Stability of equilibrium solutions of autonomous and periodic Hamiltonian systems in the case of multiple resonances

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We consider the Hamiltonian system in $\mathbb{R}^{2n}$ with n degrees of freedom

$$\dot{\mathbf{q}} = \frac{\partial H}{\partial \mathbf{p}}, \quad \dot{\mathbf{p}} = -\frac{\partial H}{\partial \mathbf{q}}, \tag{1}$$

such that the origin is an equilibrium point.

$H = H(\mathbf{q}, \mathbf{p})$ in the autonomous case or
$H = H(\mathbf{q}, \mathbf{p}, t) = H(\mathbf{q}, \mathbf{p}, t + 2\pi)$ in the 2π-periodic case.

H is an analytic function in $(\mathbf{q}, \mathbf{p}) = (q_1, \cdots, q_n, p_1 \cdots, p_n)$ in a neighborhood of the origin.

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Preliminaries

Using Taylor's series of H in a neighborhood of the origin, we have

$$H = H_2 + H_3 + \cdots + H_j + \cdots , \tag{2}$$

where H_j are homogeneous polynomials of degree j in $(\mathbf{q}, \mathbf{p})$, that is,

$$H_j = \sum_{|\mathbf{k}|+|\mathbf{l}|=j} h_{\mathbf{kl}}\, \mathbf{q}^{\mathbf{k}}\mathbf{p}^{\mathbf{l}}, \tag{3}$$

with $\mathbf{k} = (k_1, \cdots, k_n) \in \mathbb{Z}^n$, $\mathbf{l} = (l_1, \cdots, l_n) \in \mathbb{Z}^n$
$|\mathbf{k}| = |k_1| + \cdots + |k_n|$, $|\mathbf{l}| = |l_1| + \cdots + |l_n|$,

$h_{\mathbf{kl}} = h_{k_1 \cdots k_n l_1 \cdots l_n}$,

$\mathbf{q}^{\mathbf{k}} = q_1^{k_1} \cdots q_n^{k_n}$ and $\mathbf{p}^{\mathbf{l}} = p_1^{l_1} \cdots p_n^{l_n}$.

Note that in the 2π-periodic case $h_{\mathbf{kl}} = h_{\mathbf{kl}}(t) = h_{\mathbf{kl}}(t + 2\pi)$.

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Note that in the 2π-periodic case $h_{\mathbf{kl}} = h_{\mathbf{kl}}(t) = h_{\mathbf{kl}}(t + 2\pi)$.

We will assume that the eigenvalues (respectively, the characteristic exponents in the periodic case) are pure imaginary.

We denote them by: $\pm\omega_1 i, \cdots, \pm\omega_n i$,

H_2 is not sign definite in the autonomous case.

Also we assume that we have normalized the quadratic part, so that

$$H_2 = \frac{\omega_1}{2}(q_1^2 + p_1^2) + \cdots + \frac{\omega_n}{2}(q_n^2 + p_n^2). \tag{4}$$

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Definition

The system (1) presents a resonance relation if there exists an integer vector $\mathbf{k} = (k_1, \ldots, k_n) \in \mathbb{Z}^n \setminus \{0\}$ *such that*

$$k_1\omega_1 + \cdots + k_n\omega_n = 0, \quad \textit{in the autonomous case} \tag{5}$$

or

$$k_1\omega_1 + \cdots + k_n\omega_n \in \mathbb{Z}, \quad \textit{in the periodic case.} \tag{6}$$

The number $|\mathbf{k}| = |k_1| + \cdots + |k_n|$ *is called order of the resonance. On the other hand, if*

$$k_1\omega_1 + \cdots + k_n\omega_n \neq 0 \;\; (\textit{resp.} \notin \mathbb{Z})\;. \tag{7}$$

holds for all integer vectors (except for the null vector) $\mathbf{k} = (k_1, \cdots, k_n) \in \mathbb{Z}^n$ *satisfying* $|\mathbf{k}| = j$, *for* $j = 1, \cdots, s$, *we say that the system (1) does not present resonance relations up to order* s, *inclusively.*

Construction of the $\mathbb{Z}$-Module M_ω

$$M_\omega = \{\mathbf{k} = (k_1, \cdots, k_n) \in \mathbb{Z}^n;\ \mathbf{k}\cdot\omega = k_1\omega_1 + \cdots + k_n\omega_n = 0(\text{resp. } \in \mathbb{Z})\}.$$

associated the frequencies $\omega_1, \cdots, \omega_n$.

1. $M_\omega = \{0\}$, if and only if, $\omega_1, \cdots, \omega_n$ are L.I. on $\mathbb{Q}$ in the autonomous case, or $\omega_1, \cdots, \omega_n, -1$ are L.I. on $\mathbb{Q}$ in the periodic case.
2. $M_\omega = \{0\}$, if and only if, the system (1) do not possess resonances.

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PROPERTIES OF M_ω

M_ω is a submodule of the finitely generated module $\mathbb{Z}^n$ and as $\mathbb{Z}$ is a principal domain, we have that M_ω is finitely generated.

Thus there are vectors $\mathbf{k}^1, \ldots, \mathbf{k}^s \in M_\omega$ (L.I. and minimal) such that

$$\begin{aligned} M_\omega = & \ \mathbf{k}^1\mathbb{Z} + \cdots + \mathbf{k}^s\mathbb{Z} \\ = & \ \{j_1\mathbf{k}^1 + \cdots + j_s\mathbf{k}^s;\ j_1, \ldots, j_s \in \mathbb{Z};\ \mathbf{k}^1, \cdots, \mathbf{k}^s \in M_\omega\}, \end{aligned} \tag{8}$$

with $s < n$ in the autonomous case and $s \leq n$ in the periodic case.

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Definition

Assume that $M_\omega \neq \{0\}$. If M_ω is cyclic (or equivalently, $s = 1$) we say that system (1) possesses single resonances, in the opposite case (or equivalently, $s > 1$) we say that the system possesses multiple resonances.

The case of multiple resonances can appear in autonomous Hamiltonian systems with $n > 2$ or in periodic Hamiltonian systems with $n \geq 2$.

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Examples of Hamiltonian with multiple Resonances

Consider the autonomous Hamiltonian function with 3-degrees of freedom in action-angles

$$H = r_1 - 2r_2 + 3r_3 + H_3 + \dots .$$

Then, $\omega = (\omega_1, \omega_2, \omega_3) = (1, -2, 3)$, so

$$M_\omega = (2,1,0)\mathbb{Z} + (3,0,-1)\mathbb{Z},$$

and $s = 2$. Then it has multiple resonances.

Considerer now the periodic Hamiltonian function with 2-degrees of freedom in action-angles variables

$$H = r_1 + 2r_2 + H_3 + \dots .$$

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Definition

Assume that $M_\omega \neq \{0\}$ and there are multiple resonances, that is, $s > 1$. We say that two vectors of resonances $\mathbf{k}^{\alpha_1}$ and $\mathbf{k}^{\alpha_2}$ with $\alpha_1, \alpha_2 \in \{1, \ldots, s\}$, $\alpha_1 \neq \alpha_2$ do not have interactions if

$$k_1^{\alpha_1} k_1^{\alpha_2} = \cdots = k_n^{\alpha_1} k_n^{\alpha_2} = 0.$$

It is said that the set of vectors $\mathbf{k}^{\alpha_1}, \ldots, \mathbf{k}^{\alpha_m}$ with $\alpha_1, \ldots, \alpha_m \in \{1, \ldots, s\}$ do not have interactions if $\mathbf{k}^{\alpha_i}$ and $\mathbf{k}^{\alpha_j}$ do not have interaction for all $i, j \in \{1, \ldots, m\}$, $i \neq j$.

Multiple resonances without interactions can appear only in autonomous Hamiltonian systems with $n \geq 4$

In periodic Hamiltonian systems with $n \geq 3$.

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Resonances with/without Interaction

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Example of Hamiltonian function with multiple resonances without interactions

Considerer the autonomous Hamiltonian functions with 4-degrees of freedom

$$H = r_1 - 2r_2 + \pi r_3 - 3\pi r_4 + H_3 + \dots .$$

Here $\omega = (\omega_1, \omega_2, \omega_3, \omega_4) = (1, -2, \pi, -3\pi)$, then

$$M_\omega = (2, 1, 0, 0)\mathbb{Z} + (0, 0, 3, 1)\mathbb{Z},$$

thus $\mathbf{k}^1 = (2, 1, 0, 0)$ and $\mathbf{k}^2 = (0, 0, 3, 1)$. Therefore, we have multiple resonances without interactions

Examples of Hamiltonian function with multiple resonances with interactions

Consider the autonomous Hamiltonian function with 3-degrees of freedom in action-angles

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Then, $\omega = (\omega_1, \omega_2, \omega_3) = (1, -2, 3)$, so

$$M_\omega = (2, 1, 0)\mathbb{Z} + (3, 0, -1)\mathbb{Z},$$

and $s = 2$. Then it has multiple resonances with interactions.

In order to put the Hamiltonian in its Lie-Deprit normal form, we write the Hamiltonian in the form

$$H = H(q,p) = H_0^0 + \varepsilon H_1^0 + \frac{\varepsilon^2}{2!} H_2^0 + \cdots .$$

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For us it is just a way to keep track of the homogeneous polynomials at each order.

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Normal form of Lie-Deprit

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It is introduced the generating function W

$$W = W_1 + \frac{\varepsilon}{1!}W_2 + \frac{\varepsilon^2}{2!}W_3 + \cdots$$

to generate the Lie transformation, this gives us news functions via

$$H_j^i = H_{j+1}^{i-1} + \sum_{k=0}^{j} \binom{j}{k} L_{W_{k+1}} H_{j-k}^{i-1}.$$

The relation between these functions is easily illustrated by means of the Lie triangle

$$\begin{array}{ccccc} H_0^0 & & & & \\ \downarrow & & & & \\ H_1^0 & \rightarrow & H_0^1 & & \\ \downarrow & & \downarrow & & \\ H_2^0 & \rightarrow & H_1^1 & \rightarrow & H_0^2 \\ \downarrow & & \downarrow & & \downarrow \end{array}$$

The Lie derivative $L_W K$ is given by the standard Poisson bracket

$$L_W K = \sum_{j=1}^{n} \frac{\partial K}{\partial q_j} \frac{\partial W}{\partial p_j} - \frac{\partial K}{\partial p_j} \frac{\partial W}{\partial q_j}.$$

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In each column W_{k+1} is determined in such way that H_0^k is so simple as possible. The transformed function (the function in the normal form) is given by

$$\tilde{H} = \tilde{H}(q, p) = H_0^0 + \varepsilon H_0^1 + \frac{\varepsilon^2}{2!} H_0^2 + \cdots$$

with $\varepsilon = 1$.

Theorem

Let $H_0^T(x) = \frac{1}{2}x^T Rx$ be the Hamiltonian function associated the system $\dot{x} = A^T x$. Then there exists a formal transformation in symplectic coordinates and 2π-periodic, $x = \phi(y, t) = y + \ldots$, which transform the original Hamiltonian in

$$H^*(y, t) = \sum_{i=0}^{\infty} H_0^i(y, t), \tag{9}$$

where H^i is a homogeneous polynomial of degree $i + 2$ in y, and 2π-periodic in t that satisfies

$$\{H_0^T, H_0^i\} + \frac{\partial H_0^i}{\partial t} = 0,\ i = 0, 1, \ldots. \tag{10}$$

Considerations

We assume that the non-degenerate and isolated equilibrium point is at the origin $(0,0)$ of system (1) and is linearly stable. Thus without loss of generality we suppose that

$$H_2 = \frac{\omega_1}{2}(q_1^2 + p_1^2) + \cdots + \frac{\omega_n}{2}(q_n^2 + p_n^2). \tag{11}$$

H^m represents the truncated Hamiltonian function which is truncated up to terms of order $m > 2$, that is,

$$H^m = H_2 + \cdots + H_m. \tag{12}$$

$H^m(\mathbf{r}, \varphi, t)$ is the truncated Hamiltonian function written in its Lie normal form up to order m inclusively, i.e., we have applied the Lie normal form process (in a finite order, then the process is convergent) to the function H up to order m, inclusively.

$(\mathbf{r}, \varphi) = (r_1, \ldots, r_n, \varphi_1, \ldots, \varphi_n)$ action-angles variables, i.e.,

$$q_j = \sqrt{2r_j}\cos\varphi_j, \quad p_j = \sqrt{2r_j}\sin\varphi_j. \tag{13}$$

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1. If $M_\omega = \{0\}$, then

$$H^m = H^m(\mathbf{r}).$$

2. If M_ω is cyclic with $M_\omega = \mathbf{k}\mathbb{Z}$ for some $\mathbf{k} \in M_\omega$, then

$$H^m = H^m(\mathbf{r}, \mathbf{k} \cdot \varphi + \gamma t)$$

with $\gamma = 0$ in the autonomous case or $\gamma = \mathbf{k} \cdot \omega$ in the periodic case.

3. If $M_\omega = \mathbf{k}^1\mathbb{Z} + \ldots \mathbf{k}^s\mathbb{Z}$, then

$$H^m = H^m(\mathbf{r}, \mathbf{k}^1 \cdot \varphi, \cdots, \mathbf{k}^s \cdot \varphi)$$

in the autonomous case, and

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Resonances and Lie normal form

1. If $M_\omega = \{0\}$, then

$$H^m = H^m(\mathbf{r}).$$

2. If M_ω is cyclic with $M_\omega = \mathbf{k}\mathbb{Z}$ for some $\mathbf{k} \in M_\omega$, then

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with $\gamma = 0$ in the autonomous case or $\gamma = \mathbf{k}\cdot\omega$ in the periodic case.

3. If $M_\omega = \mathbf{k}^1\mathbb{Z} + \dots \mathbf{k}^s\mathbb{Z}$, then

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Normal form for multiple resonances without interactions

If $\mathbf{k}^1, \ldots, \mathbf{k}^s$ do not have interactions then for all $m \geq 2$ there exist functions $H^{m,0}(\mathbf{r})$ and $H^{m,j}(\mathbf{r}, \mathbf{k}^j \cdot \varphi)$, $j = 1, \ldots s$, such that

$$H^m(\mathbf{r}, \mathbf{k}^1 \cdot \varphi, \ldots, \mathbf{k}^s \cdot \varphi) = H^{m,0}(\mathbf{r}) + H^{m,1}(\mathbf{r}, \mathbf{k}^1 \cdot \varphi) + \cdots + H^{m,s}(\mathbf{r}, \mathbf{k}^s \cdot \varphi).$$

Autonomous case (similar in the periodic case).

Definition

We say that an equilibrium in (1) is Lie-stable if there exists $m > 2$ *such that the truncated Hamiltonian system in Lie normal form associated to* H^j *is stable (in the sense of Lyapunov) for any* $j \geq m$ *(arbitrary).*

Remarks

It is important to observe that the stability of the equilibrium solution of the truncated Hamiltonian system in its Lie normal form does not possess in general any relation with the stability of the truncated Hamiltonian system.

Remarks

As an example we assume the existence of a single resonance vector of order 4 given by $\mathbf{k} = (1, 1, 2)$, thus $\omega_1 + \omega_2 + 2\omega_3 = 0$. It is clear that the origin of the Hamiltonian system with 3-degrees of freedom associated to

$$H_2 = \frac{\omega_1}{2}(q_1^2 + p_1^2) - \frac{\omega_2}{2}(q_2^2 + p_2^2) + \omega_3(q_3^2 + p_3^2),$$

is stable.

We can verify that the origin of

$$H = H_2 + (q_1^2 + p_1^2)^2,$$

is Lie-stable.

While for the Hamiltonian

$$\overline{H} = H_2 + 2[q_1q_2q_3^2 - q_1q_2p_3^2 - q_3^2p_1p_2 + p_1p_2p_3^2 - 2q_1q_3p_2p_3 - 2q_2q_3p_1p_3]$$

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Remarks

Since in the previous examples we have $\{H, H_2\} = 0$ and $\{\overline{H}, H_2\} = 0$, then both Hamiltonian functions H and $\overline{H}$ are in its Lie normal form.

Remarks

We emphasize that in general the problem of stability in the periodic case is no equivalent to the autonomous case.
For example, for the 2π-periodic Hamiltonian with 1-degree of freedom

$$H = \frac{1}{2}(q^2 + p^2) + 2u_{30}(q^3 - 3qp^2) + 2v_{30}(p^3 - 3qp^2),$$

where $u_{30} = x_{30}\cos(3t) - y_{30}\sin(3t)$,
$v_{30} = x_{30}\sin(3t) + y_{30}\cos(3t)$, with $x_{30}^2 + y_{30}^2 \neq 0$, it is verified that the origin is unstable in the Lyapunov sense.

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We will assume the existence of s multiple resonances with vectors of resonance $\mathbf{k}^1, \ldots, \mathbf{k}^s$.

Let $I_j = \mathbf{a}_j \cdot \mathbf{r}$, $(j = 1, \ldots, n - s)$ with $\mathbf{a}_j \cdot \mathbf{k}^1 = \cdots = \mathbf{a}_j \cdot \mathbf{k}^s = 0$

$$S = \{\mathbf{r}; \quad I_1(\mathbf{r}) = \cdots = I_{n-s}(\mathbf{r}) = 0\},$$

and

$$S^m = \{\mathbf{r}; H^m(\mathbf{r}, \varphi) = 0, \forall \varphi\}.$$

Statement of the Problem

We will assume the existence of s multiple resonances with vectors of resonance $\mathbf{k}^1, \dots, \mathbf{k}^s$.

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<u>First case: we assume that</u> $S = \{\mathbf{r} = \mathbf{0}\}$.

Proposition

If $S = \{\mathbf{r} = 0\}$, *then the null solution of (1) is Lie-stable.*

Proof: Since the function

$$W = I_1^2 + \cdots + I_{n-s}^2,$$

is a positive definite first integral of the truncated Hamiltonian system in its Lie normal form for any arbitrary order.

Corollary

If there exists $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{R}^n$, *with* $a_1, \ldots, a_n > 0$ *such that*

$$\mathbf{k}^1 \cdot \mathbf{a} = \cdots = \mathbf{k}^s \cdot \mathbf{a} = 0,$$

then the null solution of (1) is Lie-stable.

Proof: The existence of $\mathbf{a}$ as above implies that $S = \{\mathbf{r} = \mathbf{0}\}$.

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Remark

A necessary condition for the existence of **a** as in Corollary 1 is that for all $\alpha \in \{1, \ldots, s\}$ there exist $i, j \in \{1, \ldots, n\}$, $i \neq j$, such that $k_i^\alpha k_j^\alpha < 0$.

The condition of Proposition 1 depends only on the conditions about the vectors of resonances $\mathbf{k}^1, \ldots, \mathbf{k}^s$, and it is independent of the terms of order greater than two of the Hamiltonian function.

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Example

The autonomous Hamiltonian system with four degrees of freedom whose Hamiltonian function in action-angle variables is given by

$$H = r_1 + 2r_2 + 3r_3 - \pi r_4 + H_3 + \dots$$

has the origin as a Lie-stable solution independently of what happens with the terms greater than two.

In fact, in this case $M_\omega = (2, -1, 0, 0)\mathbb{Z} + (3, 0, -1, 0)\mathbb{Z}$ and taking $\mathbf{a} = (1, 2, 3, \pi)$ we have
$\mathbf{a} \cdot (2, -1, 0, 0) = \mathbf{a} \cdot (3, 0, -1, 0) = 0$, or equivalently, $(2, -1, 0, 0)$ and $(3, 0, -1, 0)$ has components of different signs.

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Second case: we assume that $S \neq \{\mathbf{r} = \mathbf{0}\}$.

If $\mathbf{r} \in S$, then

$\mathbf{a}_1 \cdot \mathbf{r} = 0, \ldots, \mathbf{a}_{n-s} \cdot \mathbf{r} = 0$

and as the vectors $\mathbf{a}_1, \ldots, \mathbf{a}_{n-s}$ are L.I. in $\mathbb{R}^n$, solving the previous system, we find subindex $j_1, \ldots, j_s \in \{1, \ldots, n\}$ such that

$$\mathbf{r} = \mathbf{r}(r_{j_1}, \ldots, r_{j_s}),$$

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Now we define the auxiliary function

$$F_m = H^m|_{S\times\mathbb{R}^s} = F_m(r_{j_1},\ldots,r_{j_s},\mathbf{k}^1\cdot\varphi,\ldots,\mathbf{k}^s\cdot\varphi). \qquad (14)$$

Theorem (Theorem of Stability)

Under the previous notations, if $S=\{\mathbf{r}=\mathbf{0}\}$ *or* $S\neq\{\mathbf{r}=\mathbf{0}\}$ *and there is* $m>2$ *such that* $F_m(r_{j_1},\ldots,r_{j_s},\mathbf{k}^1\cdot\varphi,\ldots,\mathbf{k}^s\cdot\varphi)\neq 0$ *for all* $(r_{j_1},\ldots,r_{j_s},\mathbf{k}^1\cdot\varphi,\ldots,\mathbf{k}^s\cdot\varphi)$ *with* $r_{j_1},\ldots,r_{j_s}>0$ *sufficiently small, then the null solution of (1) is Lie-stable.*

2) $S \neq \{\mathbf{r} = \mathbf{0}\}$ and consider the function

$$V = I_1^2 + \cdots + I_{n-s}^2 + (H^m)^2.$$

This function is clearly a first integral of the Hamiltonian system associated to H^m for every $m \geq 2$. On the other hand, we have that $V = 0$, if and only if, $\mathbf{r} \in S$ and $F_m(r_{j_1}, \ldots, r_{j_s}, \mathbf{k}^1 \cdot \varphi, \ldots, \mathbf{k}^s \cdot \varphi) = 0$. Assuming that there exists $m > 2$ such that $F_m(r_{j_1}, \ldots, r_{j_s}, \mathbf{k}^1 \cdot \varphi, \ldots, \mathbf{k}^s \cdot \varphi) \neq 0$ for all $(r_{j_1}, \ldots, r_{j_s}, \mathbf{k}^1 \cdot \varphi, \ldots, \mathbf{k}^s \cdot \varphi)$ with $r_{j_1}, \ldots, r_{j_s} > 0$ sufficiently small, we must have that $\mathbf{r} = \mathbf{0}$.

Hypothesis: We assume that for each $m > 2$ there is an angle vector φ^* such that $F_m(r_{j_1}, \ldots, r_{j_s}, \mathbf{k}^1 \cdot \varphi^*, \ldots, \mathbf{k}^s \cdot \varphi^*) = 0$ for all $r_{j_1}, \ldots, r_{j_s} > 0$, sufficiently small.

Natural question: Is the previous condition sufficient to guarantee that the null solution of system (1) to be unstable in the Lyapunov sense ?

In our work we will analyze two situations:

1. There exist at least two resonances of different order.
2. There exist resonances of the same order and without interaction.

Results about instability

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1. There exist at least two resonances of different order.
2. There exist resonances of the same order and without interaction.

First situation: We suppose that

$$|\mathbf{k}^1| < |\mathbf{k}^2| \le \cdots \le |\mathbf{k}^s|, \text{ and } \quad 2|\mathbf{k}^1| - 2 < |\mathbf{k}^2|$$

that is, there are at least two resonance of different order.
$3|\mathbf{k}^1| - 2 < |\mathbf{k}^2|$, in the periodic case.

Let $\eta = |\mathbf{k}^1|$, then the truncated Hamiltonian function $H^{2\eta-2}$ in its Lie normal form has the form (in the autonomous case)

$$H^{2\eta-2} = H_2(\mathbf{r}) + \cdots + H_{2l}(\mathbf{r}) + H_\eta(\mathbf{r}, \mathbf{k}^1 \cdot \varphi) + \cdots + H_{2\eta-2}(\mathbf{r}, \mathbf{k}^1 \cdot \varphi),$$

in the autonomous case, and

$$H^{2\eta-2} = H_4(\mathbf{r}) + \cdots + H_{2l}(\mathbf{r}) + H_\eta(\mathbf{r}, \mathbf{k}^1 \cdot \varphi) + \cdots + H_{2\eta-2}(\mathbf{r}, \mathbf{k}^1 \cdot \varphi),$$

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In the case $k_1^1, \ldots, k_n^1 \geq 0$ with $k_1^1 > 0$, define the auxiliary function

$$\Psi(\phi) = \left(\frac{1}{k_1^1}\right)^{\eta/2} H_\eta(\mathbf{k}^1, \phi), \tag{15}$$

where $\phi = \mathbf{k}^1 \cdot \varphi = k_1^1\varphi_1 + \cdots + k_n^1\varphi_n$.

Theorem of Instability in the first situation

Theorem

Under the previous notations, if $k_1^1, \ldots, k_n^1 \geq 0$, $k_1^1 > 0$, $H_4(\mathbf{k}^1) = \cdots = H_{2l}(\mathbf{k}^1) = 0$ *and there is* ϕ^* *such that* $\Psi(\phi^*) = 0$ *and* $\Psi'(\phi^*) \neq 0$, *then the null solution of (1) is unstable in the Lyapunov sense.*

By simplicity and without loss of generality we will suppose that

$$\mathbf{k}^1 = (k_1^1, \ldots, k_{\alpha_1}^1, 0, \ldots, 0)$$

with $k_1^1, \ldots, k_{\alpha_1}^1 > 0$, $\alpha_1 \leq n$. Next we consider the following convenient vectors in $\mathbb{Z}^n$

$$\begin{aligned} \mathbf{b}_1 &= (k_2^1, -k_1^1, 0, \ldots, 0), \\ \mathbf{b}_2 &= (k_3^1, 0, -k_1^1, 0, \ldots, 0), \\ &\vdots \\ \mathbf{b}_{\alpha_1-1} &= (k_{\alpha_1}^1, 0, \ldots, 0, -k_1^1, 0, \ldots, 0), \end{aligned} \tag{16}$$

under this construction we have that the $n-1$ functions

$V_1 = \mathbf{b}_1 \cdot \mathbf{r}$, $V_2 = \mathbf{b}_2 \cdot \mathbf{r}$, $\ldots$, $V_{\alpha_1-1} = \mathbf{b}_{\alpha_1-1} \cdot \mathbf{r}$, $V_{\alpha_1} = r_{\alpha_1+1}$, $\ldots$, $V_{n-1} = r_n$

are first integrals of the truncated Hamiltonian function associated to $H^{2\eta-2}$.

$$V_n = H(\mathbf{r}, \varphi) - H_2(\mathbf{r})$$

in the autonomous case and

$$V_n = H(\mathbf{r}, \varphi, t)$$

in the periodic case.

Idea of the proof

$$V_n = H(\mathbf{r}, \varphi) - H_2(\mathbf{r})$$

in the autonomous case and

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Now we define the Chetaev function

$$V = V_1^2 + V_2^2 + \cdots + V_{n-1}^2 + V_n^2 - \delta^2 r_1^\eta$$

where δ is chosen conveniently.
Let

$$\Omega_a = \{V \ \leq 0, \quad r_1 < a\},$$

$$\Omega_a^+ = \{V \ \leq 0, \quad r_1 < a, \quad \Psi'(\phi) > 0\}$$

and

$$\Omega_a^- = \{V \leq 0, \quad r_1 < a, \quad \Psi'(\phi) < 0\},$$

where a is a convenient positive real number that we will choose conveniently.

There exist real bounded functions $f_j = f_j(r_1, r_j)$, $j = 1, \dots, n-1$ with $|f_j| \le 1$ such that

$$\begin{aligned} r_{j+1} &= \frac{k_j^1}{k_1^1} r_1 + |\delta| r_1^{\eta/2} f_j(r_1, r_j), \quad j = 1, \dots, \alpha_1 - 1, \\ r_k &= |\delta| r_1^{\eta/2} f_k(r_1, r_k), \quad k = \alpha_1 + 1, \dots, n. \end{aligned} \tag{17}$$

The derivative of V through the solutions of the Hamiltonian system associated to H:

$$\dot{V} = \{V, H\} + \frac{\partial V}{\partial t} = \sum_{j=1}^{n-1} 2V_j\{V_j, H\} + 2V_n\{V_n, H\} - \delta^2 \eta r_1^{\eta-1}\{r_1, H\} + \frac{\partial V}{\partial t}. \tag{18}$$

Since V_j, $j = 1, 2, \ldots, n-1$ are first integrals for the truncated Hamiltonian system of order $2\eta - 2$ in both cases autonomous and periodic, then

$$\{V_j, H\} = O(r_1^{\eta-1/2}), \quad j = 1, \ldots, n-1,$$

in Ω_a.

Also H_2 is a first integral for the Hamiltonian system associated to $H^{2\eta-2}$, then

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In the autonomous case $\frac{\partial V}{\partial t} = 0$ and
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We conclude that

$$\dot{V} = -\delta^2 \eta k_1^1 \Psi'(\phi) r_1^{3\eta/2-1} + O(r_1^{3\eta/2-1/2}),$$

that it is used to prove that V is a Chetaev function in the region Γ, in both cases autonomous and periodic.

The second situation consists in to assume the existence of multiple resonance such that

$$\eta := |\mathbf{k}^1| = \cdots = |\mathbf{k}^\mu|,\, 2\eta < |\mathbf{k}^{\mu+1}| \leq \cdots \leq |\mathbf{k}^s|,$$

with $s \geq \mu \geq 2$ and

$$\mathbf{k}^1, \ldots, \mathbf{k}^\mu \quad \text{do not have interactions.}$$

In this case the Lie normal form of $H^{2\eta}$ assumes the form

$$\begin{aligned} H^{2\eta} = \;& H_2(\mathbf{r}) + \cdots + H_{2l}(\mathbf{r}) + H_\eta(\mathbf{r}, \mathbf{k}^1 \cdot \varphi, \ldots, \mathbf{k}^\mu \cdot \varphi) + \cdots + \\ & H_{2\eta}(\mathbf{r}, \mathbf{k}^1 \cdot \varphi, \ldots \mathbf{k}^\mu \cdot \varphi), \end{aligned}$$

where $2l$ is a natural number less than η.

Because of the no-interactions, there exist functions $H_i^j(\mathbf{r}, \mathbf{k}^j \cdot \varphi)$, and $H_i^0(\mathbf{r})$, $j = 1, \ldots, \mu$, $i = 2, \ldots, 2\eta$, such that

$$H_i(\mathbf{r}, \mathbf{k}^1 \cdot \varphi, \ldots, \mathbf{k}^\mu \cdot \varphi) = H_i^0(\mathbf{r}) + H_i^1(\mathbf{r}, \mathbf{k}^1 \cdot \varphi) + \cdots + H_i^\mu(\mathbf{r}, \mathbf{k}^\mu \cdot \varphi).$$

If $k_1^j, \ldots, k_n^j \geq 0$ and $k_1^j > 0$, we define the auxiliaries functions

$$\Psi_j(\phi_j) = \left(\frac{1}{k_1^j}\right)^{\eta/2} H_\eta^j(\mathbf{k}^j, \phi_j), \quad j = 1, \ldots, \mu. \qquad (19)$$

where $\phi_j = \mathbf{k}^j \cdot \varphi = k_1^j \varphi_1 + \cdots + k_n^j \varphi_n$.

Theorem of Instability for the Second situation

Theorem

Under the previous conditions, if there exist $j \in \{1, \ldots, \mu\}$ *such that* $k_1^j, \ldots, k_n^j \geq 0$, $k_1^j > 0$, $H_4(\mathbf{k}^j) = \cdots = H_{2l}(\mathbf{k}^j) = 0$ *and there are* ϕ_j^* *such that* $\Psi_j(\phi_j^*) = 0$ *and* $\Psi_j'(\phi_j^*) \neq 0$, *then the null solution of (1) is unstable in the Lyapunov sense.*

By simplicity we will suppose that $j = 1$, $\mu = 2$ and

$$\begin{aligned} \mathbf{k}^1 &= (k_1^1, \ldots, k_{\alpha_1}^1, 0, \ldots, 0), \\ \mathbf{k}^2 &= (0, \ldots, 0, k_{\alpha_1+1}^2, \ldots, k_{\alpha_1+\alpha_2}^2, 0, \ldots, 0), \end{aligned} \tag{20}$$

where $k_1^1, k_2^1, \ldots, k_{\alpha_1}^1 > 0$. The other cases can be proved similarly.

$$H^{2\eta} = \quad H_2(\mathbf{r}) + \cdots + H_{2l}(\mathbf{r}) + H_\eta^1(\mathbf{r}, \mathbf{k}^1 \cdot \varphi) + H_\eta^2(\mathbf{r}, \mathbf{k}^2 \cdot \varphi) + \cdots +$$

Here we introduce the notation

$$F_\eta = H_\eta^1(\mathbf{r}, \mathbf{k}^1 \cdot \varphi)$$

and

$$G_\eta = H_\eta^2(\mathbf{r}, \mathbf{k}^2 \cdot \varphi).$$

Considering the vectors in $\mathbb{Z}^n$

$$
\begin{aligned}
\mathbf{b}_1 &= (k_2^1, -k_1^1, 0, \ldots, 0), \\
\mathbf{b}_2 &= (k_3^1, 0, -k_1^1, 0, \ldots, 0), \\
&\vdots \\
\mathbf{b}_{\alpha_1 - 1} &= (k_{\alpha_1}^1, 0, \ldots, 0, -k_1^1, 0, \ldots, 0), \\
\mathbf{c}_1 &= (0, \ldots, 0, k_{\alpha_1+1}^2, -k_{\alpha_1}^2, 0, \ldots, 0) \\
\mathbf{c}_2 &= (0, \ldots, 0, k_{\alpha_1+2}^2, 0, -k_{\alpha_1}^2, 0, \ldots, 0) \\
&\vdots \\
\mathbf{c}_{\alpha_2 - 1} &= (0, \ldots, 0, k_{\alpha_1+\alpha_1}^2, 0, \ldots, 0, -k_{\alpha_1}^2, 0, \ldots, 0)
\end{aligned}
\tag{21}
$$

We define the $n-2$ functions $V_1 = \mathbf{b}_1 \cdot \mathbf{r}$, $V_2 = \mathbf{b}_2 \cdot \mathbf{r}$, $\ldots$, $V_{\alpha_1 - 1} = \mathbf{b}_{\alpha_1 - 1} \cdot \mathbf{r}$, $W_1 = \mathbf{c}_1 \cdot \mathbf{r}$, $W_2 = \mathbf{c}_2 \cdot \mathbf{r}$, $\ldots$, $W_{\alpha_2 - 1} = \mathbf{c}_{\alpha_2 - 1} \cdot \mathbf{r}$, $E_1 = r_{\alpha_1 + \alpha_2 + 1}$, $\ldots$, $E_{n-\alpha_1-\alpha_2} = r_n$. They are first integrals of the Hamiltonian function associated to $H^{2\eta}$. Now, We define

$$\begin{aligned} V = \; & V_1^2 + \cdots + V_{\alpha_1-1}^2 + W_1^2 + \cdots + W_{\alpha_2-1} + E_1^2 + \cdots + \\ & E_{n-\alpha_1-\alpha_2}^2 + (F_\eta(r_1, \ldots, r_{\alpha_1}, \mathbf{k}^1 \cdot \varphi))^2 + \\ & (G_\eta(r_{\alpha_1+1}, \ldots, r_{\alpha_1+\alpha_1}, \mathbf{k}^2 \cdot \varphi))^2 - \delta^2 r_1^\eta . \end{aligned}$$

It is verified that

$$\dot{V} = -\delta^2 \eta k_1^1 \Psi_1'(\mathbf{k^1} \cdot \varphi) r_1^{3\eta/2-1} + O(r_1^{3\eta/2-1/2}).$$