Instability of high dimensional Hamiltonian Systems: multiple resonances do not impede diffusion[1]

HAMSYS 2014
in honor of Professor Clark Robinson's 70th birthday
CRM, Bellaterra, June 2 to 6, 2014

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June 5th, 2014

[1]Preprint in arXiv, mp_arc

Instability for a priori unstable Hamiltonian systems

We consider a periodic in time perturbation of n pendula and a d-dimensional rotor described by the non-autonomous Hamiltonian,

$$
\begin{aligned}
H(p,q,I,\varphi,t,\varepsilon) &= P(p,q) + h(I) + \varepsilon Q(p,q,I,\varphi,t,\varepsilon),\\
P(p,q) = \sum_{j=1}^{n} P_j(p_j,q_j), &\quad P_j(p_j,q_j) = \pm\left(\frac{1}{2}p_j^2 + V_j(q_j)\right),
\end{aligned}
\tag{1}
$$

where $I \in \mathcal{I} \subset \mathbb{R}^d, \varphi \in \mathbb{T}^d$, $\mathcal{I}$ an open set, $p, q \in \mathbb{R}^n$, $t \in \mathbb{T}^1$, and $P_j(p_j, q_j)$ is a *pendulum* for the saddle variables p_j, q_j.

For $\varepsilon = 0$, the d-dimensional action I remains constant.

Main result: If Q is a trigonometric polynomial in the variables (φ, t), under general, verifiable, non-degeneracy assumptions, there are orbits of the system in which the I variables can perform largely arbitrary excursions in a set $\mathcal{I}^* \subset \mathcal{I}$ of size of order 1 (that is, independent of ε as $\varepsilon \to 0$).

Elementary and regularity assumptions

We will assume:

- **H1** The functions h, V_j, Q are C^r , $r \geq r_0$ sufficiently large.
- **H2** The potentials V_j have non-degenerate local maxima, say at $q_j = 0$, each of which gives rise to homoclinic orbit $(p_j^*(t), q_j^*(t))$ of the pendulum P_j:

$$\frac{d}{dt} q_j^*(t) = p_j^*(t); \quad \frac{d}{dt} p_j^*(t) = -V_j'(q_j^*(t)); \qquad (2)$$
$$\lim_{t \to \pm\infty} (p_j^*(t), q_j^*(t)) = (0, 0).$$

- **H3** The mapping $I \to \omega(I) := \frac{\partial}{\partial I} h(I)$ is a local diffeomorphism from some domain $\mathcal{I}^* \subset \mathcal{I}$ to its image.

Simplifying assumption

We will furthermore assume the simplifying hypothesis:

- **H4** The function Q in (1) is a trigonometric polynomial on (φ, t):

$$Q(p, q, I, \varphi, t, \varepsilon) = \sum_{(k,l)\in\mathcal{N}_Q} Q_{k,l}(p, q, I, \varepsilon) \exp(2\pi\ i(k \cdot \varphi + lt)) \qquad (3)$$

with $\mathcal{N}_Q \subset \mathbb{Z}^d \times \mathbb{Z}$ a finite set, and $Q_{k,l} \not\equiv 0$ in $\mathcal{I} \times \mathcal{U}$, if $(k, l) \in \mathcal{N}_Q$.

Remark

*For the case $d = 1$, $n = 1$, **H4** appeared in [D-Llave-S06], and was eliminated in [D-Huguet09], [Gidea-Llave06], under generic assumptions. Similar improvements are clearly possible in this higher dimensional case.*

Non-degeneracy assumptions

- **H5** Consider the set of integer indexes $\mathcal{N}^{[\leq 2]} = \mathcal{N}_1 \cup \mathcal{N}_2 \subset \mathbb{Z}^{d+1}$ where $\mathcal{N}_1$ is the support of the Fourier series of the perturbation $Q(I, \varphi, p, q, t; 0)$, $\mathcal{N}_2 = (\mathcal{N}_1 + \mathcal{N}_1) \cup \bar{\mathcal{N}}$, where $\bar{\mathcal{N}}$ is the support of the Fourier series of $\frac{\partial Q}{\partial \varepsilon}(I, \varphi, p, q, t; 0)$.
  Then we assume that, for any $(k, l) \neq (0, 0) \in \mathcal{N}^{[\leq 2]}$, the set

  $$\{I \in \mathcal{I}^*, Dh(I)k + l = 0, \quad k^\top D^2 h(I) k = 0\} \tag{4}$$

  is empty or a manifold of codimension two in $\mathcal{I}^*$.
  Note: If $\tilde{h}(I_0, I) := I_0 + h(I)$ is a quasi convex function, assumption **H5** is true for any perturbation Q.
- **H6** Assume that the perturbation Q satisfies some non-degeneracy conditions, stated later, in the connected domain $\mathcal{I}^* \times \mathbb{T}^{d+1}$, related to the existence of whiskered tori for the averaged Hamiltonian.

Melnikov potential

$$L(\tau, I, \varphi, s) = -\int_{-\infty}^{\infty} \Big[Q(p^*(\tau+\sigma), q^*(\tau+\sigma) I, \varphi + \sigma\omega(I), s+\sigma, 0) \\ - Q(0,0,I,\varphi+\sigma\omega(I), s+\sigma, 0) \Big] \, d\sigma \tag{5}$$

where $\tau = (\tau_1, \dots, \tau_n)$, $p^*(\tau+\sigma) = (p_1^*(\tau_1+\sigma), \dots, p_n^*(\tau_n+\sigma))$, $q^*(\tau+\sigma) = (q_1^*(\tau_1+\sigma), \dots, q_n^*(\tau_n+\sigma))$.

- **H7** Assume that, for any value of $I \in \mathcal{I}^*$, there exists a non-empty set $\mathcal{J}_I \subset \mathbb{T}^{d+1}$, with the property that when $(I, \varphi, s) \in H_-$, where

  $$H_- = \bigcup_{I \in \mathcal{I}^*} \{I\} \times \mathcal{J}_I \subset \mathcal{I}^* \times \mathbb{T}^{d+1}, \tag{6}$$

  the system of equations $\dfrac{\partial}{\partial \tau} L(\tau, I, \varphi, s) = 0$ admits a non degenerate solution $\tau = \tau^*(I, \varphi, s)$ with τ^* a smooth function.

Poincaré reduced function

- **H8** Define the auxiliary functions (related to the so-called scattering map):

$$\mathcal{L}(I,\varphi,s)=L(\tau^*(I,\varphi,s),I,\varphi,s),\quad \mathcal{L}^*(I,\theta)=\mathcal{L}(I,\theta,0) \qquad (7)$$

  Assume that the reduced Poincaré function $\mathcal{L}^*(I,\varphi-\omega(I)s)$ satisfies some non-degeneracy conditions, stated later, on in the domain H_-.

Remark

Assumption **H8** *is simplified by the very simple condition:*

- **H8'**: $\forall I\in\mathcal{I}^*$, *the reduced Poincaré function* $\mathcal{L}^*(I,\theta)$ *defined in* (7) *has non-degenerate critical points.*

Main Result

Theorem

Let H be a Hamiltonian of the form (1) satisfying the elementary assumptions **H1, H2**, *the regularity assumption* **H3**, *the simplifying assumption* **H4** *and the non-degeneracy assumptions* **H5, H6, H7, H8**. *Then, for every $\delta > 0$, there exists $\varepsilon_0 > 0$, such that for every $0 < |\varepsilon| < \varepsilon_0$, given $I_\pm \in \mathcal{I}^*$, there exists an orbit $\tilde{x}(t)$ of (1) and $T > 0$, such that:*

$$\begin{aligned} |I(\tilde{x}(0)) - I_-| &\leq C\delta \\ |I(\tilde{x}(T)) - I_+| &\leq C\delta. \end{aligned} \tag{8}$$

Remark

Actually, we can show that given a largely arbitrary path $\gamma(s) \subset \mathcal{I}^*$, we can find orbits $\tilde{x}(t)$ such that $I(\tilde{x}(t))$ is δ-close to $\gamma(\Psi(t))$ for some reparameterization Ψ.

- One can forget about δ and prescribe arbitrary paths on a set $\mathcal{J}^* \subset \mathcal{I}^*$. This set $\mathcal{J}^*$ is described precisely in the course of the proof, and is determined by the non-degeneracy assumptions **H5**-**H8**. The main idea is that $\mathcal{J}^*$ is obtained from the open set $\mathcal{I}^*$ described in **H5** (where the intersection of stable and unstable manifolds of a normally hyperbolic invariant manifold is transversal), just eliminating some sets of codimension 2, like double resonances,
- All the conditions **H5**-**H8** are generic: C^2 open and hold except in sets of infinite codimension. The only non-generic hypothesis is the assumption **H4**, maintained here to simplify the exposition.
- The new point: The dimension of the actions $d \geq 2$. [Treschev12] establishes diffusion far from *strong resonances* for the case $n = 1$, $d \geq 2$, using the method of the (Shil'nikov) separatrix map.
- For $d \geq 2$, codimension 2 objects do not separate the space of actions and can be contoured (compare with [Gelfreich-Simó-Vieiro] numerical results).

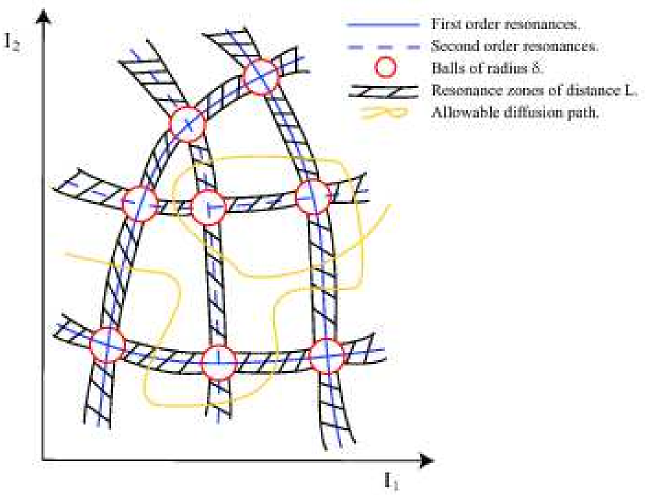


Figure : paths of diffusion

- The model used is usually called a-priori-unstable [Chierchia-Gallavotti94]. This distinction makes sense for analytic models depending only on one parameter.
- The results could be applied just as well for μV_i instead of V_i and $0 < \mu \ll 1$, but require to choose ε very small (even exponentially small) with respect to μ. In particular, one could use this method to produce systems that present instability but which are as close to integrable as desired. This procedure was pioneered in [Arnold64].
- Hamiltonian (1) can be considered as a simplified model of what happens in a neighborhood of a resonance of multiplicity n in a near integrable Hamiltonian. The averaging method shows that near a resonance, one can reduce the system to a Hamiltonian of the form

$$h(I) + \sum_{i=1}^{n} \frac{p_i^2}{2} + \varepsilon V(q_1, \ldots, q_n, I) + O(\varepsilon^2). \tag{9}$$

- The assumption that the averaged system is given by uncoupled pendula is not general, but is made often [Holmes-Marsden82, Haller97]. It is a *generic* assumption only for $n = 1$.
- Hamiltonian (1) is a simplified model in a neighborhood of a resonance of multiplicity n in a near integrable Hamiltonian. For a more general model

$$h(I) + \sum_{i=1}^{n} \frac{p_i^2}{2} + \varepsilon V(q_1, \ldots, q_n, I) + O(\varepsilon^2). \tag{10}$$

we would need for our analysis to assume the existence of homoclinic orbits to a hyperbolic equilibrium point of the mechanical system $\sum_{i=1}^{n} \frac{p_i^2}{2} + \varepsilon V(q_1, \ldots, q_n, I)$ [Bolotin78] which are also transversal.

An explicit example

$$H(I_1, I_2, \varphi_1, \varphi_2, p, q, t, \varepsilon) = \pm\left(\frac{p^2}{2} + \cos q - 1\right) + h(I_1, I_2) + \varepsilon \cos q \; g(\varphi_1, \varphi_2, t) \quad (11)$$

where

$$h(I_1, I_2) = \Omega_1 \frac{I_1^2}{2} + \Omega_2 \frac{I_2^2}{2},$$

and

$$g(\varphi_1, \varphi_2, t) = a_1 \cos\varphi_1 + a_2 \cos\varphi_2 + a_3 \cos(\varphi_1 + \varphi_2 - t).$$

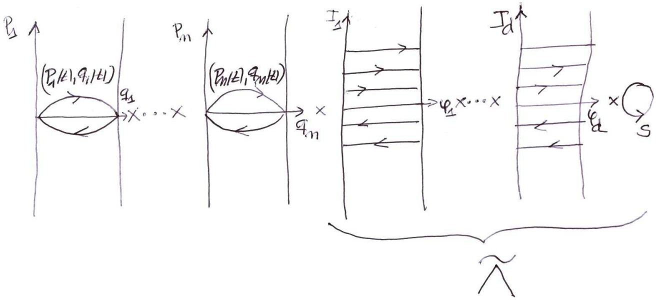
Example

Assume that a_0, a_1, a_2, Ω_1, Ω_2, $\Omega_1 + \Omega_2$, $4\Omega_1 + \Omega_2$ and $\Omega_1 + 4\Omega_2$ are non zero. Then Hamiltonian (11) verifies hypotheses **H1** to **H8** of Theorem 1.

Sketch of the proof

- **Part I**: Existence of a normally hyperbolic invariant manifold with associated stable and unstable manifolds.
- **Part II**: Outer dynamics.
- **Part III**: Inner dynamics.
- **Part IV**: Combination of both dynamics.

$\varepsilon = 0$



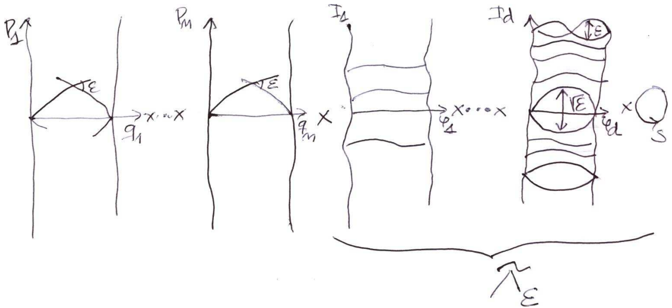
- Normally hyperbolic invariant manifold $(2d+1)$-dimensional:

$$\tilde{\Lambda} = \{(0, 0, I, \varphi, s) : (I, \varphi, s) \in \mathcal{I} \times \mathbb{T}^{d+1}\}$$

- Invariant manifolds $(2d+n+1)$-dimensional which coincide:

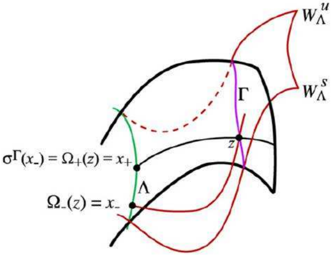
$$\Gamma = W^s\widetilde{\Lambda} = W^u\widetilde{\Lambda} = \{(p^*(\tau), q^*(\tau), I, \varphi, s) : \tau \in \mathbb{R}^n, (I, \varphi, s) \in \mathcal{I} \times \mathbb{T}^{d+1}$$

$0 < \varepsilon \ll 1$



- $\widetilde{\Lambda}$ persists to $\widetilde{\Lambda}_\varepsilon$, which is ε-close to $\widetilde{\Lambda}$.
- $W^s\widetilde{\Lambda}_\varepsilon$ and $W^u\widetilde{\Lambda}_\varepsilon$ are ε-close to the unperturbed ones.
- Using assumption **H7** for the Melnikov potential $L(\tau, I, \varphi, s)$, one has that $W^s\widetilde{\Lambda}_\varepsilon \pitchfork W^u\widetilde{\Lambda}_\varepsilon$ along a homoclinic manifold $\widetilde{\Gamma}_\varepsilon$.

Scattering map (outer map)



- Scattering map (outer map):

$$\begin{array}{rcccc} S_\varepsilon : & H_- \subset \widetilde{\Lambda}_\varepsilon & \to & H_+ \subset \widetilde{\Lambda}_\varepsilon \\ & x_- & \mapsto & x_+ \end{array}$$

defined by $x_+ = S_\varepsilon(x_-) \Leftrightarrow \exists\ z \in \widetilde{\Gamma}_\varepsilon$, such that

$$\mathrm{dist}(\Phi_t(z), \Phi_t(x_\pm)) \to 0 \qquad \text{for} \quad t \to \pm\infty$$

- S_ε is exact symplectic [D-Llave-S08].

Perturbative computation of the Scattering map

- Consider the reduced Poincaré function $\mathcal{L}^*(I,\theta)$ given in assumption **H8** .
- Up to first order in ε, S_ε is the ε-time flow of the Hamiltonian flow of Hamiltonian $-\mathcal{L}^*(I,\theta)$, where $\theta = \varphi - s\omega(I)$:

$$S_\varepsilon(I,\varphi,s) = (I,\varphi,s) + \varepsilon \mathrm{J}\nabla\left(\mathcal{L}^*(I,\varphi - s\omega(I))\right) + \mathcal{O}(\varepsilon^2), \qquad (12)$$

- The scattering map can jump distances of $\mathcal{O}(\varepsilon)$ along the trajectories of the Hamiltonian $\mathcal{L}^*(I,\theta)$.

- We need now to study the inner dynamics in $\widetilde{\Lambda}_\varepsilon$ and more precisely its invariant tori to construct a transition chain along $\widetilde{\Lambda}_\varepsilon$, i.e., a sequence of whiskered tori $\{\mathcal{T}_i\}_{i=1}^N$ such that

$$W^u_{\mathcal{T}_i} \pitchfork W^s_{\mathcal{T}_{i+1}}$$

- Standard shadowing methods [Fontich-Martin00] provide orbits connecting arbitrary small neighborhoods of the tori $\mathcal{T}_i$, $i = 1 \dots N$.
- The key point is to use the property

$$S_\varepsilon(\mathcal{T}_i) \pitchfork_{\widetilde{\Lambda}_\varepsilon} \mathcal{T}_{i+1} \Rightarrow W^u_{\mathcal{T}_i} \pitchfork W^s_{\mathcal{T}_{i+1}}$$

to choose convenient transition chains.

Inner Dynamics

The flow restricted to the NHIM $\tilde{\Lambda}_\varepsilon$ is a hamiltonian flow of Hamiltonian:

$$K_\varepsilon(I, \varphi, s) = h(I) + \sum_{i=1}^{N} \varepsilon^i K^i(I, \varphi, s) + O(\varepsilon^{N+1}),$$

- K^i are trigonometric polynomials in φ, s easily computable.
- We avoid multiple resonances which are of codimension two.
- Standard averaging far from resonances and close to single secular resonances (ressonances which appear in the first or second step of averaging) provide adequate approximations for KAM tori.

Non resonant KAM tori

- In the non-resonant region where $k\omega(I)+\ell\neq 0$ for $(k,\ell)\in\mathcal{N}^{[\leq 2]}$ one can do at least two steps of averaging:

$$K_\varepsilon(I,\varphi,s)=\bar{K}(I,\varepsilon)+O(\varepsilon^3),$$

- The tori of the averaged system are called *primary tori*.
  They are given by the level sets of a d-dimensional vector function $F=(F_1,\ldots,F_d)$ of the form

$$F(I,\varphi,s)=I+\mathcal{O}(\varepsilon).$$

Primary and secondary tori in the resonant region

- Resonances: Values of I such that $\omega(I) \cdot k + \ell = 0$, for some $(k, \ell) \in \mathbb{Z}^{d+1}$. It is a hypersurface in $\mathcal{I}^*$. We avoid the intersection of two such surfaces, which are called multiple resonances.
- Resonant tori (corresponding to *resonances*) are typically destroyed by the perturbation, creating gaps in the foliation of the persistent primary tori of size up to $\mathcal{O}(\sqrt{\varepsilon})$ centered around resonances.
- Other invariant objects are created inside these gaps, like secondary tori, which are (d+1)-dimensional invariant KAM tori contractible to d-dimensional invariant tori.

Primary and secondary tori in the resonant region

- Given any $(k, \ell) \in \mathbb{Z}^{d+1}$, $k \neq 0$, determining a single resonant region around $\omega(I) \cdot k + \ell = 0$, for simplicity of notation, assume $k_d \neq 0$ and write $k = (\hat{k}, k_d)$ with $\hat{k} \in \mathbb{Z}^{d-1}$.
- By averaging theory, the invariant tori in this resonant region can be approximated by the level sets of a vector function $F = (F_1, \dots, F_d) = (\hat{F}, F_d)$ where

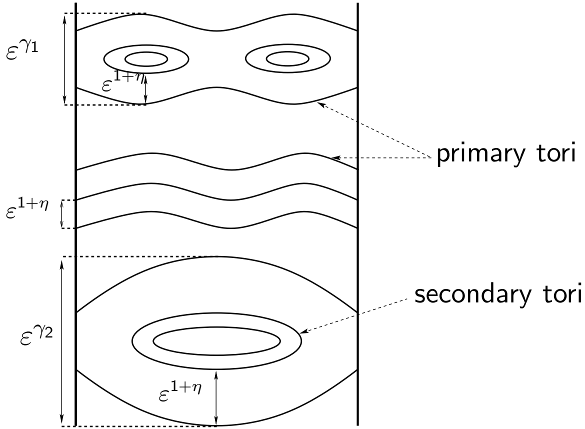
$$\begin{aligned} \hat{F} &= \hat{I} - \frac{I_d}{k_d} \hat{k} \\ F_d &= \bar{K}_\varepsilon(I, k\varphi + \ell s; \varepsilon) + \frac{\ell}{k_d} I_d . \end{aligned}$$

Once fixed the value of $\hat{F}$, thanks to assumption **H6**, F_d is the Hamiltonian of a pendulum.

Which of the tori obtained in the averaged system survive and at what distance when we add the perturbation term?

KAM theorem (Quantitative version)
For $I \in \mathcal{I}^*$, there exists in $\widetilde{\Lambda}_\varepsilon$ a discrete sequence of invariant tori $\mathcal{T}_i$ (some primary and some secondary) which are $\varepsilon^{1+\eta}$-closely spaced, with $0 < \eta \ll 1$. They are given by the leaves $L_E^{F^*}$ of a foliation $\mathcal{F}_{F^*}$, with F^* close to F.

Invariant objects in the NHIM $\widetilde{\Lambda}_\varepsilon$



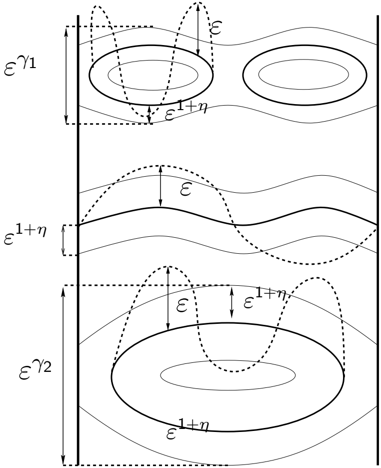
- The image under the scattering map S_ε of a leaf $L_E^{F^*}$ satisfies

$$F^* \circ S_\varepsilon^{-1} = F^* - \varepsilon\{F^*, \mathcal{L}^*\} + h.o.t$$

- For any $j = 1, \dots, d$, at points

$$\{F_j, \mathcal{L}^*\} < 0 \qquad (13)$$

  the scattering map increases the value of F_j by order ε.
- The non-degeneracy assumption **H6-H7** provide explicit conditions to ensure that $\{\{F_j, \mathcal{L}^*\}, \mathcal{L}^*\} \not\equiv 0$ when $\{F_j, \mathcal{L}^*\} = 0$. For instance, for F_i, $l = 1, \dots, d-1$, i.e., for $\hat{F}$, they amount to $\det\left(\frac{\partial^2 \mathcal{L}^*}{\partial \hat{\theta}^2}\right) \not\equiv 0$.



The end of the proof

- By the hypotheses of the theorem, for any $I_\pm \in \mathcal{I}^*$ and for every $\delta > 0$, there exists a path from I_- to I_+ in the set $\mathcal{I}^* = \mathcal{I}_\delta$ at a distance δ of the codimension 2 sets where the hypotheses of the Main Theorem are fulfilled.
  By the construction presented, there exists $\varepsilon_0 > 0$ such that for any $0 < |\varepsilon| < \varepsilon_0$, this path is in an ε-neighborhood of (primary and secondary) transition tori $\mathcal{T}_i$ forming a transition chain, so there exists an orbit $\tilde{x}(t) = (p(t), q(t), I(t), \varphi(t))$ of (1) which shadows the transition chain, so that, for some $T > 0$:

$$
\begin{aligned}
|I(0) - I_-| &\leq C\delta \\
|I(T) - I_+| &\leq C\delta
\end{aligned}
$$

Illustration of the transport of invariant tori under the scattering map

$$H_\varepsilon(p,q,I,\varphi,t) = \pm\left(\frac{p^2}{2} + \cos q - 1\right) + \frac{I^\top \cdot I}{2} + \varepsilon \cos q \sum_{|k|+|l|=1} a_{kl}\cos(k\cdot\varphi + lt).$$

Red curves: Invariant tori (primary and secondary) around $I = 0$

Green curves: Images of these invariant tori under the scattering map.

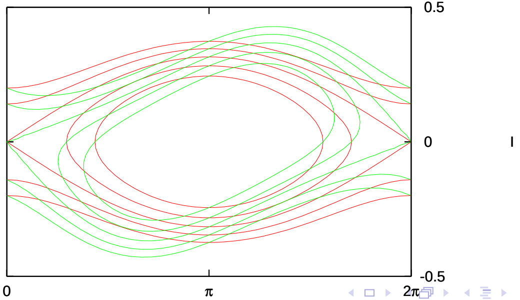


Illustration of how to combine the two dynamics to cross the big gaps region. Invariant tori for the inner dynamics (red curves) and invariant sets for the outer dynamics (blue curves). Inner dynamics is represented by dashed lines whereas outer dynamics is represented by solid lines.