Poincaré Maps And Dynamics In Restricted Planar (n + 1)-Bodies Problems

Antonio García [1]

[1] Departamento de Matemáticas
Universidad Autónoma Metropolitana-Iztapalapa

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Program:

1. Setting
    - The $n-$center problem
2. Fundamental region
    - Definition of the fundamental region
    - Differential geometry of the fundamental region
    - Symplectic elliptic change of coordinates
3. Billiard maps
4. Restricted $(n+1)-$body problem

Main Problems

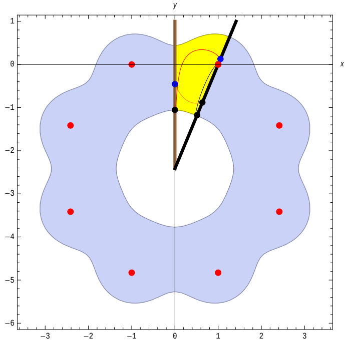
n particles (the primaries) with mass 1 in a regular polygon.

An infinitesimal mass particle (the secondary). Let $\mathbf{q} = (x, y)$ be the position and $\mathbf{p} = (X, Y)$ be the velocity of the secondary.

We study the following problems:

1. The primaries are fixed: (The n–center problem).
2. The primaries rotate at uniform angular velocity ω. (The restricted $(n+1)$–body problem).

Objective: To study the motion of the secondary.



Case $n = 8$.

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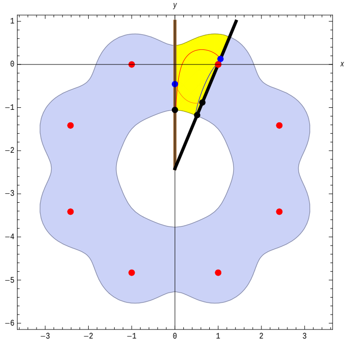

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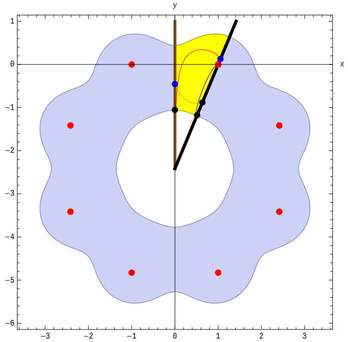

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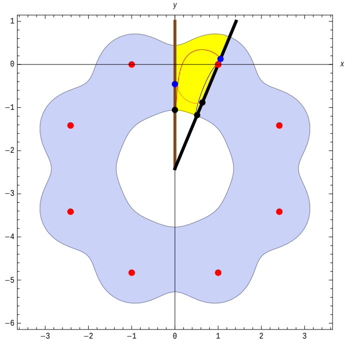

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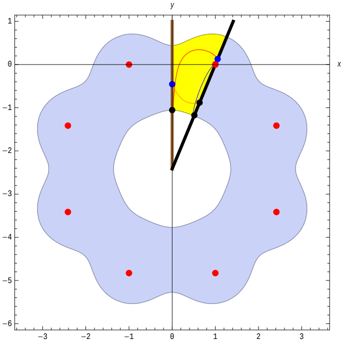

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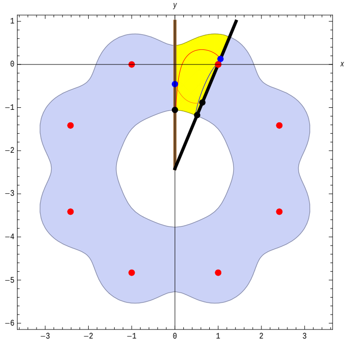

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The n–center problem

Theorem

The n–center problem has the following properties:

- *The Hamiltonian is:* $H(\mathbf{q},\mathbf{p}) = \frac{1}{2}\mathbf{p}\cdot\mathbf{p} - U(\mathbf{q})$, *where* $U(\mathbf{q}) = \sum_{k=0}^{n-1} \frac{1}{|\mathbf{q}-(A_k,B_k)|}$.
- *The only singularities are:* (A_k, B_k), $k = 0, \ldots, n-1$, *the positions of the fixed points.*
- $h = H(\mathbf{q},\mathbf{p})$ *is an integral.*
- D_n*–symmetry.*

Theorem

If $-h$ is a regular value of $U(\mathbf{q})$ thus

- *The set $A = \{\mathbf{q} : h + U(\mathbf{q}) \geq 0\} \subset \mathbb{R}^2$ is a manifold with boundary $\partial A = \{\mathbf{q} : h + U(\mathbf{q}) = 0\}$ and interior $A^\circ = \{\mathbf{q} : h + U(\mathbf{q}) > 0\}$.*
- *For certain values of h the set A has a ring shape.*

Definition

A is the Hill region for the value of energy h.

Values: $n = 8$, $h = 3.3$.

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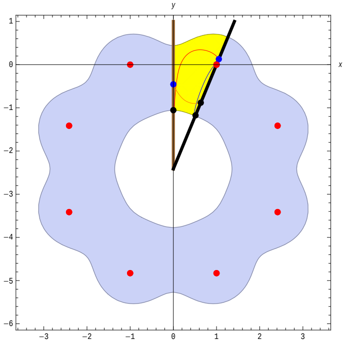
Values: $n = 8$, $h = 3.3$.

Let L_a be the Y-axis and L_b be the line that joins the center of mass and the singularity (A_1, B_1), L_a and L_b are symmetry axis of the motion.
Let S be the region between them. It is a fundamental region of the problem (Using the D_n symmetry).

S is simply connected. The boundary has 4 components. The only singularity: (A_1, B_1), is in one of them.



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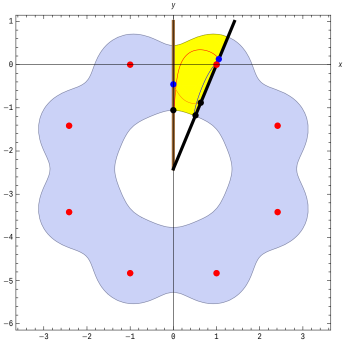

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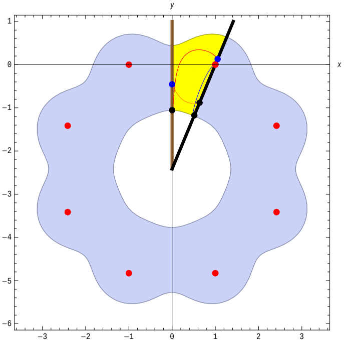

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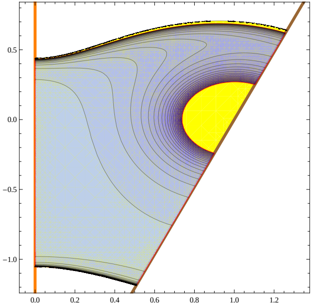

Definition

The mechanical or Jacobi metric on A is: $\tilde{g} = 2\left(h + U\left(\mathbf{q}\right)\right) g$.

Meaning: $\tilde{g}\left(\left(\mathbf{q}, \mathbf{v}\right), \left(\mathbf{q}, \mathbf{w}\right)\right) = 2\left(h + U\left(\mathbf{q}\right)\right) \mathbf{v} \cdot \mathbf{w}$

Theorem

We have the following properties:

- *The mechanical and the standard metric are conformal.*
- *The mechanical curvature is:*

$$K_h(x,y) = \frac{-\left[\left(\sum_{k=0}^{n-1}\frac{x-A_k}{|(x,y)-(A_k,B_k)|^3}\right)^2 + \left(\sum_{k=0}^{n-1}\frac{y-B_k}{|(x,y)-(A_k,B_k)|^3}\right)^2 + (h+U((x,y)))\sum_{k=0}^{n-1}\frac{1}{|(x,y)-(A_k,B_k)|^3}\right]}{2(h+U((x,y)))} < 0$$

$K_h(x,y) = -\infty$ *in* ∂A *and in* (A_1, B_1).

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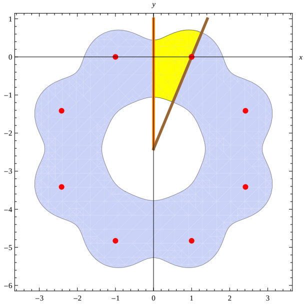
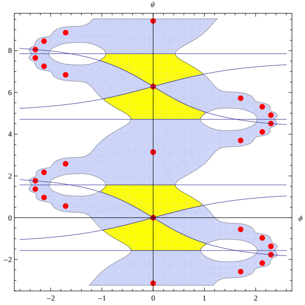
Symplectic elliptic change of coordinates (Birkhoff).





$$x_1 = \cosh\phi\cos\psi, \qquad X_1 = 2\frac{\Psi\sinh\phi\cos\psi - \Phi\cosh\phi\sin\psi}{\cosh(2\phi)-\cos(2\psi)},$$

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Fundamental domain: $\phi \in (-\infty,\infty)$, $\psi \in [0,\pi]$.

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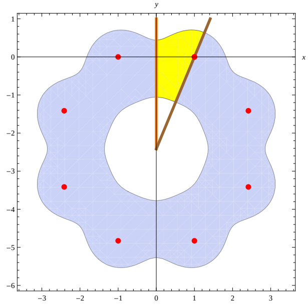

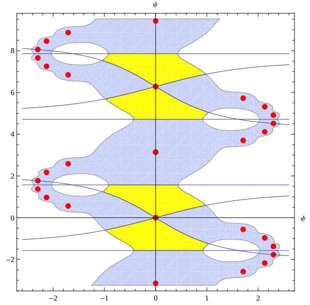

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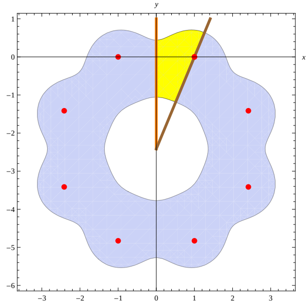

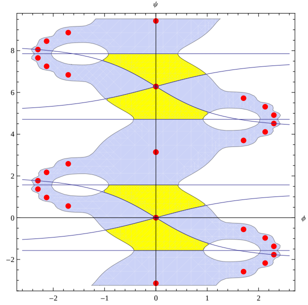

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The Hamiltonian becomes:

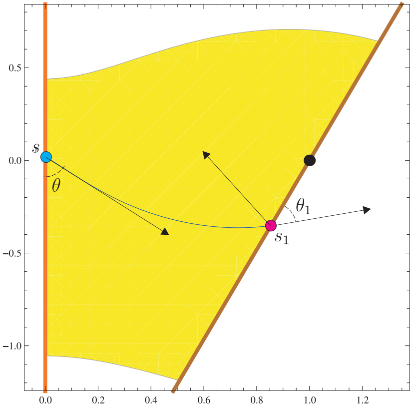
$$H(\phi,\psi,\Phi,\Psi) = [\cos(2\psi) - \cosh(2\phi)]^{-1} A(\phi,\psi,\Phi,\Psi)$$

The term $g(\phi,\psi) = \cos(2\psi) - \cosh(2\phi)$ is related to the singularities $(\pm 1, 0)$. $A(\phi,\psi,\Phi,\Psi)$ has singularities associated to the remaining collisions.
Let h be a level of energy, and define the function:

$$\widehat{H}(\phi,\psi,\Phi,\Psi) = g(\phi,\psi)(H(\phi,\psi,\Phi,\Psi) - h)$$

The flows associated to $H(\phi,\psi,\Phi,\Psi)$ in the set $H = h$ and $\widehat{H}(\phi,\psi,\Phi,\Psi)$ in the level set $\widehat{H} = 0$ are conjugated (except in the points $\phi = 0$ and $\psi = k\pi$, $k \in \mathbb{Z}$). The flow of $\widehat{H}$ on S is smooth.

Definition of the billiard map



The Poincaré map is $P(s, \theta) = (s_1, \theta_1)$.

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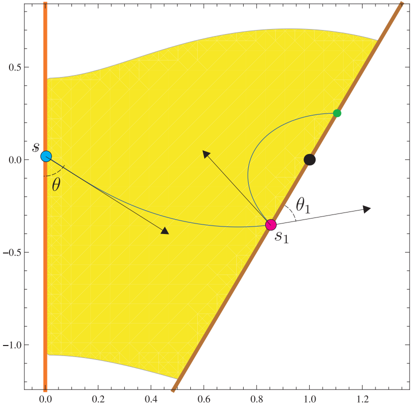

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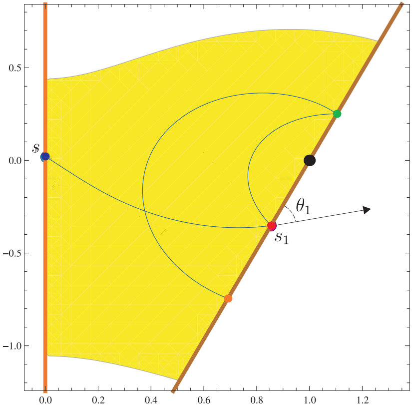

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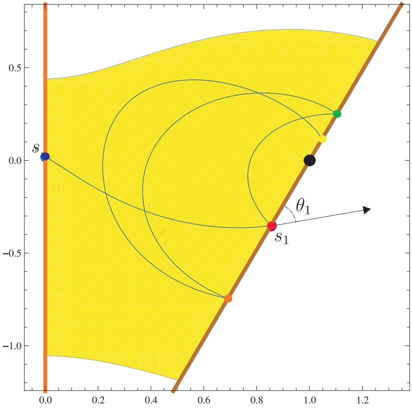

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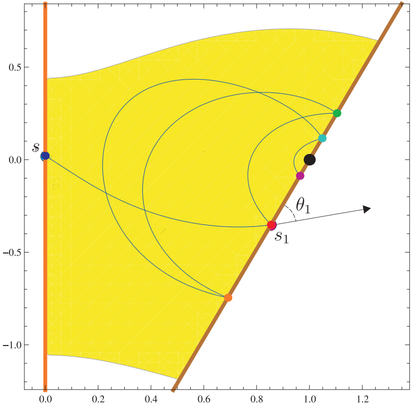

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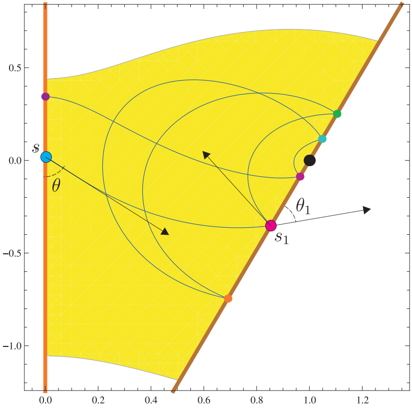

The Poincaré map is $P(s,\theta) = (s_1,\theta_1)$.

Theorem

There are three types of orbits on the fundamental region S:
- *Orbits that start in L_a, point to the interior of S, and reach L_b.*
- *Orbits that start in L_b, point to the interior of S, and reach L_b again.*
- *Orbits that start in L_b, point to the interior of S, and reach L_a.*

Corollary

The flow defines a geodesic billiard, it consists in the three types of Poincaré maps:

$$\begin{aligned} P_{ab} &: L_a \times (0,\pi) \to L_b \times (0,\pi) \\ P_{ba} &: L_b \times (0,\pi) \to L_a \times (0,\pi) \qquad (s,\theta) \to (s_1,\theta_1) \\ P_{bb} &: L_b \times (0,\pi) \to L_b \times (0,\pi) \end{aligned}$$

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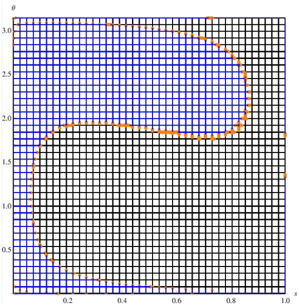

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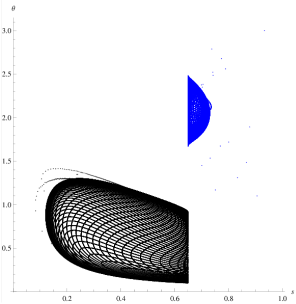

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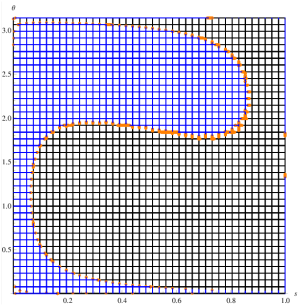

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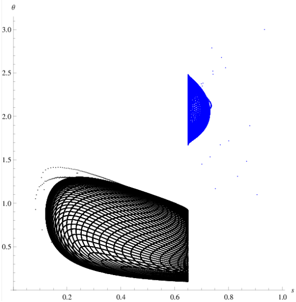

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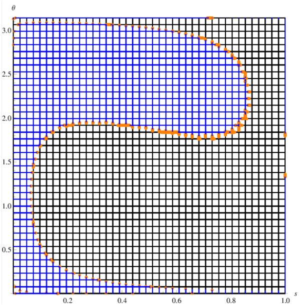

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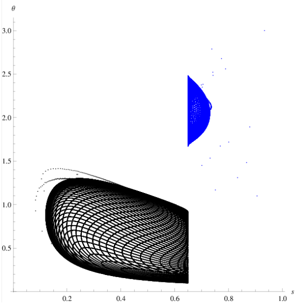

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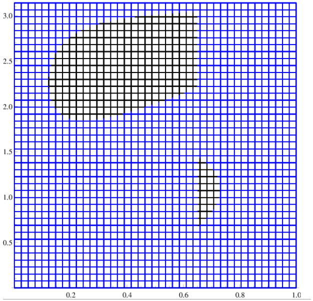

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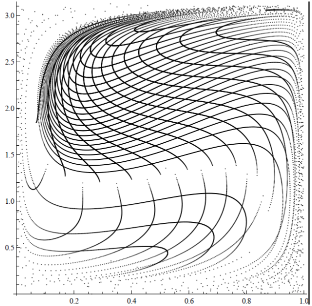

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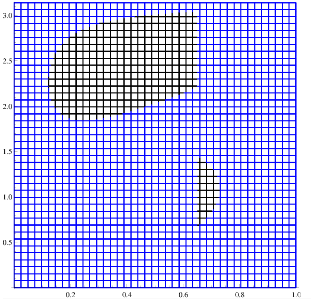

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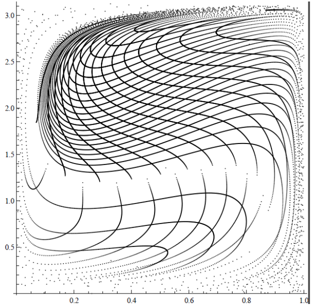

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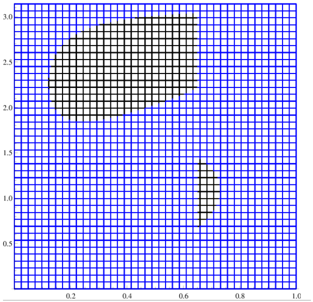

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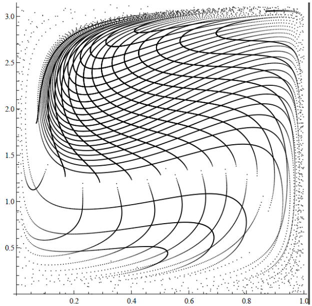

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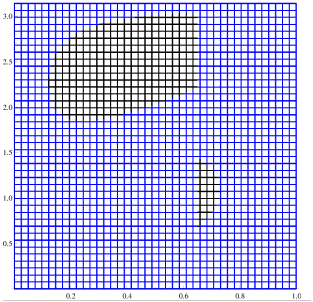
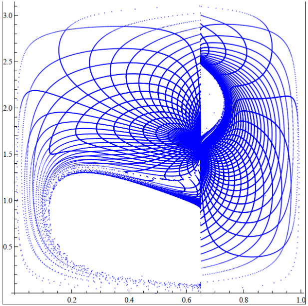
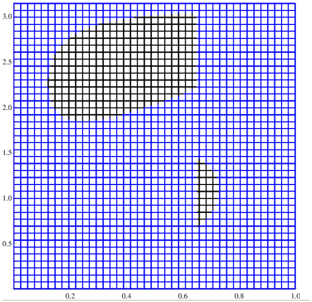


Image

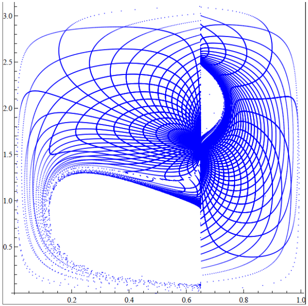


$$P_{bb} : L_b \times (0, \pi) \to L_b \times (0, \pi)$$

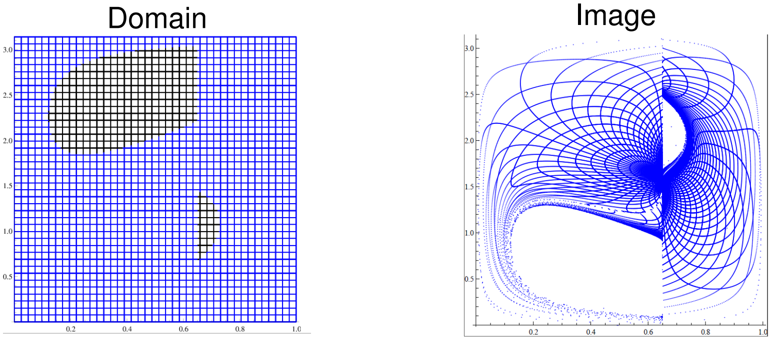
Domain



Image

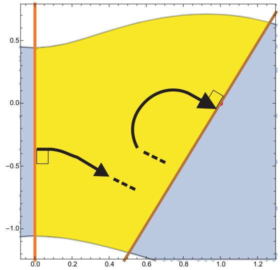


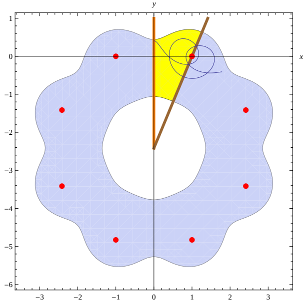
$$P_{bb} : L_b \times (0, \pi) \to L_b \times (0, \pi)$$

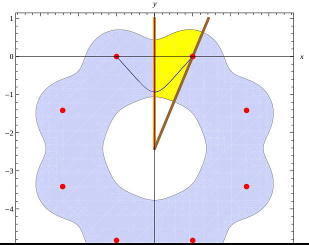


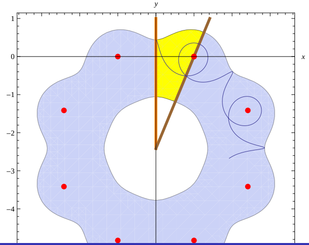
$$P_{bb} : L_b \times (0, \pi) \to L_b \times (0, \pi)$$

A simple way of finding periodic orbits:









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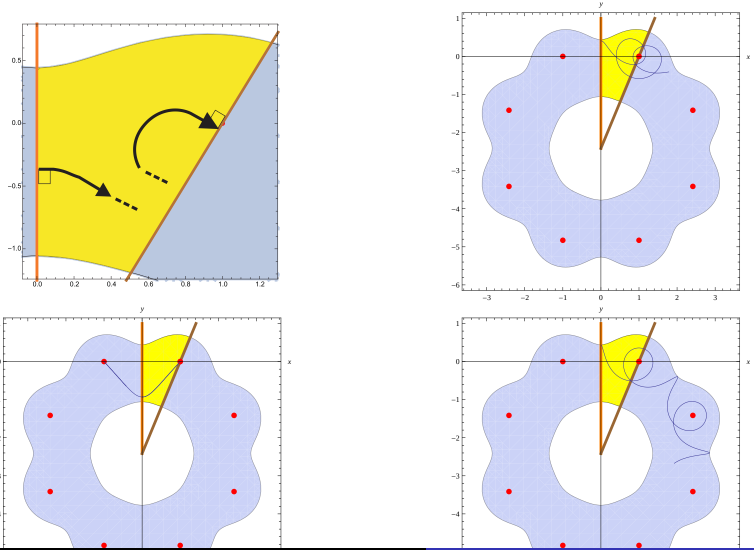

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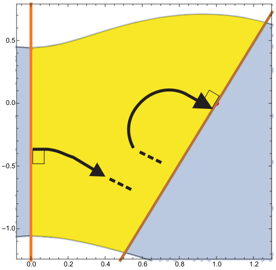

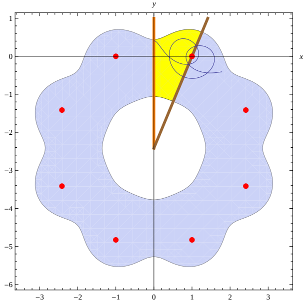

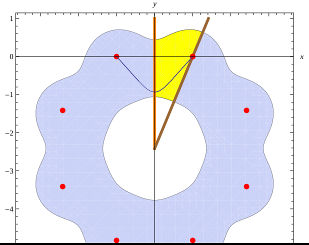

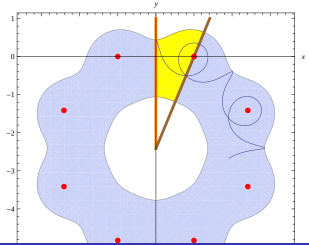

The restricted $(n+1)-$body problem

Now the primaries are subject to their mutual attraction and move in a regular polygon at uniform angular velocity ω. Using rotating coordinates, we fix the primaries. Two of them are in $(\pm 1, 0)$. The Hamiltonian associated to the motion of the secondary is

$$H(\mathbf{x}, \mathbf{X}) = \frac{1}{2c_1}|\mathbf{X}|^2 + \omega\left[x_2 X_1 - x_1 X_2 + \cot\left(\frac{\pi}{n}\right) X_1\right] - U(\mathbf{x}),$$

$$U(\mathbf{x}) = \frac{1}{c_2|1-\mathbf{x}|} + \frac{1}{c_2|1+\mathbf{x}|} + \sum_{k=2}^{n-1} \frac{1}{c_2|A_k + iB_k - \mathbf{x}|}.$$

Main differences between the Hamiltonians

1. The Hamiltonian is not mechanical. (Extra term: $\omega\left[x_2 X_1 - x_1 X_2 + \cot\left(\frac{\pi}{n}\right) X_1\right]$.
2. The system is not reversible, but has a rotational symmetry $\mathbb{Z}_n$.

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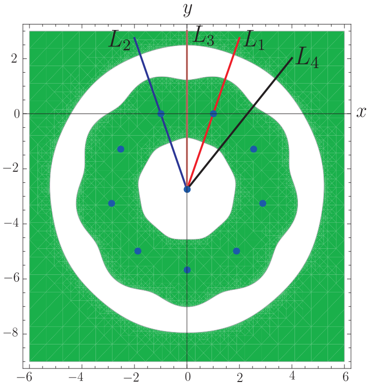
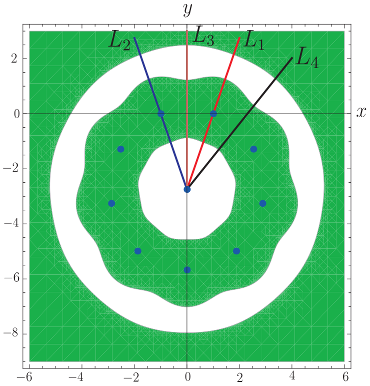


Fundamental region: Between the lines L_2 y L_4. There are two singularities: $(\pm 1, 0)$.

$$P_{L_3 L_1} : L_3 \times \mathbb{S}^1 \to L_1 \times \mathbb{S}^1$$
$$P_{L_3 L_3} : L_3 \times \mathbb{S}^1 \to L_3 \times \mathbb{S}^1$$
$$P_{L_3 L_2} : L_3 \times \mathbb{S}^1 \to L_2 \times \mathbb{S}^1$$
$$P_{L_1 L_3} : L_1 \times \mathbb{S}^1 \to L_3 \times \mathbb{S}^1$$
$$P_{L_1 L_1} : L_1 \times \mathbb{S}^1 \to L_1 \times \mathbb{S}^1$$
$$P_{L_1 L_4} : L_1 \times \mathbb{S}^1 \to L_4 \times \mathbb{S}^1.$$



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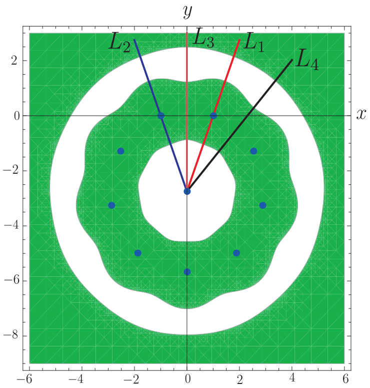
$$P_{L_3L_2} : L_3 \times \mathbb{S}^1 \to L_2 \times \mathbb{S}^1$$
$$P_{L_1L_3} : L_1 \times \mathbb{S}^1 \to L_3 \times \mathbb{S}^1$$
$$P_{L_1L_1} : L_1 \times \mathbb{S}^1 \to L_1 \times \mathbb{S}^1$$
$$P_{L_1L_4} : L_1 \times \mathbb{S}^1 \to L_4 \times \mathbb{S}^1.$$



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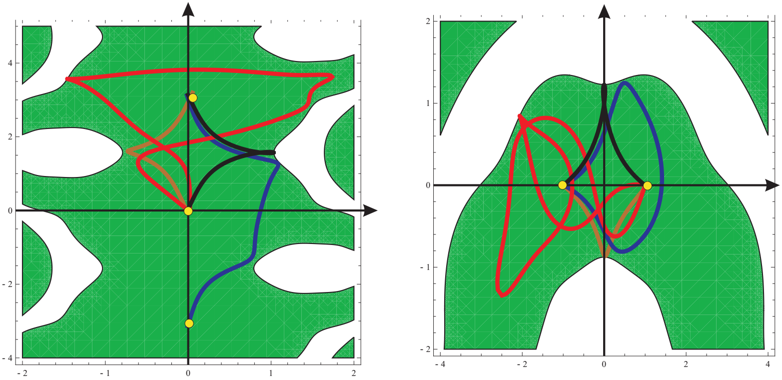

Some collition-collition orbits.

For Further Reading

- M. Alvarez-Ramírez, A. García, *Poincaré maps and near-collision dynamics for a restricted planar (n+1)-body problem.* App. Math. and Comp. (233), 2014, 328-337.
- 0. Chong-Pin, *Curvature and Mechanics* Adv. in Math. (15), 1975, 269-311.
- N. Soave, S. Terracini *Symbolic dynamics for the N-centre problem at negative energies* Disc. and Cont. Dyn. Sys. (32), 2012, 3245-3245.

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