On Central Configurations of Twisted Rings

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Outline

Planar Central Configuration of κn Bodies

We consider

- κ groups of n bodies: $\mathbf{q}_{ji} \in \mathbb{R}^2$, $i = 1, \ldots, n$, $j = 1, \ldots, \kappa$
- All bodies in the same group have equal mass m_j, $j = 1, \ldots, \kappa$
- $\displaystyle\sum_{j=1}^{\kappa} m_j(\mathbf{q}_{j1} + \cdots + \mathbf{q}_{jn}) = 0$

A central configuration is $\mathbf{q} = (\mathbf{q}_{11}, \mathbf{q}_{12}, \ldots, \mathbf{q}_{\kappa n}) \in \mathbb{R}^{2\kappa n}$ such that, for some value of w, satisfies the algebraic equation

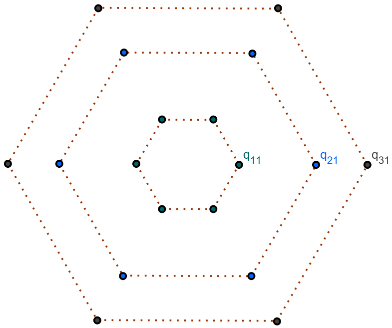
$$\nabla U(\mathbf{q}) + w^2 M \mathbf{q} = 0$$

$$U(\mathbf{q}) = \sum_{j=1}^{\kappa} \sum_{i=1}^{n} \left(\sum_{l=i+1}^{n} \frac{m_j^2}{||\mathbf{q}_{ji} - \mathbf{q}_{jl}||} + \sum_{l=j+1}^{n} \sum_{m=1}^{n} \frac{m_j m_l}{||\mathbf{q}_{ji} - \mathbf{q}_{lm}||} \right)$$

Planar Central Configuration of κn Bodies

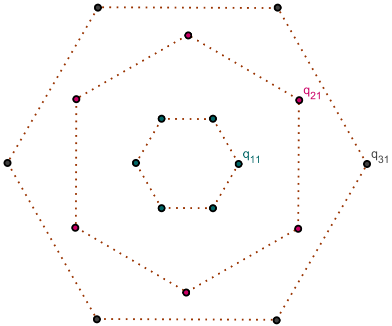
The particles within the same group are located at the vertices of an n-gon.

Nested rings



Vertices align

Twisted rings



Rotation of π/n

Previous Results

- Moeckel and Simó (1995)
  Two nested rings for any n: for every ratio m_1/m_2, there are exactly two planar central configurations, one with the ratio of the sizes of the two polygons less than 1, and the other one greater than 1.

- Zhang and Zhou (2002)
  Two rings: they take all the masses different, imposing CC equations, the masses within the same group must be equal.
  Corollary (MacMillan-Bartky): in the case of two twisted rings of two bodies, the ratio of the sizes of the two polygons $a \in (1/\sqrt{3}, \sqrt{3})$.

- Yu and Zhang (2012)
  CC of two rings: if two rings, one rotated with respect the other an angle θ, are a CC then $\theta = 0$ (nested) or $\theta = \pi/n$ (twisted)

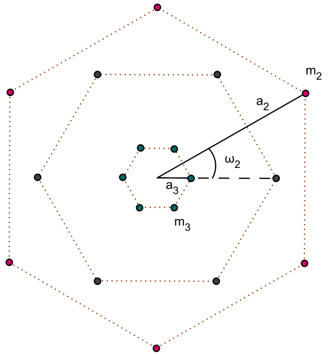
Previous Results

- Corbera, Delgado, Llibre (2009)
  Nested κ n-gons: there exist planar central configurations of κ nested rings of n-gons
- Llibre, Melo (2009)
  Particular cases of twisted rings: existence of central configurations of twisted rings for $\kappa = 3$ and $n = 2, 3$ and $\kappa = 4$ and $n = 2$.
- Roberts (1999 PhD), Sekiguchi (2004), Lei and Santoprete (2006)
  CC of two rings and a central mass: number of central configurations and bifurcations varying the value of the central mass

Equations (1)

- Size of the n-gon: a_j, $j = 1, \ldots, \kappa$
- Argument of the first particle in a group: ϖ_j, $j = 1, \ldots, \kappa$
- First group: $m_1 = 1$, $a_1 = 1$ and $\varpi_1 = 0$

$$\mathbf{q}_{j1} = a_j e^{i\varpi_j},\ \mathbf{q}_{jk} = \mathbf{q}_{j1} e^{i2\pi(k-1)/n}, \quad k = 2, \ldots, n,\ j = 1, \ldots, \kappa.$$



Equations (2)

Using the symmetry of the problem, the CC equation reduces to

$$(\mathrm{Eq}_j): \quad \frac{\partial U}{\partial \mathbf{q}_{j1}} + w^2 m_j \mathbf{q}_{j1} = 0 \quad \longrightarrow \quad \begin{cases} \Re(\mathrm{Eq}_j) = 0 \\ \Im(\mathrm{Eq}_j) = 0 \end{cases}$$

for all $j = 1, \ldots, \kappa$.

$\Im(\mathrm{Eq}_j) = 0$ if and only if:

$$\sum_{\substack{l=1 \\ l \neq j}}^{\kappa} m_l a_l \sum_{k=1}^{n} \frac{\sin(\varpi_l - \varpi_j + 2\pi k/n)}{(a_l^2 + a_j^2 - 2a_l a_j \cos(\varpi_l - \varpi_j + 2\pi k/n))^{3/2}} = 0$$

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Equations (3)

Fixed $\{\varpi_j\}_{j=1,\ldots,\kappa}$ with $\varpi_j \in \{0, \pi/n\}$,

$$(C_{j1} - S_n a_j)m_1 + \sum_{\substack{l=2 \\ l\neq j}}^{\kappa} (C_{jl} - a_j C_{1l})m_l + \left(\frac{S_n}{a_j^2} - a_j C_{1j}\right) m_j = 0,$$

$$j = 2, \ldots, \kappa$$

$\kappa - 1$ equations with $2\kappa - 2$ unknowns: $m_2, \ldots, m_\kappa$, $a_2, \ldots, a_\kappa$

$$S_n = \frac{1}{4}\sum_{k=1}^{n-1} \frac{1}{\sin(k\pi/n)},$$

$$C_{jl} = C_{jl}(a_j, a_l) = \sum_{k=1}^{n} \frac{a_j - a_l \cos(\varpi_j - \varpi_l + 2k\pi/n)}{(a_j^2 + a_l^2 - 2a_j a_l \cos(\varpi_j - \varpi_l + 2k\pi/n))^{3/2}}.$$

Outline

Two twisted rings

The unknowns are $m = m_2$ and $a = a_2$

$$(C_{21}(a) - S_n a) + \left(\frac{S_n}{a^2} - aC_{12}(a)\right) m = 0$$

where

$$C_{21}(a) = -\phi'(a), \qquad C_{12}(a) = \phi(a) + a\phi'(a)$$

and

$$\phi(a) = \sum_{k=1}^{\kappa} \frac{1}{(1 + a^2 - 2a\cos\frac{(2k-1)\pi}{n})^{1/2}}$$

Two twisted rings

Inverse problem

$$m = H(a) = \frac{a^2(S_n a - C_{21}(a))}{S_n - a^3 C_{12}(a)}, \quad a > 0.$$

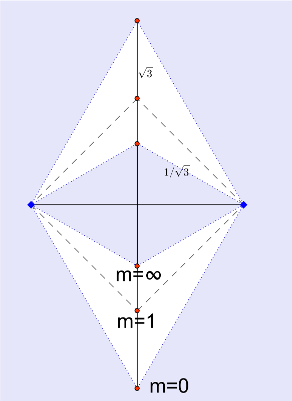
1. $H(1) = 1$. If the two n-gons have the same size, all the masses are equal.
2. $H(1/a) = 1/H(a)$.
   If (m, a) corresponds to a CC $\longrightarrow$ $(1/m, 1/a)$ is also a CC
3. The CC is convex if

$$\cos\left(\frac{\pi}{n}\right) \leq a \leq \frac{1}{\cos\left(\frac{\pi}{n}\right)}$$

Results for $n = 2$

For each value of the mass ratio m, there exist only one central configuration of two twisted 2-gons of the 4-body problem (rhomboidal central configuration)

- Admissible sizes $a \in (1/\sqrt{3}, \sqrt{3})$
- $a = 1/\sqrt{3} \rightarrow m = \infty$
- $a = \sqrt{3} \rightarrow m = 0$
- All the CC are convex



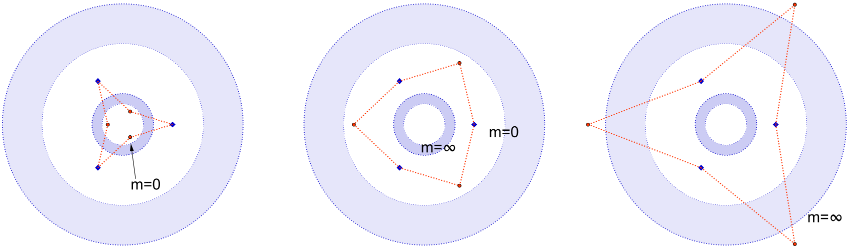
Results for $n = 3$

Theorem

For $n = 3$, the set of admissible values of a such that there exist a CC is

$$(0, z_1) \cup (1/z_2, z_2) \cup (1/z_1, \infty),$$

where $z_1 < 1 < z_2$ are the only two positive zeros of $m = H(a) = 0$, and $z_1 z_2 < 1$



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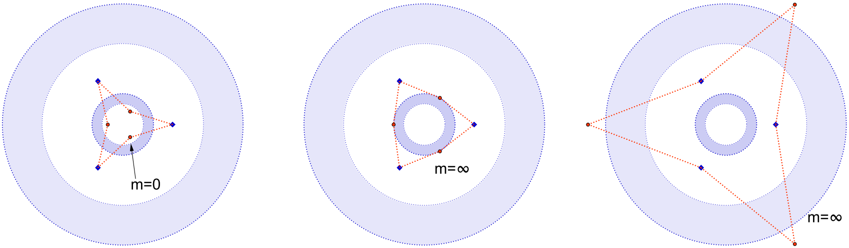

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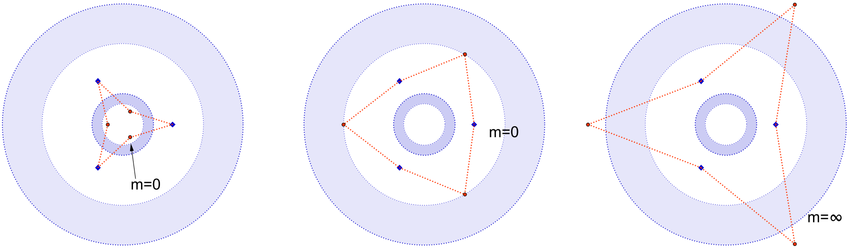


convex

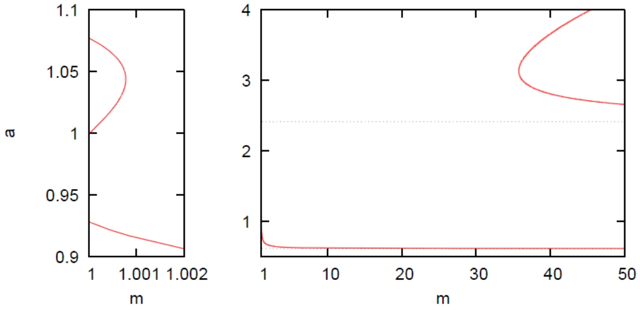
Results for $n = 3$

Theorem

Given a value of the mass ratio $m \geq 1$, the number of different CC are

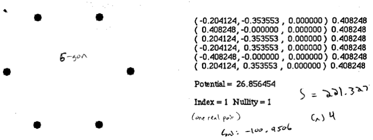
- three for any $m \in (1, m^*) \cup (m^{**}, \infty)$
- one for any $m \in (m^*, m^{**})$
- two for $m = 1, m^*, m^{**}$.

$$m^* \simeq 1.000768, \qquad m^{**} \simeq 35.700177$$

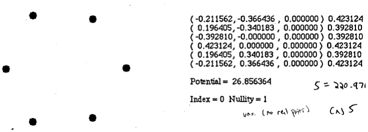


Convex CC for the 6-body problem

- Moeckel: numerical exploration of CC for the N-body problem with equal masses ($N = 4, \ldots, 8$). In particular, for $N = 6$ there exist two convex CC:



$a = 1$



$a \simeq 1.077171777$

- For any $m \in [1, 1.000768)$ there exist two convex CC of the 6-body problem

Results for $n \geq 4$

Proposition

There exist at least two positive solutions, z_1 and z_2, of $m = H(a) = 0$ with $z_1 < 1 < z_2$ and and $z_1 z_2 > 1$.

Conjecture

For $n \geq 4$ the equation $H(a) = 0$ has only two solutions z_1, z_2 in $(0, \infty)$.

Results for $n \geq 4$

Theorem

Suppose that the Conjecture is true. Then, for $n \geq 4$, the set of admissible values of a such that there exist a CC of two twisted n-gons is

$$(0, 1/z_2) \cup (z_1, 1/z_1) \cup (z_2, \infty)$$

where $z_1 < 1 < z_2$ are the only two positive zeros of $m = H(a) = 0$, and $z_1 z_2 > 1$

Notice that $m \in (0, \infty)$ for a in any of the three admissible intervals

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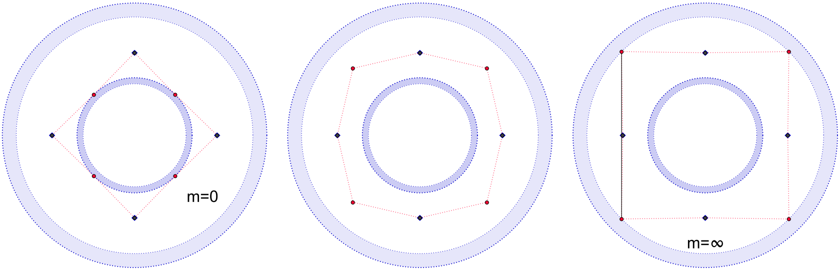
Notice that $m \in (0, \infty)$ for a in any of the three admissible intervals

Theorem

For $n > 4$ any CC of two twisted n-gons with $a \in (z_1, 1/z_1)$ is convex

Convex CC for $n = 4$



$(0.69738051, 0.707106781)$ $(0.707106781, 1.414213563)$ $(1.414213563, 1.433937407)$

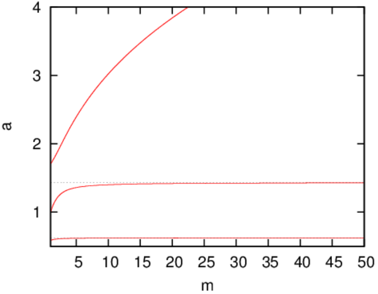
convex

- not all the configurations $a \in (z_1, 1/z_1)$ are convex.
- convex configurations correspond to $m \in (0.062278912, 16.05679941)$
- there are not convex CC of twisted rings for certain values of m

Results for $n \geq 4$

Theorem

Suppose that the Conjecture is true. Then, for two nested rings with $n \geq 4$ bodies in each ring, and for any positive value of the mass ratio m there exists at least three different central configurations.



Outline

Introduction

CC of two twisted rings

CC of three twisted rings

Three twisted rings

Fixed values for ϖ_2 and ϖ_3, the unknowns are m_2, m_3 and a_2, a_3.

$$\begin{aligned}(S_n - a_2^3 C_{12}(a_2))\, m_2 + a_2^2(C_{23}(a_2,a_3) - a_2 C_{13}(a_3))\, m_3 &= a_2^2(S_n a_2 - C_{21}(a_2)),\\ a_3^2(C_{32}(a_2,a_3) - a_3 C_{12}(a_2))\, m_2 + (S_n - a_3^3 C_{13}(a_3))\, m_3 &= a_3^2(S_n a_3 - C_{31}(a_3)).\end{aligned}$$

- Nested: $\varpi_2 = \varpi_3 = 0$
- Twisted: one of the rings is rotated with respect the other two. It is enough to restrict the study to

$$\varpi_2 = \pi/n, \varpi_3 = 0, \qquad \text{and} \qquad 0 < a_2,\ 0 < a_3 < 1$$

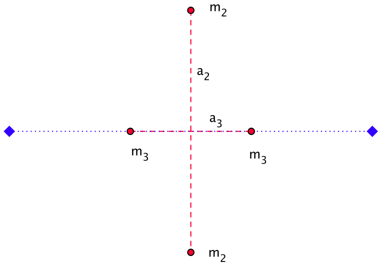
Results for $n = 2$

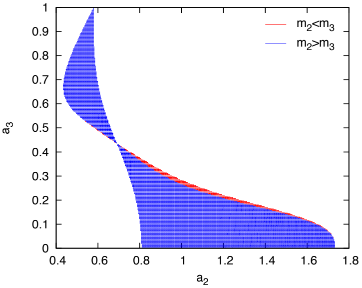
Theorem

Consider three rings of two bodies in each ring, all of them with equal masses. Then, if they are in a twisted central configuration, then $a_3 < a_2 < 1$, that is, the twisted ring is in the middle of the other two.

Numerical results

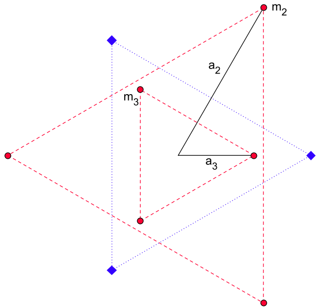
Region of admissible values of (a_2, a_3) for $n = 2$

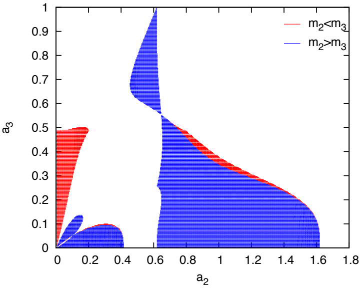




Numerical results

Region of admissible values of (a_2, a_3) for $n = 3$





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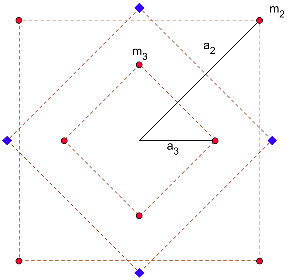
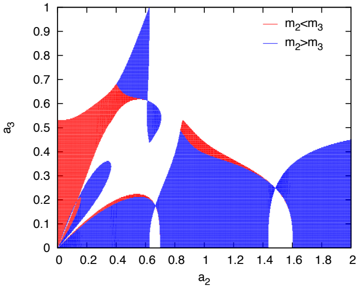
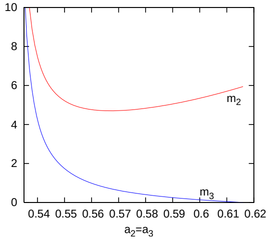




Numerical results for $a_2 = a_3$

Conjecture If $a_2 = a_3$ then $m_2 > m_3$

$n = 2$



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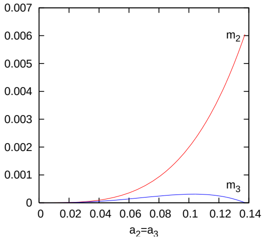

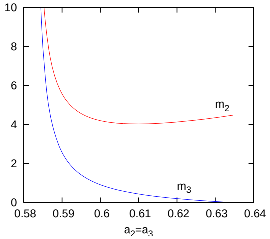

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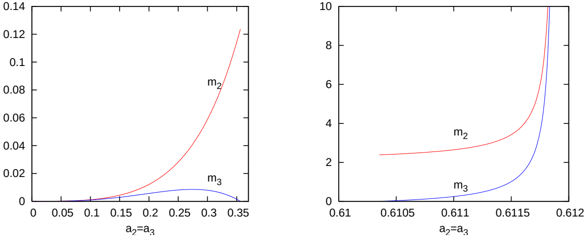

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