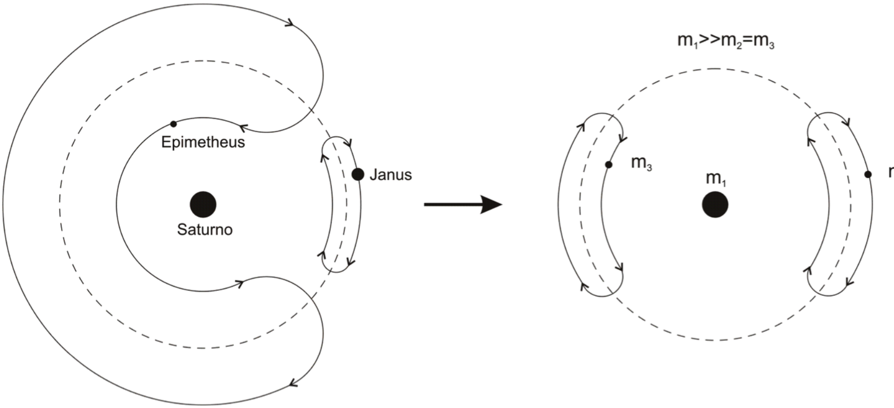


A closer look at the encounter

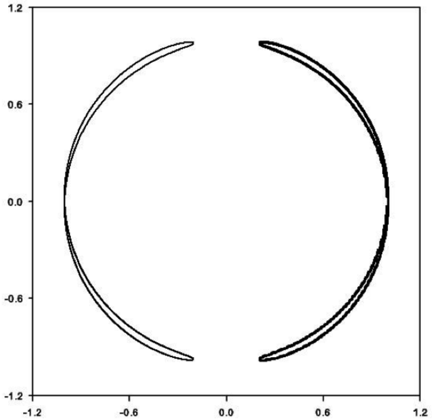


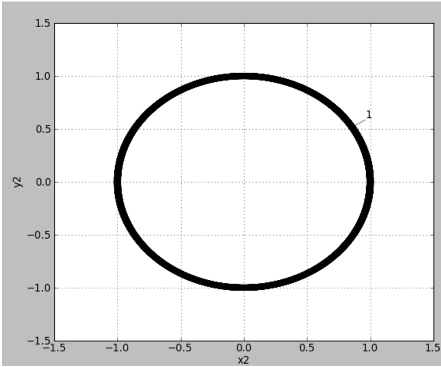
- The inner body **does not overtake** the outer one but they interchange orbits; the inner becomes outer and vice-versa.
- The defining property of a exchange orbit orbit is this no overtaking condition.
- In a **rotating frame** the orbits resemble a **horseshoe**.

Exchange orbits in a rotating frame; scheme

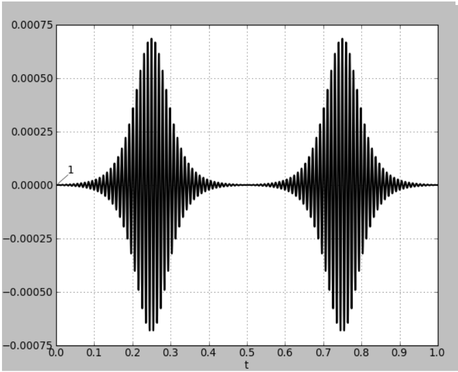


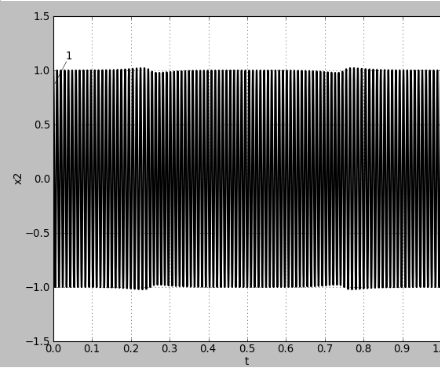
Exchange orbits in a rotating and fixed frame; calculated



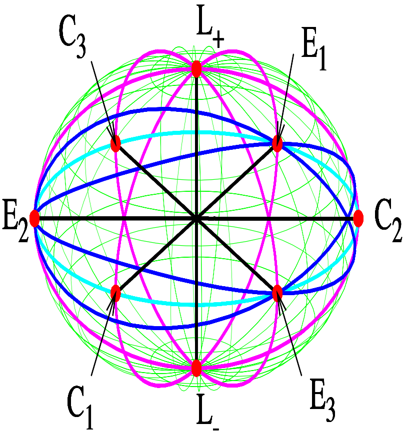


Exchange orbits in a fixed frame; calculated





Exchange orbits in the shape sphere (help welcome!)



Objectives

- Classification of the exchange orbits family of solutions
- Stability and bifurcations
- Connections with other families
- Generalizations to more than 3 bodies $(2k + 1)$

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Tools

- Computation of an initial solution by reduction, shooting methods or Lyapunov Center Theorem.
- Numerical continuation of solutions.
- Computation of Floquet multipliers.

Continuation of solutions in conservative systems

$$\begin{aligned}
\ddot{\mathbf{x}}_1 &= -m_2 \frac{\mathbf{x}_1 - \mathbf{x}_2}{|\mathbf{x}_1 - \mathbf{x}_2|^3} - m_3 \frac{\mathbf{x}_1 - \mathbf{x}_3}{|\mathbf{x}_1 - \mathbf{x}_3|^3} \,, \\
\ddot{\mathbf{x}}_2 &= -m_1 \frac{\mathbf{x}_2 - \mathbf{x}_1}{|\mathbf{x}_1 - \mathbf{x}_2|^3} - m_3 \frac{\mathbf{x}_2 - \mathbf{x}_3}{|\mathbf{x}_2 - \mathbf{x}_3|^3} \,, \\
\ddot{\mathbf{x}}_3 &= -m_2 \frac{\mathbf{x}_3 - \mathbf{x}_2}{|\mathbf{x}_3 - \mathbf{x}_2|^3} - m_1 \frac{\mathbf{x}_3 - \mathbf{x}_1}{|\mathbf{x}_1 - \mathbf{x}_3|^3} \,,
\end{aligned}$$

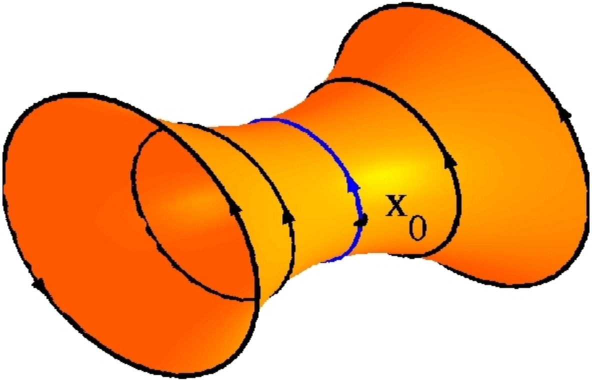
7 (4 in $\mathbb{R}^2$) first integrals H, $\mathbf{P}$ and $\mathbf{J}$, permutations (if $m_2 = m_3$)

$$(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3) \mapsto (\mathbf{q}_1, \mathbf{q}_3, \mathbf{q}_2, \mathbf{p}_1, \mathbf{p}_3, \mathbf{p}_2).$$

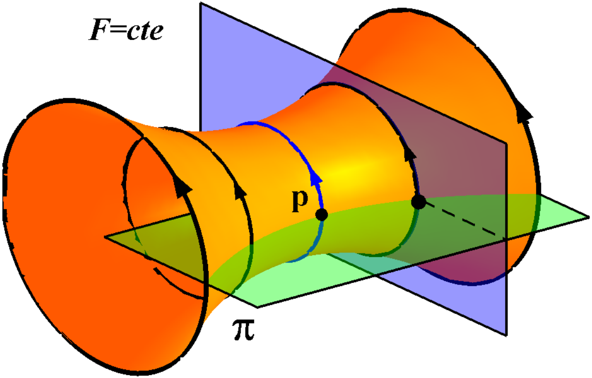
Orbital symmetry (scaling)

$$(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3) \mapsto (\lambda^{-2}\mathbf{q}_1, \lambda^{-2}\mathbf{q}_2, \lambda^{-2}\mathbf{q}_3, \lambda\mathbf{p}_1, \lambda\mathbf{p}_2, \lambda\mathbf{p}_3)$$

Geometrical picture: Cylinder Theorem



Geometrical picture: Reduction



Theory: BVP Formulation

$$u' = T(J\nabla H(u(t)) + \alpha \nabla H(u(t))), \qquad u(1) = u(0). \quad (1)$$

with u, α and T as unknowns. Finding a T-periodic orbit of $u' = J\nabla H(u)$ is equivalent to finding a solution of (1) if $\alpha = 0$. We have to include a phase condition to fix the time origin.

$$(u(0) - u_0(0))^* u_0'(0) = 0. \quad (2)$$

Continuation theorem with 1 conserved quantity

Theorem

Let $u_0(t)$ be a periodic solution with period $0 < T_0 < +\infty$ whose monodromy matrix has 1 as an eigenvalue with **geometric multiplicty one** or **algebraic multipicity two**. Then, there existis a unique branch of solutions of (**??**) and (**??**) in a neighbourhood of $(u, T, \alpha) = (u_0, T_0, 0)$. Moreover, along the branch $\alpha = 0$.

- The proof is a direct application of IFT and the fact that $H(u(t))$ is constant along the periodic orbit.

Generalization to several conserved quantities

- Let $\mathcal{W}_{\mathbf{p}} = \{\nabla F(\mathbf{p}) \,:\, F$ first integral of $\dot{x} = f(x)\}$, $\dim(\mathcal{W}_{\mathbf{p}}) = k$, $\varphi_t(\mathbf{x}, \boldsymbol{\alpha})$ the flow and $\mathrm{orb}_{\varphi}(\mathbf{p})$ the orbit.
- $\dot{x} = f(x) \quad \rightarrow \quad \dot{x} = f(x) + \alpha_1 \nabla F_1(x) + \ldots + \alpha_k \nabla F_k(x),$

Proposition

Let $\mathbf{p} \in \mathbb{R}^n$ s. t. $\mathrm{orb}_{\varphi}(\mathbf{p})$ be $T-$periodic. It holds that $\mathrm{Im}(D\varphi_T(\mathbf{p}) - I) + \mathbb{R}f(\mathbf{p}) \subseteq \mathcal{W}_{\mathbf{p}}^{\perp}$.

General results

Definition (Normal periodic orbit)

Let $\mathbf{p} \in \mathbb{R}^n$ such that the orbit $\mathrm{orb}_\varphi(\mathbf{p})$ is periodic with period $T > 0$ and $\mathbf{p}$ is not an equilibrium of $\dot{\mathbf{z}} = f(\mathbf{z})$. We say that $\mathrm{orb}_\varphi(\mathbf{p})$ is a normal periodic orbit of e $\dot{\mathbf{z}} = f(\mathbf{z})$ if

$$\mathrm{Im}(D\varphi_T(\mathbf{p}) - I) + \mathbb{R}f(\mathbf{p}) = \mathcal{W}_{\mathbf{p}}^{\perp}.$$

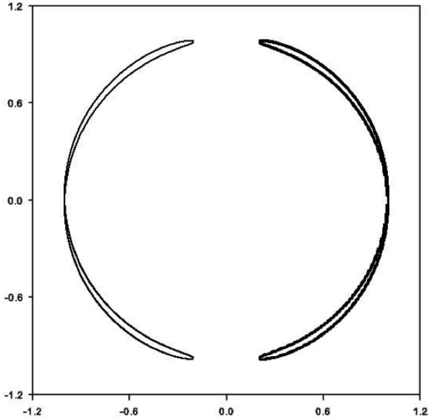
Theorem (Continuation with k conserved quantities)

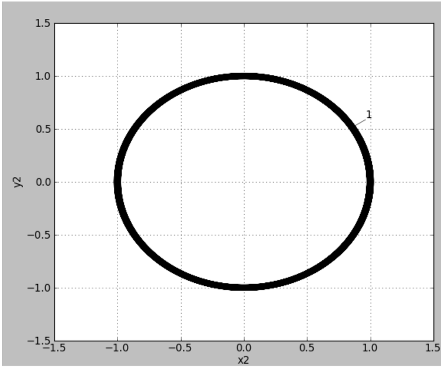
Let $\mathbf{p} \in \mathbb{R}^n$ be a point that generates a normal periodic orbit of $\dot{\mathbf{x}} = f(\mathbf{x})$ with period $T > 0$. Then there exists a neighborhood of $T > 0$ such that the set of points that generate periodic orbits whose period is in that neighborhood of T is locally a submanifold at $\mathbf{p}$.

Numerical Implementation

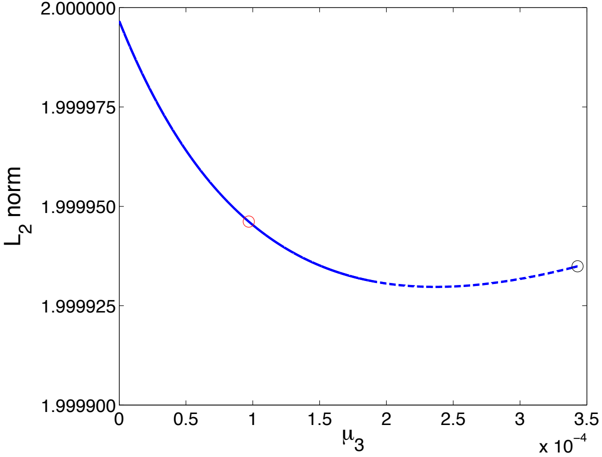
- We make use of the a boundary value based general technique to continue solutions in conservative systems. [Physica D **181** 1 (2003) and Celest. Mech. D. A. **97** 17 (2007)]
- We choose two relevant parameters $\mu_2 = m_2/m_1$ and $\mu_3 = m_3/m_1$
- The initial solution is taken from Bengochea et al Astrophys. Space Sci. **333** 399 (2011) [$\mu_2 = \mu_3 = 3.4 \times 10^{-4}$]
- The orbit includes around 100 revolutions around the planet.
- We can continue the **full** periodic orbit or just an **arc** [periodic and relative periodic orbits] and exploit the reversibility properties.

Initial Orbit

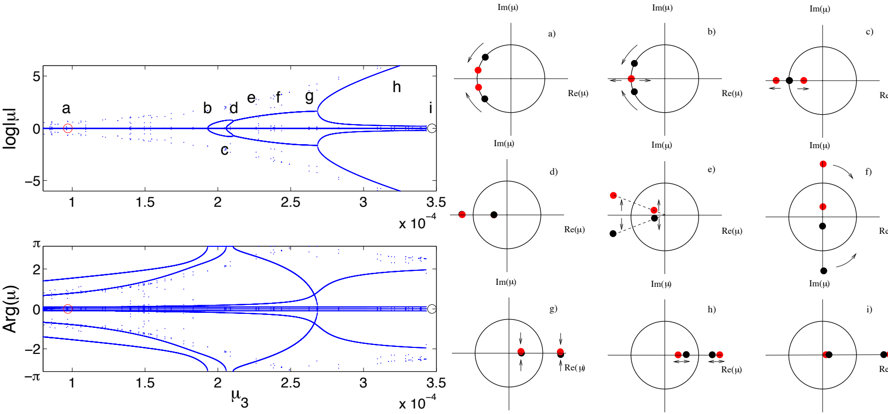




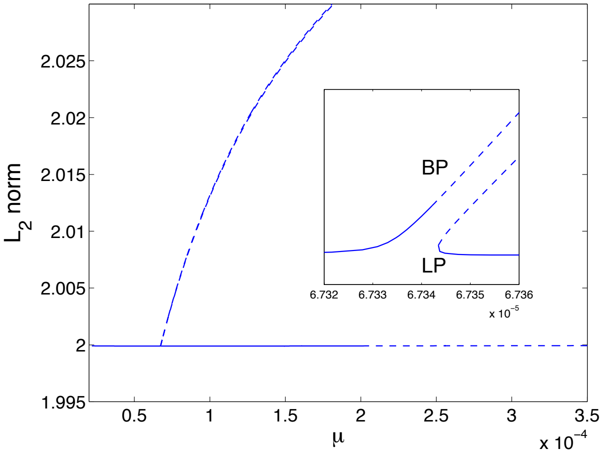
Stability of the exchange orbit $\mu_2 \neq \mu_3$



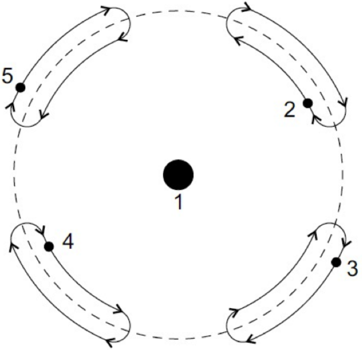
Floquet Multipliers

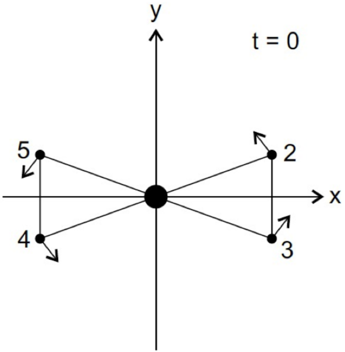


Case $\mu_2 = \mu_3$


L2 norm
2.025
2.02
2.015
2.01
2.005
2
1.995
0.5
1
1.5
2
2.5
3
3.5
μ
x 10−4
BP
LP
6.732
6.733
6.734
6.735
6.736
x 10−5

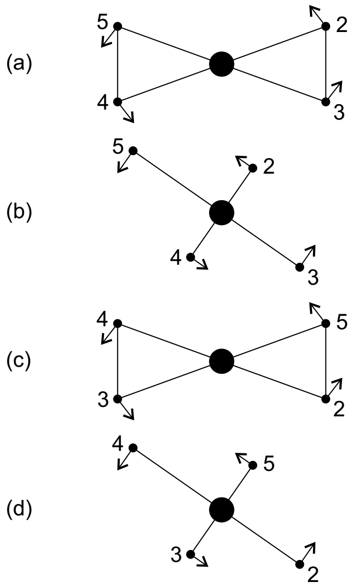
2k+1 exchange orbit solution [hot dog?]





No overtaking condition

5 body exchange orbit



5 body exchange orbit (existence theorem)

Theorem 2 *Consider a solution* $\mathbf{u}(t)$ *of the* $4k+1$*–body problem with* $4k$ *equal masses. Suppose that* $\mathbf{u}$ *is invariant under* Υ_π *and passes trough* RI *at* $t=0$ *and* RO *at* $t=T_0$. *Let* θ *be the angle measured from* $\mathbf{r}_2(0)$ *to* $\mathbf{r}_2(T_0)+\mathbf{r}_3(T_0)$ *in the counterclockwise sense. The orbit is periodic if and only if*

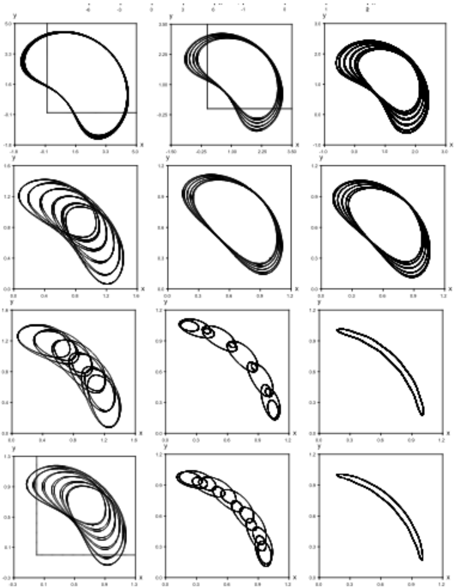
$$\theta = \frac{p}{q}\pi, \text{ for some } p, q \in \mathbb{N}.$$

The period T *of the orbit, supposing* p *and* q *are relative primes, is*

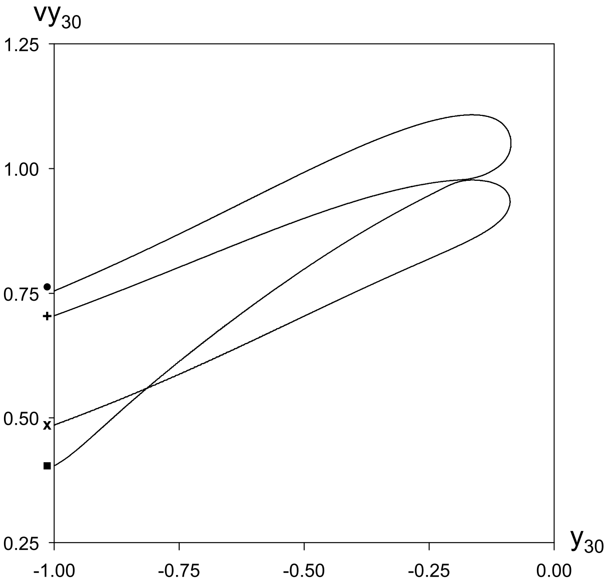
$$T = \begin{cases} qT_\theta, & \text{if } q = 8i-4 \text{ for some } i \in \mathbb{N}, \\ 2qT_\theta, & \text{if } q = 8i \text{ for some } i \in \mathbb{N}, \\ 4qT_\theta, & \text{if } q = 4i-2 \text{ for some } i \in \mathbb{N}, \\ 8qT_\theta, & \text{if is odd.} \end{cases}$$

The theorem also provides a numerical method to compute the initial conditions.

5 body exchange orbits



5 body exchange orbit connected to Euler-like solution



Conclusions

- The numerical continuation of periodic orbits in the three (and 2k+1) body is a challenging but feasible problem.
- The reduction procedure has been used to construct the initial exchange orbit solution and prove its existence.
- However, for the continuation we have made used of the **reversibility properties** or continued the full planar 3 body problem.
- The branching of exchange orbits form the Euler solutions could be proved by continuation on the eccentricity.
- A systematic classification of the exchange orbits families is still pending.
- The 2k+1 body exchange orbit and Saari's conjecture.