Rigidity of Planar Central Configurations

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Acknowledgments

Happy bithday to professor Clark Robinson!

Newtonian n–body problem

The *Planar Newtonian n–body problem* consists of studying the motion of n punctual bodies with positive masses $m_1, \ldots, m_n$ interacting according to Newton's gravitational law. In other words, the equations of motion are given by

$$\ddot{r}_i = - \sum_{j=1,\, j\neq i}^{n} m_j \frac{r_i - r_j}{r_{ij}^3}, \qquad (1)$$

for $i = 1, 2, \ldots, n$.

Here, $r_j \in \mathbb{R}^2$, are the position vectors of the bodies;

$r_{ij} = |r_i - r_j|$ is the Euclidean distance between the bodies i and j.

Remarks

Henceforth consider:

- r_i lies in the *inertial barycentric system*, that is,

$$\sum_{i=1}^{n} m_i r_i = 0;$$

- The *configurations* $r = (r_1, \ldots, r_n) \in \mathbb{R}^{2n}$ are out of the collision set;
- Two configurations $r = (r_1, \ldots, r_n)$ and $r' = (r'_1, \ldots, r'_n)$ are *similar* if we can pass from one configuration to the other by a dilation or a rotation of $\mathbb{R}^2$. Thus configuration means its equivalence class.

Central configurations

At a given t_0 a *central configuration* ($c.c.$) is a configuration, such that the acceleration vector for each body is a common scalar multiple of its position vector (referred to the inertial barycentric system), that is

$$\ddot{r}_j = \lambda r_j, \quad \lambda \neq 0,$$

for all $j = 1, \ldots, n$.

Finding c.c.'s is an algebraic problem, since by equation (1), we have

$$\lambda r_i = -\sum_{j=1, j\neq i}^{n} \frac{m_j}{r_{ij}^3}(r_i - r_j), \tag{2}$$

for $i = 1, 2, \ldots, n$.

The first examples of Central Configurations

The first three c.c.'s were found in 1767 by Euler in the 3–body problem, for which three bodies are *collinear*.

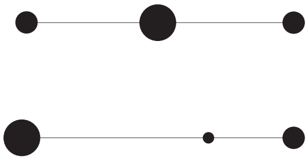

Figura: Euler's collinear c.c..

L. Euler, *De moto rectilineo trium corporum se mutuo attahentium*, Novi Comm. Acad. Sci. Imp. Petrop., **11** (1767), 144–151.

The first examples of Central Configurations

In 1772 Lagrange found two additional c.c.'s in the 3–body problem, where the three bodies are at the vertices of an *equilateral triangle*.

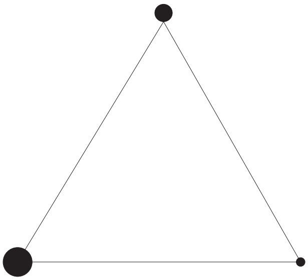

Figura: Lagrange's equilateral c.c..

J.L. Lagrange, *Essai sur le problème de trois corps*, Ouvres, vol 6, Gauthier–Villars, Paris, 1873.

Perpendicular Bisector Theorem

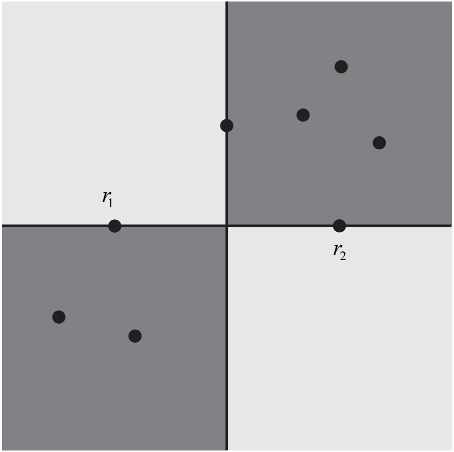

Figura: Not a central configuration.

Andoyer´s equations

For the planar central configurations instead of working with equation (2) we shall do with the Andoyer´s equations

$$f_{ij} = \sum_{k=1, k\neq i,j}^{n} m_k \left(R_{ik} - R_{jk}\right) \Delta_{ijk} = 0, \qquad (3)$$

for $1 \leq i < j \leq n$, where

$$R_{ij} = \frac{1}{r_{ij}^3}, \quad \Delta_{ijk} = (r_i - r_j) \wedge (r_i - r_k).$$

H. Andoyer, *Sur l'equilibre relatif de n corps*, Bulletin Astronomique 23 (1906) pp. 50–59.

A simple application of Andoyer´s equation

What are the non–collinear central configurations with 3 bodies?

$$\begin{cases} f_{12} = m_3\,(R_{12} - R_{23})\,\Delta_{123} = 0 \\ f_{13} = m_2\,(R_{12} - R_{32})\,\Delta_{132} = 0 \\ f_{23} = m_1\,(R_{21} - R_{31})\,\Delta_{321} = 0 \end{cases}$$

This equations are verified if, and only if,

$$R_{12} = R_{13} = R_{23} \Rightarrow r_{12} = r_{13} = r_{23},$$

That is, the bodies are at the vertices of an equilateral triangle.

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This equations are verified if, and only if,

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That is, the bodies are at the vertices of an equilateral triangle.
Note that this configurations does not depend on the values of the masses.

Question

Consider n bodies with position vectors $r_1, r_2, \ldots, r_n$ and positive masses $m_1, m_2, \ldots, m_n$ in a planar central configuration. Consider also $0 < k < n$. Is it possible to change the values of the masses

$$\bar{m}_{n-k} \neq m_{n-k}, \bar{m}_{n-k+1} \neq m_{n-k+1}, \ldots, \bar{m}_n \neq m_n$$

keeping fixed all the position vectors $r_1, r_2, \ldots, r_n$ and the values of the other masses $m_1, m_2, \ldots, m_{n-k-1}$ such that the n bodies are still in a central configuration?

Theorem case $k = 1$

Theorem. Consider the planar non–collinear n–body problem with $n \geq 4$. Then the central configurations for which is possible to change the value of one mass keeping fixed all the positions and the values of the other $n - 1$ masses and still have a central configuration are the central configurations with n bodies such that $n - 1$ bodies are in a co–circular central configuration and one body of arbitrary mass is at the center of the circle.

Proof for the case $k = 1$

Without loss of generality, suppose that the mass to be change is m_n. The proof of Theorem is divided into two lemmas.

Lemma 1. In order to have a central configuration in which the mass m_n can be changed it is necessary that the other $n-1$ bodies must be in a co–circular configuration with center at r_n.

Proof for the case $k = 1$

The planar Andoyer equations (3) must be satisfied for the n bodies. Consider the Andoyer equations with $i \neq n$ and $j \neq n$. These equations can be written as

$$f_{i,j} = \sum_{k \neq i,j,n} m_k \left(R_{i,k} - R_{j,k}\right) \Delta_{i,j,k} + m_n \left(R_{i,n} - R_{j,n}\right) \Delta_{i,j,n} = 0,$$

for all indices i and j, such that $0 < i < j < n$. Note that in equation the part under summation does not depend on the mass m_n. So, the variation of the mass m_n implies that the part under summation and the coefficient of m_n must vanish. Thus we have

$$\left(R_{i,n} - R_{j,n}\right) \Delta_{i,j,n} = 0,$$

for all indices i and j, such that $0 < i < j < n$.

Proof for the case $k = 1$

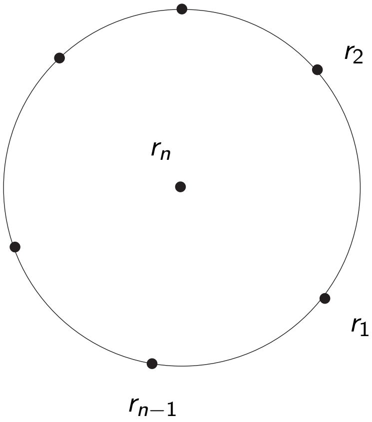

Figura: The only possible configuration with the continuous variation of m_n.

Proof for the case $k = 1$

By assumption the configuration is non–collinear, so in last equation at least one $\Delta_{i,j,n} \neq 0$. Without loss of generality, suppose $\Delta_{1,2,n} \neq 0$. Thus from

$$(R_{1,n} - R_{2,n})\,\Delta_{1,2,n} = 0$$

we have

$$R_{1,n} - R_{2,n} = 0,$$

which implies that $r_{1n} = r_{2n} = d > 0$. Therefore r_1 and r_2 belong to the circle of radius d and center at r_n.

Proof for the case $k = 1$

We can classify the other indices into two sets

$$\mathfrak{C}_1 = \{j : \Delta_{1,j,n} = 0\}\,, \quad \mathfrak{C}_2 = \{j : \Delta_{1,j,n} \neq 0\}\,,$$

that is, $\mathfrak{C}_1$ contains the indices of the bodies whose position vectors are collinear with r_1 and r_n, while $\mathfrak{C}_2$ contains the indices of the bodies whose position vectors are not collinear with r_1 and r_n. For $j \in \mathfrak{C}_2$ and from

$$(R_{1,n} - R_{j,n})\,\Delta_{1,j,n} = 0$$

we have

$$R_{1,n} - R_{j,n} = 0.$$

Thus, $r_{jn} = r_{1n} = d > 0$, for all $j \in \mathfrak{C}_2$. Then r_1, r_2 and r_j belong to the circle of radius d and center at r_n, for all $j \in \mathfrak{C}_2$.

Proof for the case $k = 1$

To complete the proof of the lemma we need to show that $\mathfrak{C}_1$ has at most one element. Suppose, by contradiction, that there exist two indices $b, c \in \mathfrak{C}_1$. So $\Delta_{1,b,n} = 0$, which implies that $\Delta_{2,b,n} \neq 0$. From

$$(R_{2,n} - R_{b,n})\,\Delta_{2,b,n} = 0,$$

we have

$$R_{2,n} - R_{b,n} = 0,$$

which implies that $r_{bn} = r_{2n} = d > 0$. Thus r_b belongs to the circle of radius d and center at r_n.

Proof for the case $k = 1$

As the central configurations are out of the collision set r_b must be diametrically opposite to r_1. Now consider the index $c \in \mathfrak{C}_1$. So $\Delta_{1,c,n} = 0$, which implies that $\Delta_{2,c,n} \neq 0$. From

$$(R_{2,n} - R_{c,n})\,\Delta_{2,c,n} = 0$$

we have

$$R_{2,n} - R_{c,n} = 0,$$

which implies that $r_{cn} = r_{2n} = d > 0$. Here we have a contradiction, since in this case r_c coincides with either r_1 or r_b.

Proof for the case $k = 1$

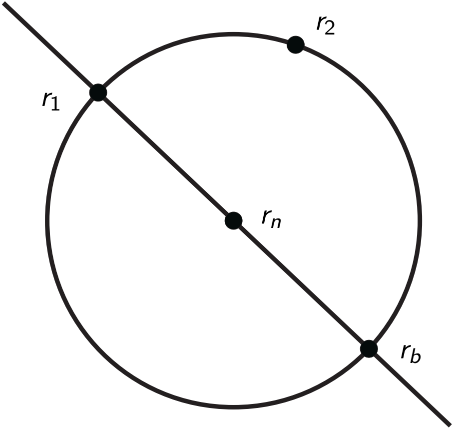

Figura: There is no position for r_c out of the collision set.

Proof for the case $k = 1$

Lemma 2. In order to have a planar central configuration in which the mass m_n can be changed it is necessary that the other $n-1$ co–circular bodies must be in a central configuration.

Note that, with $1 \leq i < j \leq n-1$

$$f_{i,j} = \sum_{k \neq i,j,n} m_k \left(R_{i,k} - R_{j,k}\right) \Delta_{i,j,k} + m_n \left(R_{i,n} - R_{j,n}\right) \Delta_{i,j,n} = 0,$$

the part under summation is exactly the Andoyer's equation for central configuration for the $n-1$ co–circular bodies.

Theorem case $k = 2$

Theorem Consider the planar non–collinear n–body problem with $n \geq 4$. There is not a central configuration for which is possible to change the values of the masses of two bodies keeping fixed all the positions and the values of the masses of the other $n-2$ bodies and still have a central configuration.

Proof for the case $k = 2$

Without loss of generality, suppose that the masses to be changed are m_{n-1} and m_n. Consider the Andoyer equations for $i = 1, 2, \ldots, n-2$ and $j = n-1$. These equations can be written as

$$f_{i,n-1} = \sum_{k \neq i, n-1, n} m_k \left(R_{i,k} - R_{n-1,k}\right) \Delta_{i,n-1,k} + m_n \left(R_{i,n} - R_{n-1,n}\right) \Delta_{i,n-1,n} = 0$$

Proof for the case $k = 2$

The change of the mass m_n implies that the coefficient of m_n must vanish, that is

$$(R_{i,n} - R_{n-1,n})\,\Delta_{i,n-1,n} = 0, \tag{4}$$

for all $0 < i < n-1$. With the same arguments for the mass m_{n-1} we have

$$(R_{i,n-1} - R_{n,n-1})\,\Delta_{i,n,n-1} = 0, \tag{5}$$

for all $0 < i < n-1$.

Proof for the case $k = 2$

From equations (4) and (5), the position vectors $r_1, \ldots, r_{n-2}$ must be either collinear with r_{n-1} and r_n or belong to the intersection of $\mathcal{C}_1$ and $\mathcal{C}_2$, where $\mathcal{C}_1$ is the circumference with center at r_{n-1} and radius $|r_n - r_{n-1}|$ and $\mathcal{C}_2$ is the circumference with center at r_n and radius $|r_n - r_{n-1}|$. Note that $\mathcal{C}_1 \cap \mathcal{C}_2$ determines two points in the plane. Since we consider non–collinear central configurations these two points must be position vectors of two bodies of the configuration, otherwise by the Perpendicular Bisector Theorem there is no such a central configuration. Without loss of generality, suppose that $\mathcal{C}_1 \cap \mathcal{C}_2 = \{r_1, r_2\}$.

Proof for the case $k = 2$

We have proved the following lemma.

Lemma. Consider the planar non–collinear n–body problem with $n \geq 4$. The following conditions are necessary in order to have a central configuration for which is possible to change the values of two masses keeping fixed all the positions and the values of the other $n-2$ masses and still have a central configuration (see Figure 6):

(a) r_1, r_2, r_{n-1} and r_n are at the vertices of a rhombus with $|r_1 - r_{n-1}| = |r_1 - r_n| = |r_2 - r_{n-1}| = |r_2 - r_n| = |r_{n-1} - r_n| \neq 0$.

(b) The other $n-4$ bodies belong to the straight line containing r_{n-1} and r_n.

Proof for the case $k = 2$

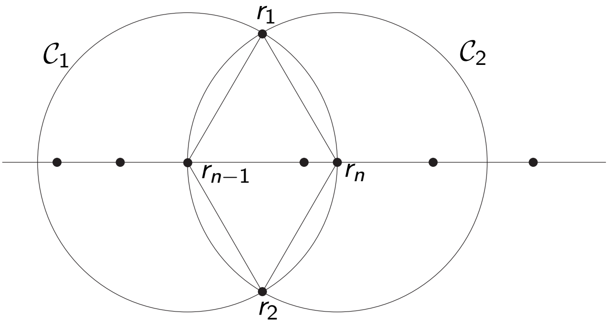

Figura: The positions of the bodies with fixed masses must be either collinear or belong to the intersection of two circumferences with centers at r_{n-1} and r_n and radius $|r_{n-1} - r_n|$.

Proof for the case $k = 2$

In the following lemma shows that the masses of the bodies at the intersections of $\mathcal{C}_1$ and $\mathcal{C}_2$ must be equal, that is $m_1 = m_2$.

Lemma. Consider the planar n–body problem, $n \geq 4$. Suppose that r_1, r_2, r_{n-1} and r_n are at the vertices of a rhombus with $|r_1 - r_{n-1}| = |r_1 - r_n| = |r_2 - r_{n-1}| = |r_2 - r_n| = |r_{n-1} - r_n| \neq 0$ and the other $n - 4$ bodies belong to the straight line containing r_{n-1} and r_n according to Figure. Then in order to have a central configuration a necessary condition is $m_1 = m_2$.

Proof for the case $k = 2$

Consider the Andoyer equations $f_{i,n} = 0$, for $i = 3, \ldots, n-2$, which can be written as

$$f_{i,n} = m_1 \left(R_{i,1} - R_{n,1}\right) \Delta_{i,n,1} + m_2 \left(R_{i,2} - R_{n,2}\right) \Delta_{i,n,2} + \sum_{k \neq 1,2,i,n} m_k \left(R_{i,k} - R_{n,k}\right) \Delta_{i,n,k} = 0. \tag{6}$$

In equation the part under summation is zero, since $\Delta_{i,n,k} = 0$. On the other hand, $\Delta_{i,n,1} = -\Delta_{i,n,2} \neq 0$, $R_{i,1} = R_{i,2}$ and $R_{n,1} = R_{n,2}$. Thus equation has the form

$$f_{i,n} = \left(m_1 - m_2\right) \left(R_{i,1} - R_{n,1}\right) \Delta_{i,n,1} = 0. \tag{7}$$

As $R_{i,1} - R_{n,1} \neq 0$ equation (7) is satisfied if and only if $m_1 = m_2$. This completes the proof of the lemma.

Proof for the case $k = 2$

Fix the following nomenclature: the masses to be changed are m_n and m_{n-1}; after the change these masses will be denoted by $M_n = m_n - \mu_n$ and $M_{n-1} = m_{n-1} - \mu_{n-1}$.

Proof for the case $k = 2$

Lemma. Consider the planar n–body problem with $n > 4$. Suppose that r_1, r_2, r_{n-1} and r_n form a rhombus with $|r_1 - r_{n-1}| = |r_1 - r_n| = |r_2 - r_{n-1}| = |r_2 - r_n| = |r_{n-1} - r_n| \neq 0$ and the other $n-4$ bodies belong to the straight line containing r_{n-1} and r_n according to Figure. Then in order to have a central configuration in which is possible to change the values of the masses m_n and m_{n-1} keeping fixed all the positions and other $n-2$ masses, the following equation must be satisfied

$$\frac{\mu_{n-1}}{\mu_n} = -\frac{(R_{1,n} - R_{i,n})\,\Delta_{1,i,n}}{(R_{1,n-1} - R_{i,n-1})\,\Delta_{1,i,n-1}}, \qquad (8)$$

for $2 < i < n-1$. Moreover, the quotient μ_{n-1}/μ_n must be positive.

Proof for the case $k = 2$

Consider the Andoyer equations $f_{1,i} = 0$, with $2 < i < n-1$.
These equations can be written as

$$f_{1,i} = \sum_{k \neq 1,i,n-1,n} m_k \left(R_{1,k} - R_{i,k}\right) \Delta_{1,i,k} \quad (9)$$

$$+ m_{n-1} \left(R_{1,n-1} - R_{i,n-1}\right) \Delta_{1,i,n-1} + m_n \left(R_{1,n} - R_{i,n}\right) \Delta_{1,i,n} = 0,$$

for $2 < i < n-1$. Consider the same equation after the change of the values of m_{n-1} and m_n

$$f_{1,i} = \sum_{k \neq 1,i,n-1,n} m_k \left(R_{1,k} - R_{i,k}\right) \Delta_{1,i,k} \quad (10)$$

$$+ M_{n-1} \left(R_{1,n-1} - R_{i,n-1}\right) \Delta_{1,i,n-1} + M_n \left(R_{1,n} - R_{i,n}\right) \Delta_{1,i,n} = 0.$$

Proof for the case $k = 2$

Note that the parts under summation in equations (9) and (10) are equal. Taking the difference of (9) and (10), we have

$$m_{n-1}\left(R_{1,n-1} - R_{i,n-1}\right)\Delta_{1,i,n-1} + m_n\left(R_{1,n} - R_{i,n}\right)\Delta_{1,i,n}$$
$$-M_{n-1}\left(R_{1,n-1} - R_{i,n-1}\right)\Delta_{1,i,n-1} - M_n\left(R_{1,n} - R_{i,n}\right)\Delta_{1,i,n} = 0,$$

which implies equation (8). We need to prove that the quotient in equation (8) is positive. Consider equation (8) written in the following form

$$\mu_{n-1}\left(R_{1,n-1} - R_{i,n-1}\right)\Delta_{1,i,n-1} + \mu_n\left(R_{1,n} - R_{i,n}\right)\Delta_{1,i,n} = 0. \quad (11)$$

Proof for the case $k = 2$

Suppose, by contradiction, that $\mu_{n-1}\mu_n < 0$. So, in order to satisfy (11) the coefficients of μ_{n-1} and μ_n must have the same sign. Suppose that there exists a body of index i out of the rhombus and located to the right of r_n. This implies that the terms $\Delta_{1,i,n-1}$, $\Delta_{1,i,n}$ are negative and the term $R_{1,n-1} - R_{i,n-1}$ is positive. So, the term $R_{1,n} - R_{i,n}$ must be positive. But this implies that $|r_1 - r_n| < |r_i - r_n|$. The same assertion holds for all bodies out of the rhombus and located to the right of r_n. All these cases lead a contradiction with the Perpendicular Bisector Theorem.

Proof for the case $k = 2$

The same argument can be used for the bodies out of the rhombus and located to the left of r_{n-1}. Thus all collinear bodies must be in the interior of the rhombus, but this also leads to a contradiction with the Perpendicular Bisector Theorem. This part of the proof implies that all bodies in the configuration must belong to the interior of the union of the sets bounded by the circumferences $\mathcal{C}_1$ and $\mathcal{C}_2$.

Proof for the case $k = 2$

An important consequence of this lemma is that in a central configuration in which is possible to change the values of two masses (m_n and m_{n-1}) keeping fixed all the positions and other $n-2$ masses, the position vectors of the collinear bodies must satisfy

$$\frac{(R_{1,n} - R_{i,n})\,\Delta_{1,i,n}}{(R_{1,n-1} - R_{i,n-1})\,\Delta_{1,i,n-1}} = \frac{(R_{1,n} - R_{k,n})\,\Delta_{1,k,n}}{(R_{1,n-1} - R_{k,n-1})\,\Delta_{1,k,n-1}} < 0, \tag{12}$$

for all i and k such that $2 < i, k < n-1$.

Proof for the case $k = 2$

Consider a system of coordinates formed by two axes, one passing through r_1 and r_2 and the other passing through the line containing the collinear bodies. See Figure 7. Without loss of generality, we assume the coordinates $r_1 = (0, \sqrt{3})$, $r_2 = (0, -\sqrt{3})$, $r_{n-1} = (-1, 0)$, $r_n = (1, 0)$ and $r_i = (r_i, 0)$ (using r_i as a scalar variable) for $i = 3, \ldots, n-2$. We study the equation

$$\frac{(R_{1,n} - R_{i,n})\,\Delta_{1,i,n}}{(R_{1,n-1} - R_{i,n-1})\,\Delta_{1,i,n-1}} = -a,$$

with $a > 0$, or equivalently the equation

$$(R_{1,n} - R_{i,n})\,\Delta_{1,i,n} + a\,(R_{1,n-1} - R_{i,n-1})\,\Delta_{1,i,n-1} = 0. \qquad (13)$$

Since r_1, r_2, r_{n-1} and r_n are fixed in our system of coordinates, equation (13) can be written as a polynomial equation of degree five in the variable r_i.

Proof for the case $k = 2$

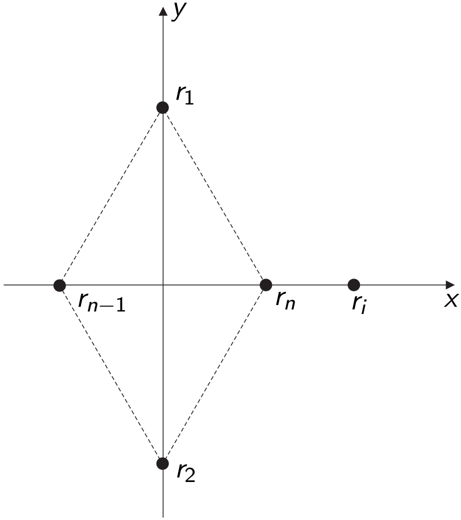

Figura: System of coordinates formed by two axes, one passing through r_1 and r_2 and the other passing through r_{n-1}, r_n and the other bodies r_i, $i = 3, \ldots, n-2$.

Proof for the case $k = 2$

By the construction of the coordinates the terms $(R_{1,n} - R_{i,n})\,\Delta_{1,i,n}$ and $(R_{1,n-1} - R_{i,n-1})\,\Delta_{1,i,n-1}$ always have the pure imaginary roots $\imath\sqrt{3}|r_1 - r_n|/2 = \imath\sqrt{3}$ and $-\imath\sqrt{3}|r_1 - r_n|/2 = -\imath\sqrt{3}$, where $\imath = \sqrt{-1}$. Therefore the polynomial equation (13) has at most three real roots, for every $a > 0$. An straightforward computation shows that a as function of r_i is strictly increasing in $(-3,-1) \cup (1,3)$ and strictly decreasing in $(-1,1)$, which are the intervals of interest in our problem.

Proof for the case $k = 2$

Note that for each value of a equation (12) is satisfied by an index i when r_i is a root of (13). Since (13) has at most three real roots, there are at most three possible positions to the collinear bodies. Moreover, for each positive value of a we have exactly one root in $(-3, 1)$, one root in $(-1, 1)$ and one root in $(1, 3)$. Thus n must be less than 8, because in the cases $n \geq 8$ collisions are always required in order to satisfy equation (12), for all indices. We have proved the following lemma.

Proof for the case $k = 2$

Lemma. Consider the planar non–collinear n–body problem with $n \geq 8$. There are no central configurations for which is possible to change the values of two masses keeping fixed all the positions and the values of the other $n - 2$ masses and still have a central configuration.

Proof for the case $k = 2$

Now we prove that there are no such kind of central configurations for the remaining cases: $n = 4$, $n = 5$, $n = 6$ and $n = 7$. We divide the proof into four lemmas.

Lemma. Consider the planar 4–body problem. Suppose that r_1, r_2, r_3 and r_4 form a rhombus with $r_{13} = r_{14} = r_{23} = r_{24} = r_{34}$ according to Figure. Then there are no positive masses for which this configuration is a central configuration.

The proof is a direct corollary of the Perpendicular Bisector Theorem.

Proof for the case $k = 2$

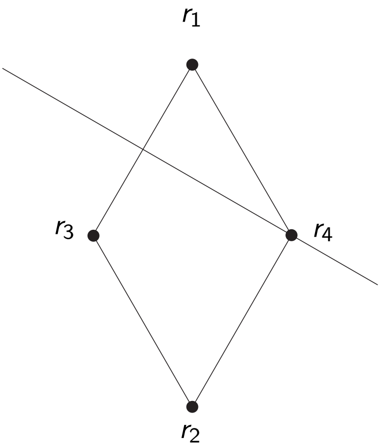

Figura: This configuration can not be a central configuration.

Proof for the case $k = 2$

Lemma. Consider the planar 5–body problem. Suppose that r_1, r_2, r_4 and r_5 form a rhombus with $r_{14} = r_{15} = r_{24} = r_{25} = r_{45}$ and r_3 belongs to the straight line containing r_4 and r_5 according to Figure. Then there are no positive masses for which this configuration is a central configuration.

Proof for the case $k = 2$

The position vector r_3 can not belong to the interior of the rhombus. This is a direct consequence of the Perpendicular Bisector Theorem. With r_3 out of the convex hull of the rhombus, consider the Andoyer equation $f_{1,4} = 0$. Taking into account the symmetries it can be written as

$$f_{1,4} = m_3 \left(R_{1,3} - R_{4,3} \right) \Delta_{1,4,3} = 0. \tag{14}$$

Equation (14) is satisfied if and only if r_3 coincides with either r_4 or r_5, but this is a contradiction.

Proof for the case $k = 2$

Lemma. Consider the planar 6–body problem. Suppose that r_1, r_2, r_5 and r_6 form a rhombus with $r_{16} = r_{15} = r_{26} = r_{25} = r_{56}$, the position vectors r_3 and r_4 belong to the straight line containing r_5 and r_6 according to Figure. Suppose also $m_1 = m_2$. Then there are no positive masses for which this configuration form a central configuration satisfying: it is possible to change the values of m_5 and m_6 keeping fixed all the positions and other four masses and still have a central configuration.

Proof for the case $k = 2$

For six bodies we have fifteen Andoyer equations (3). By our assumptions of symmetries the following equations are already verified

$$f_{1,2} = 0, \quad f_{3,4} = 0, \quad f_{3,5} = 0, \quad f_{3,6} = 0,$$

$$f_{4,5} = 0, \quad f_{4,6} = 0, \quad f_{5,6} = 0.$$

The remaining equations are $f_{1,j} = 0$ and $f_{2,j} = 0$, with $3 \leq j \leq 6$. The assumption $m_1 = m_2$ and the symmetries imply that $f_{1,j} = 0$ if and only if $f_{2,j} = 0$. So we just study the equations $f_{1,j} = 0$.

Proof for the case $k = 2$

$$f_{1,3} = m_2\left(R_{1,2} - R_{3,2}\right)\Delta_{1,3,2} + m_4\left(R_{1,4} - R_{3,4}\right)\Delta_{1,3,4}$$
$$+m_5\left(R_{1,5} - R_{3,5}\right)\Delta_{1,3,5} + m_6\left(R_{1,6} - R_{3,6}\right)\Delta_{1,3,6} = 0,$$

$$f_{1,4} = m_2\left(R_{1,2} - R_{4,2}\right)\Delta_{1,4,2} + m_3\left(R_{1,3} - R_{4,3}\right)\Delta_{1,4,3}$$
$$+m_5\left(R_{1,5} - R_{4,5}\right)\Delta_{1,4,5} + m_6\left(R_{1,6} - R_{4,6}\right)\Delta_{1,4,6} = 0,$$

$$f_{1,5} = m_2\left(R_{1,2} - R_{5,2}\right)\Delta_{1,5,2} + m_3\left(R_{1,3} - R_{5,3}\right)\Delta_{1,5,3}$$
$$+m_4\left(R_{1,4} - R_{5,4}\right)\Delta_{1,5,4} + m_6\left(R_{1,6} - R_{5,6}\right)\Delta_{1,5,6} = 0,$$

$$f_{1,6} = m_2\left(R_{1,2} - R_{6,2}\right)\Delta_{1,6,2} + m_3\left(R_{1,3} - R_{6,3}\right)\Delta_{1,6,3}$$
$$+m_4\left(R_{1,4} - R_{6,4}\right)\Delta_{1,6,4} + m_5\left(R_{1,5} - R_{6,5}\right)\Delta_{1,6,5} = 0.$$

Proof for the case $k = 2$

Equivalently the above equations can be written as

$$H\vec{M} = \vec{0}, \tag{15}$$

where,

$$\vec{M} = (m_2, m_3, m_4, m_5, m_6)^t, \qquad \vec{0} = (0, 0, 0, 0, 0)^t$$

and

$$H = \begin{bmatrix} h_{11} & 0 & h_{13} & h_{14} & h_{15} \\ h_{21} & h_{22} & 0 & h_{24} & h_{25} \\ h_{31} & h_{32} & h_{33} & 0 & 0 \\ h_{41} & h_{42} & h_{43} & 0 & 0 \end{bmatrix}$$

If H has maximum rank the solutions of (15) are parallel to the vector $\vec{T} = (T_1, -T_2, T_3, -T_4, T_5)$, where T_k is the determinant of the matrix obtained from H deleting the column k.

Proof for the case $k = 2$

Consider again equation (13) in the context of six bodies

$$\frac{(R_{1,6} - R_{3,6})\,\Delta_{1,3,6}}{(R_{1,5} - R_{3,5})\,\Delta_{1,3,5}} = \frac{(R_{1,6} - R_{4,6})\,\Delta_{1,4,6}}{(R_{1,5} - R_{4,5})\,\Delta_{1,4,5}} = -a < 0.$$

Using this relation in the matrix H we have

$$H = \left[\begin{array}{ccccc} h_{11} & 0 & h_{13} & -ah_{15} & h_{15} \\ h_{21} & h_{22} & 0 & -ah_{25} & h_{25} \\ h_{31} & h_{32} & h_{33} & 0 & 0 \\ -h_{31} & h_{42} & h_{43} & 0 & 0 \end{array}\right],$$

which implies that $\vec{T} = (T_1, -T_2, T_3, -T_4, T_5) = (0, 0, 0, aT_5, T_5)$. But in this case it is necessary that the masses m_2, m_3 and m_4 vanish, assuming that the rank of H is equal to four.

Proof for the case $k = 2$

Our last case is $n = 7$. We have the following lemma.
Lemma. Consider the planar 7–body problem. Suppose that r_1, r_2, r_6 and r_7 form a rhombus with $r_{16} = r_{17} = r_{26} = r_{27} = r_{67}$, the position vectors r_3, r_4 and r_5 belong to the straight line containing r_6 and r_7 and $m_1 = m_2$. Then there are no positive masses for which this configuration form a central configuration satisfying: it is possible to change the values of m_6 and m_7 keeping fixed all the positions and other five masses and still have a central configuration.
The proof follows in an analogous way to last Lemma.
This ends the proof of the theorem.

Proof for the case $k = 2$

Thanks for your Attention!