Proof of the Saari's conjecture
for the planar 3-body problem
with general masses
under
Newton potential
and a strong force potential

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# Saari's conjecture

If configurational measure $\mu = I^{\alpha/2} U =$ constant,
then the motion is homographic. [Donald Saari 2005].

$$I = \frac{\sum m_i m_j r_{ij}^2}{\sum m_k},\ U = \sum \frac{m_i m_j}{r_{ij}^\alpha}$$

$$r_{ij} = |q_i - q_j|$$

$$L = \frac{1}{2}\sum m_k \left|\frac{dq_k}{dt}\right|^2 + \frac{U}{\alpha}$$

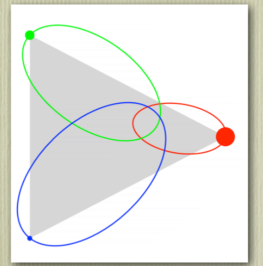
$$\mu = \left(\frac{\sum m_i m_j r_{ij}^2}{\sum_k m_k}\right)^{\alpha/2} \sum \frac{m_i m_j}{r_{ij}^\alpha}: \text{ invariant for } q_k \to \lambda e^{i\theta} q_k.$$

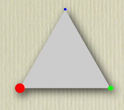
$\mu$ depends only on the shape of the $N$-body configuration.

a motion is homographic $\Rightarrow \mu =$ constant



# Lagrange solution



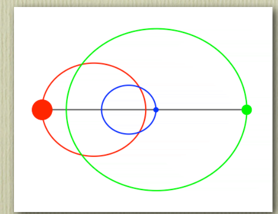


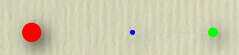
the shape is unchanged,
namely,
the motion is homographic

$$\Rightarrow \mu = \sqrt{\frac{\sum m_i m_j r_{ij}^2}{\sum_k m_k}} \sum \frac{m_i m_j}{r_{ij}} = \text{constant}$$



# Euler solution





the shape is unchanged,
namely,
the motion is homographic

$$\Rightarrow \mu = \sqrt{\frac{\sum m_i m_j r_{ij}^2}{\sum_k m_k}} \sum \frac{m_i m_j}{r_{ij}} = \text{constant}$$



# Saari's conjecture

claims that the reverse is also true.

$\mu =$ constant $\Rightarrow$ The shape is unchanged.

for 3-body problem
$\Rightarrow$ the solutions are only Lagrange & Euler.

We consider

$N = 3$,
$q_k \in \mathbb{R}^2 \sim \mathbb{C}$,
$m_k > 0$,
$\alpha = 1$ (Newton), and $\alpha = 2$ (a strong force).

$$U = \sum_{1 \le i < j \le 3} \frac{m_i m_j}{|q_i - q_j|^\alpha}$$



# Saari's conjecture

⇓

## attention to Shape variable

- what is the Shape variable ?
- equation of motion for the Shape variable ?



# Degrees of freedom

$q_1, q_2, q_3 \in \mathbb{C} \Rightarrow 6$

center of mass $\Rightarrow 2$
size $\Rightarrow 1$
rotation $\Rightarrow 1$

$\therefore$ shape $\Rightarrow 2$



# Shape variable I

January 18, 2011, I received a mail from Richard Montgomery with an unpublished preprint [2007, unpublished].

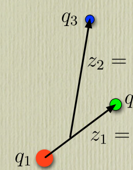
In the preprint, Moeckel and Montgomery ...

- the Shape variable for Planar 3-body,
- the Lagrangian,
- the equations of motion.



# The Shape variable I

$z_1$, $z_2$: two Jacobi vectors.



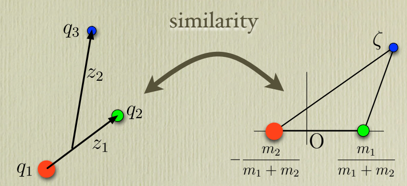

$$z_2 = q_3 - \frac{m_1 q_1 + m_2 q_2}{m_1 + m_2}$$

$$z_1 = q_2 - q_1$$

Shape variable: $\zeta = \frac{z_2}{z_1} = \left(\frac{m_1 + m_2 + m_3}{m_1 + m_2}\right)\left(\frac{q_3}{q_2 - q_1}\right).$

I learned this from Moeckel & Montgomery 2007

$\zeta$ is invariant under scaling and rotation: $q_k \to \lambda e^{i\theta} q_k$
$\Rightarrow$ depends only on shape




Shape variable: $\zeta = \frac{z_2}{z_1} = \left(\frac{m_1+m_2+m_3}{m_1+m_2}\right)\left(\frac{q_3}{q_2-q_1}\right).$

$$\eta = \sqrt{\frac{m_1 m_2 m_3}{m_1+m_2+m_3}}\left(\frac{m_1+m_2}{m_1 m_2}\right)\zeta = \sqrt{\frac{(m_1+m_2+m_3)m_3}{m_1 m_2}}\left(\frac{q_3}{q_2-q_1}\right).$$



# Lagrangian

$$\frac{K}{2} = \frac{1}{2}\sum m_k|\dot{q}_k|^2 = \frac{\dot{r}^2}{2} + \frac{r^2}{2}\left(\dot{\phi} + \frac{\eta\times\dot{\eta}}{1+|\eta|^2}\right)^2 + \frac{r^2}{2}\frac{|\dot{\eta}|^2}{(1+|\eta|^2)^2}.$$

$r = \sqrt{I}$,

$\phi$ : the rotation angle

$$\eta = \sqrt{\frac{m_1 m_2 m_3}{m_1+m_2+m_3}}\left(\frac{m_1+m_2}{m_1 m_2}\right)\zeta = \sqrt{\frac{(m_1+m_2+m_3)m_3}{m_1 m_2}}\left(\frac{q_3}{q_2-q_1}\right).$$



# Lagrangian

$$\frac{K}{2} = \frac{1}{2}\sum m_k|\dot{q}_k|^2 = \frac{\dot{r}^2}{2} + \frac{r^2}{2}\left(\dot{\phi} + \frac{\eta\times\dot{\eta}}{1+|\eta|^2}\right)^2 + \frac{r^2}{2}\frac{|\dot{\eta}|^2}{(1+|\eta|^2)^2}.$$

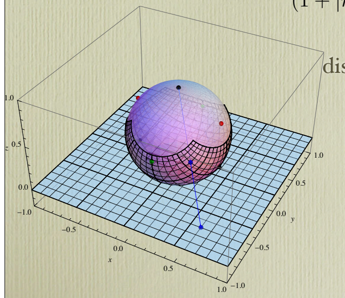
kinetic energy for: size change, rotation, the shape change

This term naturally defines the metric of the shape space ...



# The Shape Sphere

$$g_{\alpha\beta} = \frac{\delta_{\alpha\beta}}{(1+|\eta|^2)^2}$$



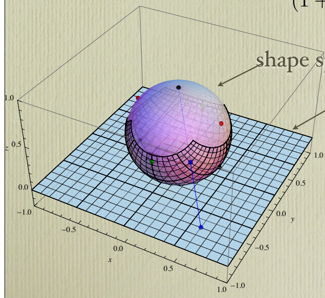
distance on the sphere:

$$ds^2 = \frac{|d\eta|^2}{(1+|\eta|^2)^2}$$



# The Shape Sphere

$$g_{\alpha\beta} = \frac{\delta_{\alpha\beta}}{(1+|\eta|^2)^2}$$



shape sphere: the Riemann sphere

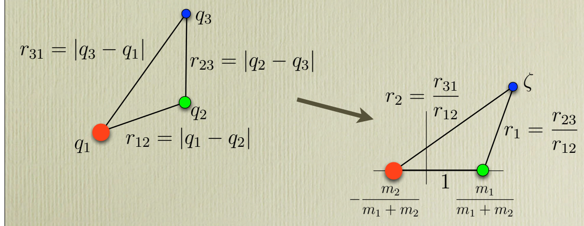
shape variable: $\eta = x^1 + ix^2$

W. Hsiang 1995, 2006,
R. Montgomery 2002,
R. Moeckel 2007,
K. H. Kuwabara & K. Tanikawa 2007



# Shape variables II $\{r_1, r_2\}$

ratio of distance between bodies






# Lagrangian

$$L = \frac{1}{2}K + \frac{1}{\alpha}U$$

$$\frac{K}{2} = \frac{1}{2}\sum m_k|\dot{q}_k|^2 = \frac{\dot{r}^2}{2} + \frac{r^2}{2}\left(\dot{\phi} + \frac{\eta\times\dot{\eta}}{1+|\eta|^2}\right)^2 + \frac{r^2}{2}\frac{|\dot{\eta}|^2}{(1+|\eta|^2)^2}.$$

$$U = \sum \frac{m_i m_j}{|q_i - q_j|^\alpha} = \frac{\mu_\alpha(\eta)}{r^\alpha},$$

$$\mu_\alpha(\zeta) = \left(\frac{m_1 m_2}{m_1+m_2}\right)^{\alpha/2}(1+|\eta|^2)^{\alpha/2}\left(m_1 m_2 + \frac{m_2 m_3}{r_1^\alpha} + \frac{m_3 m_1}{r_2^\alpha}\right)$$



# equations of motion for rotation & size

$$\frac{\partial L}{\partial \phi} = 0 \Rightarrow C = \frac{\partial L}{\partial \dot{\phi}} = r^2\left(\dot{\phi} + \frac{\eta\wedge\dot{\eta}}{1+|\eta|^2}\right) = \text{constant}$$

:the angular momentum

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = \frac{\partial L}{\partial r}$ by a few lines calculations, we get

$$\frac{d\mu}{dt} = \frac{r^{\alpha-2}}{2}\frac{d}{dt}\left(r^4\frac{|\dot{\eta}|^2}{(1+|\eta|^2)^2}\right) : \text{Saari's relation}$$

if $\frac{d\mu}{dt} = 0 \Rightarrow$ $= v^2 =$ const.

$$\Rightarrow E = \frac{\dot{r}^2}{2} + \frac{C^2+v^2}{2r^2} - \frac{\mu}{r^\alpha}$$



# Equation of motion for the Shape variable

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\eta}}\right) = \frac{\partial L}{\partial \eta}$$

↓

$$\frac{d^2\eta}{d\tau^2} = \frac{2i\left(-C + \eta\times\frac{d\eta}{d\tau}\right)}{1+|\eta|^2}\frac{d\eta}{d\tau} + r^{2-\alpha}\frac{\partial\mu}{\partial\eta}.$$

$$\frac{d}{dt} = \left(\frac{1+|\eta|^2}{r^2}\right)\frac{d}{d\tau}$$

# Summary of the results from the equations of motion

$$E = \frac{\dot r^2}{2} + \frac{C^2 + v^2}{2r^2} - \frac{\mu}{r^\alpha},$$

$$v^2 = r^4 \frac{|\dot\eta|^2}{(1+|\eta|^2)^2} = \left|\frac{d\eta}{d\tau}\right|^2, \longleftarrow$$ time dependent in general

$$\frac{d^2\eta}{d\tau^2} = \frac{2i\left(-C + \eta \times \frac{d\eta}{d\tau}\right)}{1+|\eta|^2}\frac{d\eta}{d\tau} + r^{2-\alpha}\frac{\partial\mu}{\partial\eta}.$$



We have 2 cases for

$$v^2 = r^4 \frac{|\dot\eta|^2}{(1+|\eta|^2)^2} = \left|\frac{d\eta}{d\tau}\right|^2 = \text{constant}$$

Case I: $\frac{d\eta}{dt} = 0$

$\Downarrow$

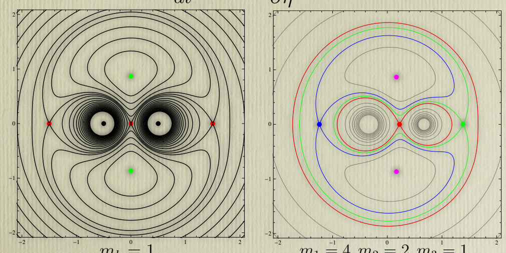
$$\frac{d^2\eta}{d\tau^2} = \frac{2i\left(-C + \eta \times \frac{d\eta}{d\tau}\right)}{1+|\eta|^2}\frac{d\eta}{d\tau} + r^{2-\alpha}\frac{\partial\mu}{\partial\eta} \quad \Rightarrow \quad \frac{\partial\mu}{\partial\eta} = 0.$$

Lagrange, Euler solution



# Lagrange, Euler solution

$$\frac{d\eta}{dt} = 0 \Leftrightarrow \frac{\partial\mu}{\partial\eta} = 0.$$



$m_k = 1$ $\quad$ $m_1 = 4, m_2 = 2, m_3 = 1$

The Saari's conjecture claims that the solutions with $\mu$ = constant are only Euler & Lagrange.



$$v^2 = r^4 \frac{|\dot\eta|^2}{(1+|\eta|^2)^2} = \left|\frac{d\eta}{d\tau}\right|^2 = \text{constant}$$

Case II: $\frac{d\eta}{dt} \neq 0$

$\Downarrow$

$$0 = \frac{d\mu}{d\tau} = \frac{d\eta}{d\tau}\cdot\frac{\partial\mu}{\partial\eta} \Rightarrow \frac{d\eta}{d\tau} = \frac{iv}{|\partial\mu/\partial\eta|}\frac{\partial\mu}{\partial\eta}$$

$\Downarrow$

Then, $$\frac{d^2\eta}{d\tau^2} = iv\frac{d}{d\tau}\left(\frac{1}{|\partial\mu/\partial\eta|}\frac{\partial\mu}{\partial\eta}\right) = \dots \overset{?}{\Leftrightarrow}$$ eq. of motion

compatibility



# compatiblity

for $\mu$ = constant & $|d\eta/dt| \neq 0$ & the equation of motion for $\eta$.

$\Downarrow$

the nessesary condition:

$$r^{2-\alpha} = \frac{-2}{|\nabla\mu|}C + \left(\frac{\lambda}{2|\nabla\mu|^4} - \frac{\Delta\mu}{|\nabla\mu|^2}\right)v^2.$$

$\eta = x + iy;$

$$|\nabla\mu|^2 = g^{ij}(\partial_i\mu)(\partial_j\mu) = (1+x^2+y^2)^2(\mu_x^2+\mu_y^2),$$

$$\Delta\mu = \frac{1}{\sqrt{|g|}}\partial_i\left(g^{ij}\sqrt{|g|}\partial_j\mu\right) = (1+x^2+y^2)(\mu_{xx}+\mu_{yy}),$$

$$\begin{aligned}\lambda &= g^{ij}(\partial_i\mu)\left(\partial_j|\nabla\mu|^2\right)\\ &= 4(1+x^2+y^2)^3(x\mu_x + y\mu_y)(\mu_x^2+\mu_y^2)\\ &\quad + 2(1+x^2+y^2)^4(\mu_x^2\mu_{xx} + 2\mu_x\mu_y\mu_{xy} + \mu_y^2\mu_{yy}).\end{aligned}$$



# compatiblity

a nessesary condition: in the invariant form

$$r^{2-\alpha} = \frac{-2}{|\nabla\mu|}C + \left(\frac{\lambda}{2|\nabla\mu|^4} - \frac{\Delta\mu}{|\nabla\mu|^2}\right)v^2.$$

$\Downarrow$

for $\alpha = 2$,

$$1 = \frac{-2}{|\nabla\mu|}C + \left(\frac{\lambda}{2|\nabla\mu|^4} - \frac{\Delta\mu}{|\nabla\mu|^2}\right)v^2.$$



# Saari's conjecture for a strong force

for $\alpha = 2$,

$$1 = \frac{-2}{|\nabla\mu|}C + \left(\frac{\lambda}{2|\nabla\mu|^4} - \frac{\Delta\mu}{|\nabla\mu|^2}\right)v^2.$$

$$U = \frac{1}{2}\sum\frac{m_i m_j}{r_{ij}^2},\ I = \left(\sum m_k\right)^{-1}\left(\sum m_i m_j r_{ij}^2\right)$$

$$\begin{aligned}\mu &= IU\\ &= \frac{1}{2}\left(\frac{m_1m_2 + m_2m_3r_1^2 + m_3m_1r_2^2}{m_1+m_2+m_3}\right)\left(m_1m_2 + \frac{m_2m_3}{r_1^2} + \frac{m_3m_1}{r_2^2}\right)\end{aligned}$$

$r_1 = r_{23}/r_{12},\ r_2 = r_{31}/r_{12}.$



$\alpha = 2$

$$1 = \frac{-2}{|\nabla\mu|}C + \left(\frac{\lambda}{2|\nabla\mu|^4} - \frac{\Delta\mu}{|\nabla\mu|^2}\right)v^2.$$

$$\Delta\mu = \frac{2(m_1m_2 + m_2m_3r_1^2 + m_3m_1r_2^2)^2(m_1r_1^4(1+r_2^4) + m_2r_2^4(1+r_1^4) + m_3(r_1^4+r_2^4))}{(m_1+m_2+m_3)r_1^4r_2^4}$$

$$\begin{aligned}|\nabla\mu|^2 = \frac{m_1m_2m_2(m_1m_2 + m_2m_3r_1^2 + m_3m_1r_2^2)^2}{(m_1+m_2+m_3)r_1^6r_2^6}\Big(&m_1^2r_1^6(r_2^4-1)^2 + m_2^2r_2^6(r_1^4-1)^2 + m_3^2(r_1^4-r_2^4)\\ &+ m_1m_2r_1^4r_2^4(r_1^4-1)(r_2^4-1)(r_1^2+r_2^2-1)\\ &+ m_2m_3(r_1^4-1)r_2^2(r_2^4 - r_1^4 + r_1^6 + r_2^6 - r_1^2r_2^4 - r_1^4r_2^2)\\ &+ m_1m_3(r_2^4-1)r_1^2(r_1^4 - r_2^4 + r_1^6 + r_2^6 - r_1^2r_2^4 - r_1^4r_2^2)\Big)\end{aligned}$$

$\lambda = \dots$ so many terms $\dots$

$$\mu = \left(\frac{m_1m_2 + m_2m_3r_1^2 + m_3m_1r_2^2}{m_1+m_2+m_3}\right)\left(m_1m_2 + \frac{m_2m_3}{r_1^2} + \frac{m_3m_1}{r_2^2}\right) = \text{const.}$$



# equal masses case

for $\alpha = 2$

$$\begin{aligned}\Delta\mu &= \frac{8(1+r_1^2+r_2^2)^2(r_1^4+r_2^4+r_1^4r_2^4)}{9r_1^4r_2^4}\\ &= \frac{8(\mu_1^2+\mu_2^2+\mu_3^2)(\mu_1\mu_2+\mu_2\mu_3+\mu_3\mu_1)^2}{9\mu_1^2\mu_2^2\mu_3^2}\\ &= 8(\mu^2 - 6\rho).\end{aligned}$$ $\longleftarrow$ symmetric polynomials

$$\mu = \frac{1}{3}(r_{12}^2 + r_{23}^2 + r_{31}^2)\left(\frac{1}{r_{12}^2} + \frac{1}{r_{23}^2} + \frac{1}{r_{31}^2}\right)$$

$$= \mu_3 + \mu_1 + \mu_2$$

elementary symmetric polynomials

$\mu = \mu_1 + \mu_2 + \mu_3,\ \rho = \mu_1\mu_2\mu_3,$

$(\nu = \mu_1\mu_2 + \mu_2\mu_3 + \mu_3\mu_1 = 3\rho)$

# equal masses case

for $\alpha = 2$

Similarly, $\Delta\mu = 8(\mu^2 - 6\rho)$,

$$|\nabla\mu|^2 = -4(\mu^2 - 2\mu^3 + 6\mu\rho + 27\rho^2),$$

$$\lambda = 16\Big(2\mu^3 - 10\mu^4 + 12\mu^5 + (36\mu^2 - 72\mu^3)\rho + (333\mu - 351\mu^2)\rho^2 + (891 + 243\mu)\rho^3\Big).$$

$$1 = \frac{-2}{|\nabla\mu|}C + \left(\frac{\lambda}{2|\nabla\mu|^4} - \frac{\Delta\mu}{|\nabla\mu|^2}\right) v^2 = f(\mu,\rho)C + g(\mu,\rho)v^2.$$

We can show that the right-hand side is not constant.
This is a proof of the Saari's conjecture for equal masses case.



# General masses case

for $\alpha = 2$

$$1 = \frac{-2}{|\nabla\mu|}C + \left(\frac{\lambda}{2|\nabla\mu|^4} - \frac{\Delta\mu}{|\nabla\mu|^2}\right) v^2.$$

$$\Delta\mu = \frac{2(m_1m_2 + m_2m_3r_1^2 + m_3m_1r_2^2)^2(m_1r_1^4(1+r_2^4) + m_2r_2^4(1+r_1^4) + m_3(r_1^4 + r_2^4))}{(m_1+m_2+m_3)r_1^4r_2^4}$$

$|\nabla\mu|^2 = ..., \lambda = ...$

Can we find a convenient variable,
something like $\rho$ in equal masses case.

$$\eta \in \mathbb{C} \leftrightarrow \{r_1, r_2\} \leftrightarrow \{\mu, \rho\}$$



# General masses case

for $\alpha = 2$

Can we find a convenient variable,
something like $\rho$ in equal masses case.

$$\eta \in \mathbb{C} \leftrightarrow \{r_1, r_2\} \leftrightarrow \{\mu, \rho\}$$

Our choice is ...

$$\tilde\mu = (m_1 + m_2 + m_3)\mu = (m_1m_2 + m_2m_3r_1^2 + m_3m_1r_2^2)\left(m_1m_2 + \frac{m_2m_3}{r_1^2} + \frac{m_3m_1}{r_2^2}\right) = \nu\,\rho,$$

$$\nu = m_1m_2 + m_2m_3r_1^2 + m_3m_1r_2^2,$$

$$\rho = m_1m_2 + \frac{m_2m_3}{r_1^2} + \frac{m_3m_1}{r_2^2}.$$



$$\eta \in \mathbb{C} \leftrightarrow \{r_1, r_2\} \leftrightarrow \{\mu, \rho\}$$

→

$$r_1^2 = \frac{\tilde\mu}{m_2m_3\rho} + \frac{m_1(m_3^2 - m_2^2)}{m_2^2m_3} + \frac{(\tilde\mu - m_2^2m_3^2)m_3}{m_2^3\tilde\mu}\rho + O(\rho^2),$$

$$r_2^2 = -\frac{m_3}{m_2} - \frac{(\tilde\mu - m_2^2m_3^2)m_3}{m_1m_2^2\tilde\mu}\rho + O(\rho^2).$$

and $r_1 \leftrightarrow r_2$, $m_1 \leftrightarrow m_2$.

$$\frac{1}{|\nabla\mu|^2} = -\frac{m_3^2(m_1+m_2+m_3)^3}{4m_1m_2^2}\left(\frac{\rho}{\tilde\mu}\right)^4 + O(\rho^5),$$

→

$$\lambda = -\frac{16\Big(3m_1m_2m_3(m_2+m_3) + 3m_1^2(m_2^2+m_3^2) - m_2^2m_3^2 + \tilde\mu\Big)}{m_3^4(m_1+m_2+m_3)^5}\left(\frac{\tilde\mu}{\rho}\right)^6 + O(1/\rho^5),$$

$$\Delta\mu = \frac{4(m_1m_2^2 + m_2^2m_3 + (m_1+m_2)m_3^2}{m_3^2(m_1+m_2+m_3)^2}\left(\frac{\tilde\mu}{\rho}\right)^2 + O(1/\rho)$$



$$1 = \frac{-2}{|\nabla\mu|}C + \left(\frac{\lambda}{2|\nabla\mu|^4} - \frac{\Delta\mu}{|\nabla\mu|^2}\right) v^2 = f(\mu,\rho)C + g(\mu,\rho)v^2$$

$$= -\frac{(m_1+m_2+m_3)v}{m_1^2m_2^2}\Big(-v\left(\tilde\mu + m_1^2\left(m_2^2 + m_2m_3 - m_3^2\right) + m_1m_2m_3^2 + m_2^2m_3^2\right) + 2iCm_1m_3\sqrt{m_2^3(m_1+m_2+m_3)}\Big)\left(\frac{\rho}{\tilde\mu}\right)^2 + O(\rho^3).$$

The right-hand side has no constant term in $\rho$ for $\rho \sim 0$.

Therefore, the neccesary condition is not satisfied.

This is a proof of the Saari's conjecture for $\alpha = 2$.



# Newton's potential $\alpha = 1$.

The necessary condition: $r = \frac{-2Cv}{|\nabla\mu|} + \frac{v^2\lambda}{2|\nabla\mu|^4} - \frac{v^2\Delta\mu}{|\nabla\mu|^2}$

$$\begin{cases} \tilde\mu = \sqrt{m_1m_2 + m_2m_3r_1^2 + m_3m_1r_2^2}\left(m_1m_2 + \frac{m_2m_3}{r_1} + \frac{m_3m_1}{r_2}\right) = \sqrt{\nu}\,\rho, \\ \nu = m_1m_2 + m_2m_3r_1^2 + m_3m_1r_2^2, \\ \rho = m_1m_2 + \frac{m_2m_3}{r_1} + \frac{m_3m_1}{r_2}. \end{cases}$$

→

$$\begin{cases} r_1 = -\frac{m_3}{m_1}\left(1 - \frac{(m_1m_3)^{3/2} - \tilde\mu}{m_1m_2\tilde\mu}\rho + \left(\frac{(m_1m_3)^{3/2} - \tilde\mu}{m_1m_2\tilde\mu}\right)^2\rho^2 + O(\rho^3)\right), \\ r_2 = \frac{1}{\sqrt{m_1m_3}}\left(\frac{\tilde\mu}{\rho} - \frac{m_2(m_1^3 + m_3^3)}{2m_1^2\tilde\mu}\rho + \frac{m_3^3((m_1m_3)^{3/2} - \tilde\mu)}{m_1^3\tilde\mu^2}\rho^2 + O(\rho^3)\right) \end{cases}$$

and other 3 similar solutions.



→ $$r = \mp\frac{3v^2((m_1m_3)^{3/2} \pm \tilde\mu)\sqrt{m_1+m_2+m_3}}{4m_1^2m_2^2}\left(\frac{\rho}{\tilde\mu}\right)^2 + O(\rho^3).$$

$$E = \frac{\dot r^2}{2} + \frac{C^2 + v^2}{2r^2} - \frac{\mu}{r} = \frac{a}{\rho^8} + O(\rho^{-7}) \text{ with } a \neq 0.$$

Therefore, $E$ is not constant.

This is a proof of the Saari's conjecture for $\alpha = 1$.



# Summary

The shape variable

$$\eta = \sqrt{\frac{(m_1+m_2+m_3)m_3}{m_1m_2}}\left(\frac{q_3}{q_2 - q_1}\right).$$

Lagrangean for $r, \phi, \eta \Rightarrow$ Equations of motion for $r, \phi, \eta$.

A necessary condition for $\mu =$ constant,

$$r^{2-\alpha} = \frac{-2}{|\nabla\mu|}C + \left(\frac{\lambda}{2|\nabla\mu|^4} - \frac{\Delta\mu}{|\nabla\mu|^2}\right)v^2$$

in $\eta \in \mathbb{C} \Rightarrow \{r_1, r_2\} \Rightarrow \{\mu, \rho\}$

For $\alpha = 2$, this condition cannot be satisfied by any $C, v, \mu, m_k$.
For $\alpha = 1$, this condition determine $r = r(\rho)$.
$\Rightarrow E \neq$ constant for any $C, v, \mu, m_k$.

This is a proof of the Saari's conjecture
for general masses, $\alpha = 1$ and 2.

Thank you