Arnold Diffusion in the Elliptic Restricted Three-Body Problem

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Honoring Professor Clark Robinson's 70th birthday

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Main result

- Planar elliptic restricted three-body problem (PERTBP)
  - Two primaries of masses $\mu, 1-\mu$ move on elliptic orbits of eccentricities ε around the center of mass
  - A third, massless particle, moves in the same plane under the gravity of the primaries
  - Model for the motion of a comet in the Sun-Jupiter system $\mu = 0.0009537$, $\varepsilon = 0.048$
- Hamiltonian system

$$H_\varepsilon(\mathbf{x}, t) = H_0(\mathbf{x}) + \varepsilon H_1(\mathbf{x}, t),$$

  where $H_0(\mathbf{x})$ is the Hamiltonian of the planar circular restricted three-body problem (PCRTBP)
- Then, there exist $\varepsilon_0 > 0$ and $\rho > 0$, s.t. for each $0 < \varepsilon < \varepsilon_0$ there exists $\mathbf{x}(t)$ s.t.

$$|H_0(\mathbf{x}(T)) - H_0(\mathbf{x}(0))| > \rho$$

  for some $T > 0$
- Remark: We use a qualitative approach – no diffusion time estimates

Some related works

- Oscillatory motions: [Sitnikov,1960], [Alekseev,1968-1969], [McGehee,1973], [Moser,1973], [Easton,McGehee,1979], [Llibre,Simó,1980], [Robinson,1984], [Martínez,Pinyol,1994], [García, Pérez-Chavela,2000], [Robinson,2008]
- Diffusion in the PERTBP (close to parabolic orbits): [Xia,1993], [Delshams,Kaloshin,de la Rosa,Seara,2014]
- Diffusion in the PERTBR (outer region, inner region): [Fejoz, Guàrdia,Kaloshin,Roldan,2014], [Urschel,Galante,2012]
- Diffusion in the PERTBR (micro-diffusion): [Capinski, Zgliczynski,2011] – near L_1 on an interval of energies of order $\varepsilon^{1/2}$
- Diffusion in the SCRTBP: [Samà,2004], [Delshams,M.G., Roldan,2013]

Relation with the Arnold diffusion problem

- [Arnold,1964]: Given

$$H_\varepsilon(I,\phi) = H_0(I) + \varepsilon H_1(I,\phi),$$

  with $(I,\phi) \in B^n \times \mathbb{T}^n$, $n \geq 3$, then for all sufficiently small ε, and for 'generic' perturbations H_1, the system has trajectories that travel 'arbitrarily far':
  - 'generic' — open and dense / residual / cusp residual in some function space (smooth or analytic)
  - 'arbitrarily far' — $\exists \varepsilon_0 > 0$, $\exists\, \rho > 0$, $\forall \varepsilon \in (0,\varepsilon_0)$, $\exists\, (I(t),\phi(t))$ s.t.

$$\|I(T) - I(0)\| > \rho$$

    for some $T > 0$
- Practical consequence: small, periodic forcing can accumulate to large effects (time as an extra variable)
- For applications: need to deal with given perturbations rather than generic ones

Relation with the Arnold diffusion problem

- Example (a priori unstable system)

$$H_\varepsilon(p,q,I,\phi,t) = \underbrace{h_0(I)}_{\textit{rotator}} + \underbrace{\sum_{i=1}^{n} \pm\left(\frac{1}{2}p_i^2 + \cos(q_i) - 1\right)}_{\textit{penduli}} + \underbrace{\varepsilon H_1(p,q,I,\phi,t)}_{\textit{perturbation}}$$

$(p,q,I,\phi,t) \in \mathbb{R}^n \times \mathbb{T}^n \times \mathbb{R}^d \times \mathbb{T}^d \times \mathbb{T}^1$

- Assume: H_1 periodic in t + generic non-degeneracy conditions
- Then, $\exists \varepsilon_0 > 0$, $\rho > 0$ s.t. $\forall \varepsilon \in (0,\varepsilon_0)$, $\exists \mathbf{x}(t)$, $T > 0$ s.t. $\|I(\mathbf{x}(T)) - I(\mathbf{x}(0))\| > \rho$.
- Some refs: [Delshams,de la Llave,Seara,2000,2006]†, [M.G.,de la Llave,2006]†, [M.G.,Robinson,2007,2009,2012]†, [Delshams,de la Llave,Seara,2013]†, [M.G.,de la Llave,Seara,2014]

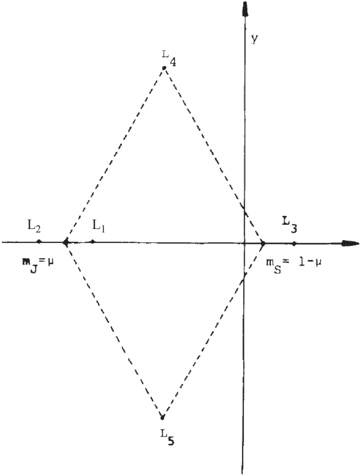
† assume that h_0 satisfies a non-degeneracy condition: $I \mapsto \partial h_0/\partial I$ is a diffeomorphism

Geometric structures in the PCRTBP

- Hamiltonian of the PERTBP

$$H_\varepsilon(\mathbf{x}, t) = H_0(\mathbf{x}) + \varepsilon G(\mathbf{x}, t) + O(\varepsilon^2)$$

- $H_0(x, y, \dot{x}, \dot{y}) = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) - \omega(x, y)$
  Hamiltonian of the PCRTBP
- Equilibrium points:
  L_1, L_2, L_3, L_4, L_5
- Choose range of energy $h \in [h_1, h_2]$,
  near Oterma's energy $h_{Oterma} \approx 1.515$



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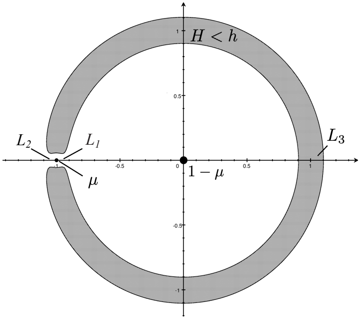
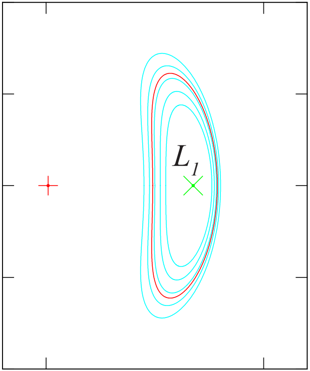


Geometric structures in the PCRTBP

- For each energy level h there exists a periodic orbit λ_h around L_1
- The periodic orbits λ_h possess stable and unstable manifolds $W^s(\lambda_h), W^u(\lambda_h)$ that intersect transversally
- These conditions have been verified numerically



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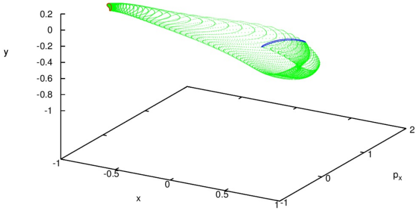

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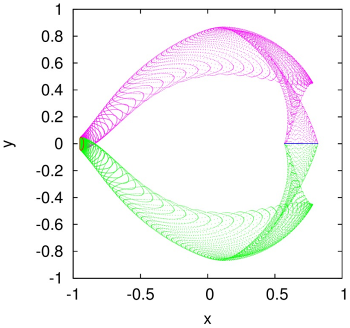
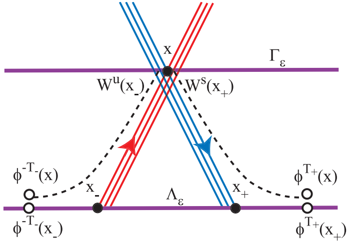


Geometric structures in the PCRTBP

- Define
  $\Lambda_0 = \{\lambda_h \,|\, h \in [h_1, h_2]\} = \{(I, \phi) \,|\, I \in [I(h_1), I(h_2)], \phi \in [0, 2\pi]\}$
- $\Lambda_0 =$ normally hyperbolic invariant manifold (NHIM)
  - $TM = T\Lambda_0 \oplus E^u \oplus E^s$
  - The expansion (contraction) rates of $D\phi_0$ on $T\Lambda_0$ are dominated by the expansion (contraction) rates of $D\phi_0$ on E^u (E^s, resp.)
- $W^u(\Lambda_0)$ ($W^s(\Lambda_0)$, resp.) foliated by $W^u(x)$ ($W^s(x)$, resp.)
- $W^u(\Lambda_0) \pitchfork W^s(\Lambda_0)$ along a homoclinic manifold Γ_0 (strong transversality condition)
- Two dynamics
  - inner dynamics: restricted to Λ_0
  - outer dynamics: along homoclinic orbits
- Remark: we will use very little information on the inner dynamics

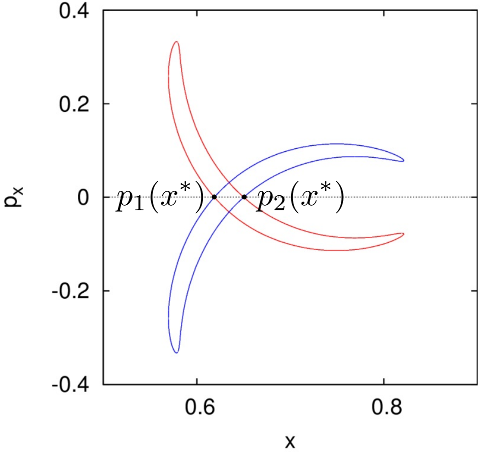
Scattering map

- Scattering map – [García,2000], [Delshams,de la Llave,Seara,2008]
    - encodes information on the outer dynamics
    - $\Omega_0^{\pm}(x) = x^{\pm}$ where $W^{s,u}(x^{\pm}) \cap \Gamma_0 = \{x\}$
    - restrict to a homoclinic channel Γ_0 s.t. $\Omega^{\pm}$ are diffeomorphisms
    - $\sigma_0 = \Omega_0^{+} \circ (\Omega_0^{-})^{-1}$
    - $\sigma_0(x^-) = x^+ \leftrightsquigarrow$ $d(\Phi^{-T_-}(x), \Phi^{-T_-}(x^-)) \to 0,$ $d(\Phi^{T_+}(x), \Phi^{T_+}(x^+)) \to 0,$ as $T_-, T_+ \to \infty$
- Properties
    - σ_0 is symplectic



Scattering map in the PCRTBP

- In the PCRTBP: $\sigma_0(I, \phi) = (I, \phi + \psi)$
- Each homoclinic intersection of the $W^u(\lambda_h), W^s(\lambda_h)$ determines, by continuation, a homoclinic manifold, and, implicitly, a scattering map
- There are many homoclinic intersections $\Rightarrow$ many scattering maps



Geometric structures in the PERTBP

- Hamiltonian in extended phase space: $\tilde{H}_\varepsilon(\mathbf{x}, t, A) = H_\varepsilon(\mathbf{x}, t) + A$
- NHIM: $\Lambda_0 \times \mathbb{T}^1 \rightsquigarrow \tilde{\Lambda}_\varepsilon$
- Scattering map: $\sigma_0 \times \mathrm{id} \rightsquigarrow \tilde{\sigma}_\varepsilon$
- Fix Poincaré section $\Sigma_{t=\tau} = \{(\mathbf{x}, t) \,|\, t = \tau\} \rightsquigarrow$ Poincaré first return map F_ε
  - NHIM: $\Lambda_\varepsilon \rightsquigarrow (F_\varepsilon)_{|\Lambda_\varepsilon}$ – inner dynamics
  - scattering map: σ_ε – outer dynamics

Scattering map in the PERTBP

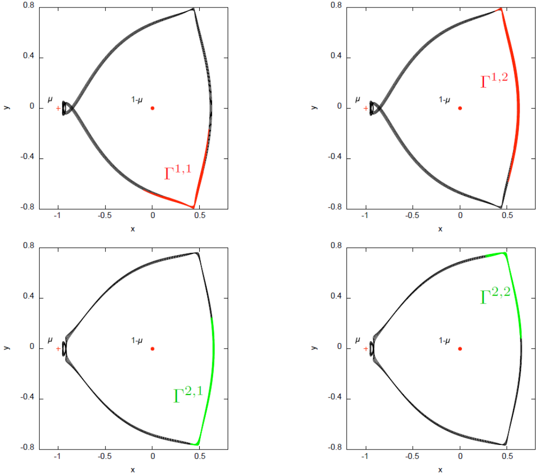
Perturbative computation

- Expansion: $\sigma_\varepsilon = \sigma_0 + \varepsilon J\nabla S_0 \circ \sigma_0 + O(\varepsilon^2)$
- S_0 =convergent integral of G along a homoclinic orbit of the PCRTBP
- $H_\varepsilon = H_0 + \varepsilon G + O(\varepsilon^2)$
- Λ_ε NHIM — parametrization $k_\varepsilon : \Lambda_0 \to \Lambda_\varepsilon$
- $s_\varepsilon = k_\varepsilon^{-1} \circ \sigma_\varepsilon \circ k_\varepsilon$ — acting on Λ_0
- $s_\varepsilon = s_0 + \varepsilon J\nabla S_0 \circ s_0 + O(\varepsilon^2)$ where

$$S_0 = \lim_{T_\pm \to \pm\infty} \int_{-T_-}^{0} \left(G \circ \Phi_{0,t} \circ (\Omega_0^-)^{-1} \circ \sigma_0^{-1} \circ k_0 - G \circ \Phi_{0,t} \circ \sigma_0^{-1} \circ k_0\right) dt + \int_0^{T_+} \left(G \circ \Phi_{0,t} \circ (\Omega_0^+)^{-1} \circ k_0 - G \circ \Phi_{0,t} \circ k_0\right) dt$$

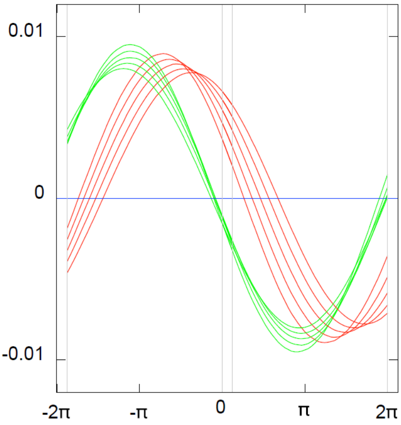
Scattering map in the PERTBP

- There exist, in fact, (at least) four distinct scattering maps $\sigma_\varepsilon^{j,k}$, $j, k = 1, 2$
- They correspond to four homoclinic channels $\Gamma^{j,k}$



Existence of diffusing orbits in the PERTBP

- For $h \in [h_1, h_2]$, $\exists \tau \in [0, 2\pi]$ s.t. $\forall (I, \phi) \in \Lambda_0$, $\exists j_1, k_1, j_2, k_2 \in \{1, 2\}$ s.t.
  $\frac{\partial}{\partial \phi} S_0^{j_1,k_1}(I, \phi) > 0$
  $\frac{\partial}{\partial \phi} S_0^{j_2,k_2}(I, \phi) < 0$
- These conditions have been verified numerically
- Then there exists an $\varepsilon_0 > 0$ s.t. for all $\varepsilon \in (0, \varepsilon_0)$ there exist pseudo-orbits of the type $x_{i+1} = \sigma_\varepsilon^{j_i,k_i}(x_i)$ from $\{I < I(h_1)\}$ to $\{I > I(h_2)\}$ and vice-versa



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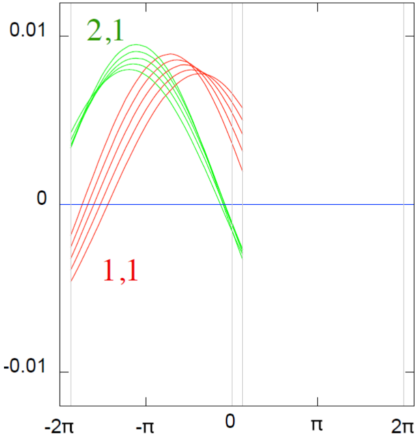

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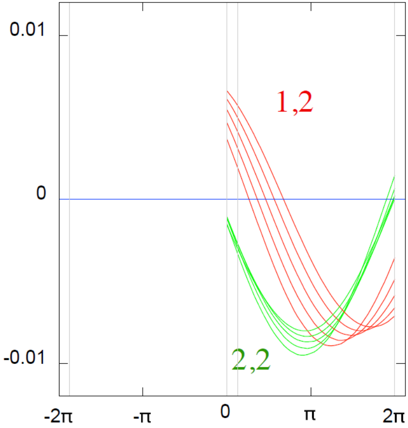
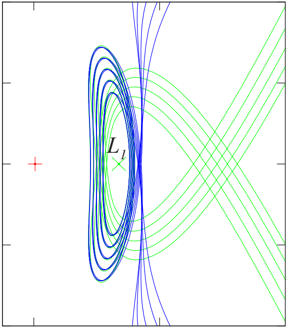


Existence of diffusing orbits in the PERTBP

- Use the Shadowing Lemma below to prove the existence of true orbits from $\{I < I(h_1)\}$ to $\{I > I(h_2)\}$ and vice-versa
- Refs. [Capinski,M.G.,de la Llave,2014]
- Remark: we do not use KAM tori, Aubry-Mather sets, etc., as in other works



Shadowing Lemma for NHIM's

Shadowing Lemma [M.G.,de la Llave,Seara,2014]

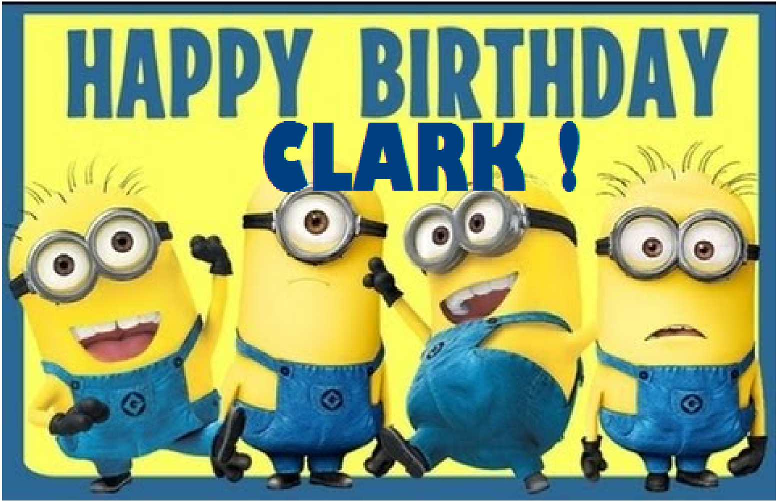
Assume:

- σ symplectic
- $\{x_i\}_{i=0,\dots,n}$ is an orbit of the scattering map in Λ, i.e. $x_{i+1} = \sigma(x_i)$ for all $i = 0, \dots, n-1$
- almost every point in Λ is recurrent for $F_{|\Lambda}$

Then, for every $\delta > 0$ there exist an orbit $z_{i+1} = F^{k_i}(z_i)$ in M, for some $k_i > 0$, s.t. $d(z_i, x_i) < \delta$ for all $i = 0, \dots, n$

- Idea of the proof: apply Poincaré Recurrence Theorem to $F_{|\Lambda}$ to return close to the x_i's
- Similar shadowing lemmas: [M.G.,Robinson,2013], [Delshams,M.G.,Roldan,2013]
- Remark: in the lemmas, one can use several scattering maps rather than a single one

Happy Birthday!



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