Newton's equations in spaces of constant curvature

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Goal: to present some results from

- F. Diacu. On the singularities of the curved n-body problem, *Trans. Amer. Math. Soc.* **363**, 4 (2011), 2249-2264.
- F. Diacu and E. Pérez-Chavela. Homographic solutions of the curved 3-body problem, *J. Differential Equations* **250** (2011), 340-366.
- F. Diacu. Polygonal homographic orbits of the curved n-body problem, *Trans. Amer. Math. Soc.* **364**, 5 (2012), 2783-2802.
- F. Diacu. *Relative equilibria in the curved N-body problem*, Atlantis Studies in Dynamical Systems, vol. I, Atlantis Press, 2012.
- F. Diacu. Relative equilibria in the 3-dimensional curved n-body problem, *Memoirs Amer. Math. Soc.* **228**, 1071 (2013), ISBN: 978-0-8218-9136-0.
- F. Diacu and S. Kordlou. Rotopulsating orbits of the curved N-body problem *J. Differential Equations* **255** (2013), 2709-2750.

History of the problem

- 1830s Nikolai Lobachevsky and János Bolyai: 2-BP in $\mathbf{H}^3$
- 1852 Lejeune Dirichlet: 2-BP in $\mathbf{H}^3$
- 1860 Paul Joseph Serret: 2-BP in $\mathbf{S}^2$
- 1870 Ernst Schering: 2-BP in $\mathbf{H}^3$
- 1873 Rudolph Lipschitz: 2-BP in $\mathbf{S}^3$
- 1885 Wilhelm Killing: 2-BP in $\mathbf{H}^3$
- 1902 Heinrich Liebmann: 2-BP in $\mathbf{S}^2$ and $\mathbf{H}^2$ also proves an analogue of Bertrand's theorem
- 1940 Erwin Schrödinger: quantum 2-BP in $\mathbf{H}^3$
- 1945 Leopold Infeld and Alfred Schild: quantum 2-BP in $\mathbf{H}^3$
- 1990s Russian school of celestial mechanics
- 2005 José Cariñena, Manuel Rañada, Mariano Santander: 2-BP in $\mathbf{S}^2$ and $\mathbf{H}^2$

Setting

The space in which the motion of the bodies takes place is:

$$\mathbb{M}^3_\kappa = \{(w, x, y, z) | w^2 + x^2 + y^2 + \sigma z^2 = \kappa^{-1} (z > 0 \text{ if } \kappa < 0)\},$$

where σ is the *signum function*

$$\sigma = \begin{cases} +1, & \text{for } \kappa > 0 \\ -1, & \text{for } \kappa < 0 \end{cases}$$

Notice that

$$\mathbb{M}^3_1 = \mathbb{S}^3 \text{ and } \mathbb{M}^3_{-1} = \mathbb{H}^3$$

Notations

Consider $m_1, \ldots, m_n > 0$ in $\mathbb{R}^4$ for $\kappa > 0$ and $\mathbb{M}^{3,1}$ (Minkowski space) for $\kappa < 0$, with positions given by

$$\mathbf{q}_i = (w_i, x_i, y_i, z_i),\ i = \overline{1, n}$$

$\mathbf{q} = (\mathbf{q}_1, \ldots, \mathbf{q}_n)$ is the configuration of the system

$\nabla_{\mathbf{q}_i} := (\partial_{w_i}, \partial_{x_i}, \partial_{y_i}, \sigma\partial_{z_i}),\ \ \nabla := (\nabla_{\mathbf{q}_1}, \ldots, \nabla_{\mathbf{q}_n})$ is the gradient

For $\mathbf{a} := (a_w, a_x, a_y, a_z), \mathbf{b} := (b_w, b_x, b_y, b_z)$,

$$\mathbf{a} \cdot \mathbf{b} := (a_w b_w + a_x b_x + a_y b_y + \sigma a_z b_z)$$

is the inner product

Potential

For $\kappa \neq 0$, the force function is

$$U_\kappa(\mathbf{q}) = \sum_{1 \le i < j \le n} \frac{m_i m_j |\kappa|^{1/2} \kappa \mathbf{q}_i \cdot \mathbf{q}_j}{[\sigma(\kappa \mathbf{q}_i \cdot \mathbf{q}_i)(\kappa \mathbf{q}_j \cdot \mathbf{q}_j) - \sigma(\kappa \mathbf{q}_i \cdot \mathbf{q}_j)^2]^{1/2}}$$

$-U_\kappa$ is the potential (a homogeneous function of degree 0).

Euler's formula for homogeneous functions:

$$\mathbf{q}_i \cdot \nabla_{\mathbf{q}_i} U_\kappa(\mathbf{q}) = 0, \quad i = \overline{1, n}.$$

Equations of motion

Using variational methods (constrained Lagrangian dynamics), we obtain the equations of motion:

$$m_i \ddot{\mathbf{q}}_i = \nabla_{\mathbf{q}_i} U_\kappa(\mathbf{q}) - m_i \kappa (\dot{\mathbf{q}}_i \cdot \dot{\mathbf{q}}_i)\mathbf{q}_i,$$
$$\mathbf{q}_i \cdot \mathbf{q}_i = \kappa^{-1},\ \mathbf{q}_i \cdot \dot{\mathbf{q}}_i = 0,\ \kappa \neq 0,\ \ i = \overline{1, n}$$

$$\nabla_{\mathbf{q}_i} U_\kappa(\mathbf{q}) = \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{m_i m_j |\kappa|^{3/2} (\kappa \mathbf{q}_j \cdot \mathbf{q}_j)[(\kappa \mathbf{q}_i \cdot \mathbf{q}_i)\mathbf{q}_j - (\kappa \mathbf{q}_i \cdot \mathbf{q}_j)\mathbf{q}_i]}{[\sigma(\kappa \mathbf{q}_i \cdot \mathbf{q}_i)(\kappa \mathbf{q}_j \cdot \mathbf{q}_j) - \sigma(\kappa \mathbf{q}_i \cdot \mathbf{q}_j)^2]^{3/2}},$$

$i = \overline{1, n}$

Elimination of κ

Coordinate and time-rescaling transformations

$$\mathbf{q}_i = |\kappa|^{-1/2}\mathbf{r}_i,\ i = \overline{1,n}\ \text{ and }\ \tau = |\kappa|^{3/4}t$$

lead to the equations of motion

$$\mathbf{r}_i'' = \sum_{j=1, j\neq i}^{n} \frac{m_j[\mathbf{r}_j - \sigma(\mathbf{r}_i\cdot\mathbf{r}_j)\mathbf{r}_i]}{[\sigma - \sigma(\mathbf{r}_i\cdot\mathbf{r}_j)^2]^{3/2}} - \sigma(\mathbf{r}_i'\cdot\mathbf{r}_i')\mathbf{r}_i,\ \ i = \overline{1,n},$$

where

$$' = \frac{d}{d\tau},\ \ \mathbf{r}_i\cdot\mathbf{r}_i = |\kappa|\mathbf{q}_i\cdot\mathbf{q}_i = |\kappa|\kappa^{-1} = \sigma$$

The positive case and the negative case

Equations of motion in $\mathbb{S}^3$:

$$\ddot{\mathbf{q}}_i = \sum_{j=1, j\neq i}^{n} \frac{m_j[\mathbf{q}_j - (\mathbf{q}_i \cdot \mathbf{q}_j)\mathbf{q}_i]}{[1-(\mathbf{q}_i \cdot \mathbf{q}_j)^2]^{3/2}} - (\dot{\mathbf{q}}_i \cdot \dot{\mathbf{q}}_i)\mathbf{q}_i,$$

$$\mathbf{q}_i \cdot \mathbf{q_i} = 1, \ \ \mathbf{q}_i \cdot \dot{\mathbf{q}}_i = 0, \ \ i = \overline{1, n}$$

Equations of motion in $\mathbb{H}^3$:

$$\ddot{\mathbf{q}}_i = \sum_{j=1, j\neq i}^{n} \frac{m_j[\mathbf{q}_j + (\mathbf{q}_i \cdot \mathbf{q}_j)\mathbf{q}_i]}{[(\mathbf{q}_i \cdot \mathbf{q}_j)^2 - 1]^{3/2}} + (\dot{\mathbf{q}}_i \cdot \dot{\mathbf{q}}_i)\mathbf{q}_i,$$

$$\mathbf{q}_i \cdot \mathbf{q_i} = -1, \ \ \mathbf{q}_i \cdot \dot{\mathbf{q}}_i = 0, \ \ i = \overline{1, n}$$

Hamiltonian form

$\mathbf{p} := (\mathbf{p}_1, \dots, \mathbf{p}_n)$, $\mathbf{p}_i := m_i \dot{\mathbf{q}}_i$, $i = \overline{1, n}$, momenta

$T(\mathbf{q}, \mathbf{p}) = \frac{1}{2} \sum_{i=1}^{n} m_i^{-1} (\mathbf{p}_i \cdot \mathbf{p}_i)(\sigma \mathbf{q}_i \cdot \mathbf{q}_i)$, kinetic energy

$H(\mathbf{q}, \mathbf{p}) = T(\mathbf{q}, \mathbf{p}) - U(\mathbf{q})$, Hamiltonian function

$$\begin{cases} \dot{\mathbf{q}}_i = \nabla_{\mathbf{p}_i} H(\mathbf{q}, \mathbf{p}) = m_i^{-1} \mathbf{p}_i, \\ \dot{\mathbf{p}}_i = -\nabla_{\mathbf{q}_i} H(\mathbf{q}, \mathbf{p}) = \nabla_{\mathbf{q}_i} U(\mathbf{q}) - \sigma m_i^{-1} (\mathbf{p}_i \cdot \mathbf{p}_i) \mathbf{q}_i, \\ \mathbf{q}_i \cdot \mathbf{q}_i = \sigma, \quad \mathbf{q}_i \cdot \mathbf{p}_i = 0, \quad i = \overline{1, n} \end{cases}$$

The wedge product

Consider the basis

$$\mathbf{e}_w = (1,0,0,0),\ \mathbf{e}_x = (0,1,0,0),\ \mathbf{e}_y = (0,0,1,0),\ \mathbf{e}_z = (0,0,0,1)$$

The wedge product of
$\mathbf{u} = (u_w, u_x, u_y, u_z), \mathbf{v} = (v_w, v_x, v_y, v_z) \in \mathbb{R}^4$ is defined as

$$\begin{aligned}\mathbf{u} \wedge \mathbf{v} := (u_w v_x - u_x v_w) e_w \wedge e_x + (u_w v_y - u_y v_w) e_w \wedge e_y + \\ (u_w v_z - u_z v_w) e_w \wedge e_z + (u_x v_y - u_y v_x) e_x \wedge e_y + \\ (u_x v_z - u_z v_x) e_x \wedge e_z + (u_y v_z - u_z v_y) e_y \wedge e_z,\end{aligned}$$

where $\mathbf{e}_w \wedge \mathbf{e}_x, \mathbf{e}_w \wedge \mathbf{e}_y, \mathbf{e}_w \wedge \mathbf{e}_z, \mathbf{e}_x \wedge \mathbf{e}_y, \mathbf{e}_x \wedge \mathbf{e}_z, \mathbf{e}_y \wedge \mathbf{e}_z$ represent the bivectors that form a canonical basis of the exterior Grassmann algebra over $\mathbb{R}^4$

Integrals of the total angular momentum

$$\sum_{i=1}^{n} m_i \mathbf{q}_i \wedge \dot{\mathbf{q}}_i = \mathbf{c},$$

where $\mathbf{c} =$
$c_{wx}\mathbf{e}_w \wedge \mathbf{e}_x + c_{wy}\mathbf{e}_w \wedge \mathbf{e}_y + c_{wz}\mathbf{e}_w \wedge \mathbf{e}_z + c_{xy}\mathbf{e}_x \wedge \mathbf{e}_y + c_{xz}\mathbf{e}_x \wedge \mathbf{e}_z + c_{yz}\mathbf{e}_y \wedge \mathbf{e}_z,$
with the coefficients $c_{wx}, c_{wy}, c_{wz}, c_{xy}, c_{xz}, c_{yz} \in \mathbb{R}$
– on components, 6 integrals:

$$\sum_{i=1}^{n} m_i(w_i\dot{x}_i - \dot{w}_i x_i) = c_{wx}, \quad \sum_{i=1}^{n} m_i(w_i\dot{y}_i - \dot{w}_i y_i) = c_{wy},$$

$$\sum_{i=1}^{n} m_i(w_i\dot{z}_i - \dot{w}_i z_i) = c_{wz}, \quad \sum_{i=1}^{n} m_i(x_i\dot{y}_i - \dot{x}_i y_i) = c_{xy},$$

$$\sum_{i=1}^{n} m_i(x_i\dot{z}_i - \dot{x}_i z_i) = c_{xz}, \quad \sum_{i=1}^{n} m_i(y_i\dot{z}_i - \dot{y}_i z_i) = c_{yz}$$

Isometries in $\mathbb{S}^3$

In some suitable basis, rotations can be written as

$$A = \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\phi & -\sin\phi \\ 0 & 0 & \sin\phi & \cos\phi \end{pmatrix}, \theta, \phi \in [0, 2\pi)$$

– simple rotations (elliptic): lead to new solutions
– double rotations (elliptic-elliptic): lead to new solutions

Isometries in $\mathbb{H}^3$

In some suitable basis, rotations can be written as

$$B = \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cosh\phi & \sinh\phi \\ 0 & 0 & \sinh\phi & \cosh\phi \end{pmatrix}, \theta \in [0, 2\pi), \phi \in \mathbb{R},$$

– simple rotations (elliptic): lead to new solutions
– simple rotations (hyperbolic): lead to new solutions
– double rotations (elliptic-hyperbolic): lead to new solutions

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\xi & \xi \\ 0 & \xi & 1-\xi^2/2 & \xi^2/2 \\ 0 & \xi & -\xi^2/2 & 1+\xi^2/2 \end{pmatrix}, \xi \in \mathbb{R}.$$

– simple rotations (parabolic): lead to no solutions

Relative equilibria (RE) in $\mathbb{S}^3$

$$\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_n), \quad \mathbf{q}_i = (w_i, x_i, y_i, z_i), \ i = \overline{1, n},$$

$$[\text{positive elliptic}] : \quad \begin{cases} w_i(t) = r_i \cos(\alpha t + a_i) \\ x_i(t) = r_i \sin(\alpha t + a_i) \\ y_i(t) = y_i \ (\text{constant}) \\ z_i(t) = z_i \ (\text{constant}), \end{cases}$$

with $w_i^2 + x_i^2 = r_i^2, \ r_i^2 + y_i^2 + z_i^2 = 1, \ i = \overline{1, n}$

$$[\text{positive elliptic}-\text{elliptic}] : \quad \begin{cases} w_i(t) = r_i \cos(\alpha t + a_i) \\ x_i(t) = r_i \sin(\alpha t + a_i) \\ y_i(t) = \rho_i \cos(\beta t + b_i) \\ z_i(t) = \rho_i \sin(\beta t + b_i), \end{cases}$$

with $w_i^2 + x_i^2 = r_i^2, \ y_i^2 + z_i^2 = \rho_i^2, \ r_i^2 + \rho_i^2 = 1, \ i = \overline{1, n}$

Relative equilibria (RE) in $\mathbb{H}^3$

$$[\text{negative elliptic}]: \quad \begin{cases} w_i(t) = r_i \cos(\alpha t + a_i) \\ x_i(t) = r_i \sin(\alpha t + a_i) \\ y_i(t) = y_i \ (\text{constant}) \\ z_i(t) = z_i \ (\text{constant}), \end{cases}$$

with $w_i^2 + x_i^2 = r_i^2,\ r_i^2 + y_i^2 - z_i^2 = -1,\ i = \overline{1,n}$

$$[\text{negative hyperbolic}]: \quad \begin{cases} w_i(t) = w_i \ (\text{constant}) \\ x_i(t) = x_i \ (\text{constant}) \\ y_i(t) = \eta_i \sinh(\beta t + b_i) \\ z_i(t) = \eta_i \cosh(\beta t + b_i), \end{cases}$$

with $y_i^2 - z_i^2 = -\eta_i^2,\ w_i^2 + x_i^2 - \eta_i^2 = -1,\ i = \overline{1,n}$

$$[\text{negative elliptic}-\text{hyperbolic}]: \quad \begin{cases} w_i(t) = r_i \cos(\alpha t + a_i) \\ x_i(t) = r_i \sin(\alpha t + a_i) \\ y_i(t) = \eta_i \sinh(\beta t + b_i) \\ z_i(t) = \eta_i \cosh(\beta t + b_i), \end{cases}$$

with $w_i^2 + x_i^2 = r_i^2,\ y_i^2 - z_i^2 = -\eta_i^2$, so $r_i^2 - \eta_i^2 = -1,\ i = \overline{1,n}$

Fixed points (FP) in $\mathbb{S}^3$

– equilateral triangle on a great circle of a great sphere (equal masses, 3BP)
– any scalene acute triangle on a great circle of a great sphere (non-equal masses, 3BP)
– regular tetrahedron in a great sphere (equal masses, 4BP)
– two equilateral triangles, each on complementary great circles (equal masses, 6 BP):

$$\begin{array}{llll}
w_1 = 1, & x_1 = 0, & y_1 = 0, & z_1 = 0, \\
w_2 = -1/2, & x_2 = \sqrt{3}/2, & y_2 = 0, & z_2 = 0, \\
w_3 = -1/2, & x_3 = -\sqrt{3}/2, & y_3 = 0, & z_3 = 0, \\
w_4 = 0, & x_4 = 0, & y_4 = 1, & z_4 = 0, \\
w_5 = 0, & x_5 = 0, & y_5 = -1/2, & z_5 = \sqrt{3}/2, \\
w_6 = 0, & x_6 = 0, & y_6 = -1/2, & z_6 = -\sqrt{3}/2,
\end{array}$$

– two, not necessarily congruent, scalene acute triangles, each on one of two complementary great circles (non-equal masses, 6 BP)

Complementary circles in $\mathbb{S}^3$

Definition 1

Two great circles, C_1 and C_2, of two different great spheres of $\mathbb{S}^3$ are called *complementary* if there is a coordinate system $wxyz$ such that

$$C_1 = \mathbf{S}^1_{wx} = \{(0,0,y,z)\,|\,y^2+z^2=1\},$$
$$C_2 = \mathbf{S}^1_{yz} = \{(w,x,0,0)\,|\,w^2+x^2=1\}.$$

Complementary circles form a Hopf link in a Hopf fibration,

$$h\colon \mathbb{S}^3 \to \mathbb{S}^2,\ h(w,x,y,z) = (w^2+x^2-y^2-z^2, 2(wz+xy), 2(xz-wy)),$$

which takes circles of $\mathbb{S}^3$ to points of $\mathbb{S}^2$. Using the stereographic projection, it can be shown that the circles C_1 and C_2 are linked.

Distance between complementary circles

Since, in $\mathbb{S}^3$, the distance between two points, $\mathbf{a}$ and $\mathbf{b}$, is

$$d(\mathbf{a}, \mathbf{b}) = \cos^{-1}(\mathbf{a} \cdot \mathbf{b}),$$

it follows that if $\mathbf{a} \in C_1$ and $\mathbf{b} \in C_2$, then

$$d(\mathbf{a}, \mathbf{b}) = \pi/2 = \mathrm{constant}$$

Therefore if the body m_1 is on C_1 and the body m_2 is on C_2, the magnitude of the attraction between them is the same, no matter where each of them lies on the respective circle

Clifford tori in $\mathbb{S}^3$

A remarkable family of surfaces in $\mathbb{R}^4$ are the Clifford tori

$$\mathbf{T}^2_{r\rho} = \{(r\cos\theta, r\sin\theta, \rho\cos\phi, \rho\sin\phi) \mid r^2+\rho^2 = 1, 0 \le \theta, \phi < 2\pi\},$$

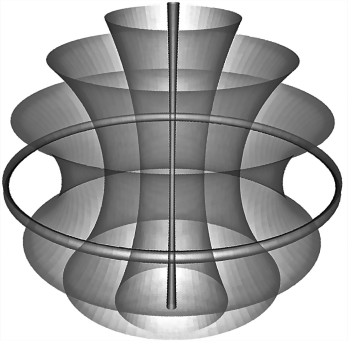
which lie in $\mathbb{S}^3$. Indeed, the Euclidean distance from the origin of the coordinate system to any point of a Clifford torus is

$$(r^2\cos^2\theta + r^2\sin^2\theta + \rho^2\cos^2\phi + \rho^2\sin^2\phi)^{1/2} = (r^2+\rho^2)^{1/2} = 1$$

Unlike the standard torus, the Clifford torus is a flat surface, which divides $\mathbb{S}^3$ into two solid tori, for which it forms the boundary

Heegaard splitting of $\mathbb{S}^3$

The Clifford torus with $r = \rho = 1/\sqrt{2}$ provides the standard genus 1 splitting of $\mathbb{S}^3$, a case in which the two solid tori are congruent.



A 3D projection of a 4D foliation of $\mathbb{S}^3$ into Clifford tori

A Lagrangian RE

$$\begin{aligned} w_1(t) &= r\cos\omega t, & x_1(t) &= r\sin\omega t,\\ y_1(t) &= y \ (\text{constant}), & z_1(t) &= z \ (\text{constant}),\\ w_2(t) &= r\cos(\omega t + 2\pi/3), & x_2(t) &= r\sin(\omega t + 2\pi/3),\\ y_2(t) &= y \ (\text{constant}), & z_2(t) &= z \ (\text{constant}),\\ w_3(t) &= r\cos(\omega t + 4\pi/3), & x_3(t) &= r\sin(\omega t + 4\pi/3),\\ y_3(t) &= y \ (\text{constant}), & z_3(t) &= z \ (\text{constant}). \end{aligned}$$

Given $m := m_1 = m_2 = m_3 > 0,\ r \in (0,1)$, and y, z with $r^2 + y^2 + z^2 = 1$, we can always find two frequencies,

$$\alpha^+ = \frac{2}{r}\sqrt{\frac{2m}{\sqrt{3}r(4-3r^2)^{3/2}}} \ \text{ and } \ \alpha^- = -\frac{2}{r}\sqrt{\frac{2m}{\sqrt{3}r(4-3r^2)^{3/2}}};$$

$$c_{wx} = 3m\omega \neq 0 \ \text{ and } \ c_{wy} = c_{wz} = c_{xy} = c_{xz} = c_{yz} = 0.$$

Stability of Lagrangian RE in $\mathbb{S}^2$

Regina Martínez and Carles Simó: On $\mathbb{S}^2$, the Lagrangian RE with masses $m_1 = m_2 = m_3 = 1$ are linearly stable for $r \in (r_1, r_2) \cup (r_3, 1)$, where $r = \sqrt{1 - z^2}$,

$$r_1 = 0.55778526844099498188467226566148375,$$

$$r_2 = 0.68145469725865414807206661241888645,$$

$$r_3 = 0.92893280143637470996280353121615412,$$

truncated to 35 decimal digits.

Regular tetrahedron RE

Place the bodies $m_1 = m_2 = m_3 = m_4$ at the vertices of a regular tetrahedron. Then m_1 and m_2 move on the Clifford torus with $r = 0$ and $\rho = 1$, which is the only Clifford torus in the class of a given foliation of $\mathbb{S}^3$ that is also a great circle of $\mathbb{S}^3$. The bodies of mass m_3 and m_4 move on the Clifford torus with $r = \frac{\sqrt{6}}{3}$ and $\rho = \frac{\sqrt{3}}{3}$:

$$w_1 = 0,\ x_1 = 0,\ y_1 = \cos(\alpha t + \pi/2),\ z_1 = \sin(\alpha t + \pi/2),$$

$$w_2 = 0,\ x_2 = 0,\ y_2 = \cos(\alpha t + b_2),\ z_2 = \sin(\alpha t + b_2),$$

with $\sin b_2 = -\frac{1}{3}$ and $\cos b_2 = \frac{2\sqrt{2}}{3}$,

Example of RE moving on Clifford tori

$$w_3 = \frac{\sqrt{6}}{3}\cos(\alpha t + 3\pi/2),\ x_3 = \frac{\sqrt{6}}{3}\sin(\alpha t + 3\pi/2),$$

$$y_3 = \frac{\sqrt{3}}{3}\cos(\alpha t + b_3),\ z_3 = \frac{\sqrt{3}}{3}\sin(\alpha t + b_3),$$

with $\cos b_3 = -\frac{\sqrt{6}}{3}$ and $\sin b_3 = -\frac{\sqrt{3}}{3}$, and

$$w_4 = \frac{\sqrt{6}}{3}\cos(\alpha t + \pi/2),\ x_4 = \frac{\sqrt{6}}{3}\sin(\alpha t + \pi/2),$$

$$y_4 = \frac{\sqrt{3}}{3}\cos(\alpha t + b_4),\ z_4 = \frac{\sqrt{3}}{3}\sin(\alpha t + b_4),$$

with $\cos b_4 = -\frac{\sqrt{6}}{3}$ and $\sin b_4 = -\frac{\sqrt{3}}{3}$. Notice that $b_3 = b_4$.

RE with fixed bodies

This is a solution of the 6-body problem with two equilateral triangles, one inscribed in a great circle of a great sphere and the other inscribed in a complementary great circle of another great sphere. The first triangle rotates uniformly, while the second triangle is fixed:

$$m_1 = m_2 = m_3 = m_4 = m_5 = m_6 =: m,$$

$$\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4, \mathbf{q}_5, \mathbf{q}_6),\ \mathbf{q}_i = (w_i, x_i, y_i, z_i),\ i \in \{1, 2, 3, 4, 5, 6\},$$

$$\begin{array}{llll}
w_1 = \cos \alpha t, & x_1 = \sin \alpha t, & y_1 = 0, & z_1 = 0, \\
w_2 = \cos(\alpha t + a), & x_2 = \sin(\alpha t + a), & y_2 = 0, & z_2 = 0, \\
w_3 = \cos(\alpha t + b), & x_3 = \sin(\alpha t + b), & y_3 = 0, & z_3 = 0, \\
w_4 = 0, & x_4 = 0, & y_4 = 1, & z_4 = 0, \\
w_5 = 0, & x_5 = 0, & y_5 = -\dfrac{1}{2}, & z_5 = \dfrac{\sqrt{3}}{2}, \\
w_6 = 0, & x_6 = 0, & y_6 = -\dfrac{1}{2}, & z_6 = -\dfrac{\sqrt{3}}{2},
\end{array}$$

where $a = 2\pi/3$ and $b = 4\pi/3$.

RE on complementary circles

In general, the orbit described below is quasiperiodic:

$$
\begin{aligned}
w_1 &= \cos \alpha t, & x_1 &= \sin \alpha t,\\
y_1 &= 0, & z_1 &= 0,\\
w_2 &= \cos(\alpha t + 2\pi/3), & x_2 &= \sin(\alpha t + 2\pi/3),\\
y_2 &= 0, & z_2 &= 0,\\
w_3 &= \cos(\alpha t + 4\pi/3), & x_3 &= \sin(\alpha t + 4\pi/3),\\
y_3 &= 0, & z_3 &= 0,\\
w_4 &= 0, & x_4 &= 0,\\
y_4 &= \cos \beta t, & z_4 &= \sin \beta t,\\
w_5 &= 0, & x_5 &= 0,\\
y_5 &= \cos(\beta t + 2\pi/3), & z_5 &= \sin(\beta t + 2\pi/3),\\
w_6 &= 0, & x_6 &= 0,\\
y_6 &= \cos(\beta t + 4\pi/3), & z_6 &= \sin(\beta t + 4\pi/3).
\end{aligned}
$$

$$c_{wx} = 3m\alpha \neq 0, \;\; c_{yz} = 3m\beta \neq 0, \;\; c_{wy} = c_{wz} = c_{xy} = c_{xz} = 0$$

Eulerian RE in $\mathbb{H}^3$

The motion described below takes place on a hyperbolic 2-sphere, and is not periodic:

$$\begin{aligned}
w_1 &= 0, \quad x_1 = 0, & y_1 &= \sinh \beta t, & z_1 &= \cosh \beta t, \\
w_2 &= 0, \quad x_2 = x \text{ (constant)}, & y_2 &= \eta \sinh \beta t, & z_2 &= \eta \cosh \beta t, \\
w_3 &= 0, \quad x_3 = -x \text{ (constant)}, & y_3 &= \eta \sinh \beta t, & z_3 &= \eta \cosh \beta t,
\end{aligned}$$

Given $m := m_1 = m_2 = m_3 > 0, x > 0, \eta > 0$ with $x^2 - \eta^2 = -1$, there exist two non-zero frequencies,

$$\beta^+ = \frac{1}{2\eta}\sqrt{\frac{1+4\eta^2}{\eta(\eta^2-1)^{3/2}}} \text{ and } \beta^- = -\frac{1}{2\eta}\sqrt{\frac{1+4\eta^2}{\eta(\eta^2-1)^{3/2}}};$$

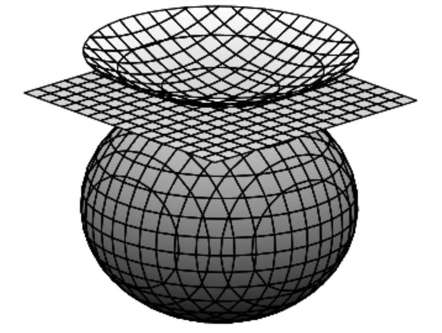
$$c_{wx} = c_{wy} = c_{wz} = c_{xy} = c_{xz} = 0, \; c_{yz} = m\beta(1-2\eta^2)$$

RE on hyperbolic cylinders

The motion described below takes place on a hyperbolic cylinder, and is not periodic:

$$
\begin{aligned}
&w_1 = 0, && x_1 = 0, && y_1 = \sinh \beta t, && z_1 = \cosh \beta t,\\
&w_2 = r\cos \alpha t, && x_2 = r\sin \alpha t, && y_2 = \eta \sinh \beta t, && z_2 = \eta \cosh \beta t,\\
&w_3 = -r\cos \alpha t, && x_3 = -r\sin \alpha t, && y_3 = \eta \sinh \beta t, && z_3 = \eta \cosh \beta t.
\end{aligned}
$$

$$c_{wx} = 2m\alpha r^2, c_{yz} = -1 - 2\beta\eta^2, c_{wy} = c_{wz} = c_{xy} = c_{xz} = 0$$



The passage from $\mathbb{S}^3_\kappa$ up to $\mathbb{R}^3$ and from $\mathbb{H}^3_\kappa$ down to $\mathbb{R}^3$ when $\kappa \to 0$.

Consider a coordinate system having the origin at the North Pole of the spheres $\mathbb{S}^3_\kappa$, i.e., the position of the body m_i is given by

$$\mathbf{r}_i = (x_i, y_i, z_i, \omega_i), \quad i = \overline{1, n}.$$

Extension of the equations to $\kappa = 0$

$$V_\kappa(\mathbf{r}) = \sum_{1\le i<j\le n} \frac{m_i m_j\left(1-\frac{\kappa r_{ij}^2}{2}\right)}{r_{ij}\left(1-\frac{\kappa r_{ij}^2}{4}\right)^{1/2}}$$

$$\ddot{\mathbf{r}}_i = \sum_{j=1, j\ne i}^{n} \frac{m_j\left[\mathbf{r}_j - \left(1-\frac{\kappa r_{ij}^2}{2}\right)\mathbf{r}_i + \frac{r_{ij}^2\mathbf{R}}{2}\right]}{r_{ij}^3\left(1-\frac{\kappa r_{ij}^2}{4}\right)^{3/2}} - (\dot{\mathbf{r}}_i\cdot\dot{\mathbf{r}}_i)(\kappa\mathbf{r}_i+\mathbf{R}), \quad i=\overline{1,n},$$

with

$$\kappa\mathbf{r}_i\cdot\mathbf{r}_i + 2|\kappa|^{1/2}\omega_i = 0, \quad \kappa\mathbf{r}_i\cdot\dot{\mathbf{r}}_i + |\kappa|^{1/2}\dot\omega_i = 0, \quad i=\overline{1,n},$$

where

$$\mathbf{R} = (0,0,0,\sigma|\kappa|^{1/2}), \ \mathbf{r}_i = (x_i, y_i, z_i, \omega_i), \ i=\overline{1,n},$$

$$r_{ij} := [(x_i-x_j)^2 + (y_i-y_j)^2 + (z_i-z_j)^2 + \sigma(\omega_i-\omega_j)^2]^{1/2}.$$

The explicit equations

$$\begin{cases} \ddot{x}_i = \sum_{j=1, j\neq i}^{n} \frac{m_j \left[x_j - \left(1 - \frac{\kappa r_{ij}^2}{2}\right) x_i\right]}{r_{ij}^3 \left(1 - \frac{\kappa r_{ij}^2}{4}\right)^{3/2}} - \kappa(\dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i) x_i \\ \ddot{y}_i = \sum_{j=1, j\neq i}^{n} \frac{m_j \left[y_j - \left(1 - \frac{\kappa r_{ij}^2}{2}\right) y_i\right]}{r_{ij}^3 \left(1 - \frac{\kappa r_{ij}^2}{4}\right)^{3/2}} - \kappa(\dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i) y_i \\ \ddot{z}_i = \sum_{j=1, j\neq i}^{n} \frac{m_j \left[z_j - \left(1 - \frac{\kappa r_{ij}^2}{2}\right) z_i\right]}{r_{ij}^3 \left(1 - \frac{\kappa r_{ij}^2}{4}\right)^{3/2}} - \kappa(\dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i) z_i \\ \ddot{\omega}_i = \sum_{j=1, j\neq i}^{n} \frac{m_j \left[\omega_j - \left(1 - \frac{\kappa r_{ij}^2}{2}\right) \omega_i + \frac{\sigma |\kappa|^{\frac{1}{2}} r_{ij}^2}{2}\right]}{r_{ij}^3 \left(1 - \frac{\kappa r_{ij}^2}{4}\right)^{3/2}} - (\dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i)[\kappa \omega_i + \sigma |\kappa|^{\frac{1}{2}}], \end{cases}$$

$$\kappa(x_i^2 + y_i^2 + z_i^2 + \sigma \omega_i^2) + 2|\kappa|^{1/2} \omega_i = 0,$$

$$\kappa(x_i \dot{x}_i + y_i \dot{y}_i + z_i \dot{z}_i + \sigma \omega_i \dot{\omega}_i) + |\kappa|^{1/2} \dot{\omega}_i = 0, \quad i = \overline{1, n}.$$

Newtonian equations

For $\kappa = 0$ we recover the Newtonian equations:

$$m_i \ddot{\mathbf{r}}_i = \sum_{j=1, j\neq i}^{n} \frac{m_i m_j (\mathbf{r}_j - \mathbf{r}_i)}{r_{ij}^3}, \quad i = \overline{1, n},$$

with $\mathbf{r}_i = (x_i, y_i, z_i, 0), i = \overline{1, n}$.

Bifurcation of the first integrals

– Integral of energy:

for all $\kappa \in \mathbb{R}$: 1 integral (no bifurcation)

– Integrals of the centre of mass:

$\kappa = 0$: 3 integrals

$\kappa \neq 0$: 0 integrals

– Integrals of the linear momentum:

$\kappa = 0$: 3 integrals

$\kappa \neq 0$: 0 integrals

– Integrals of the total angular momentum:

$\kappa = 0$: 3 integrals

$\kappa \neq 0$: 6 integrals

Happy Birthday, Clark!