Dinàmica asimptòtica d'una equació en diferències cap a un punt fixe parabòlic.

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Trobada al voltant del nostre àmic Armengol, 60è aniversari

Confluències en Sistemes Dinàmics

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Bons records!

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Plan

1. Aim of this work
2. Recalling the complex and real cases in analytical dynamics. The parabolic case
3. Main results on asymptotic dynamical properties in one dimension
4. Idea of the proof
5. An application to $2D$ difference equations
6. Taking a look in higher dimensions. Examples

Aim of this work

Consider the difference equation

$$x_{n+1} = F(x_n) = \lambda x_n + \sum_{i=1}^{\infty} a_i x_n^{k+i}, \qquad (1)$$

where $F : U \subset \mathbb{R} \to \mathbb{R}, \quad F \in \mathcal{C}^\infty(U)$, being U an open subset of the origin, $k \geq 1$, $a_1 \neq 0$, and $0 \leq |\lambda| \leq 1$.

According to the different possibilities of F, i.e. of λ

Question:

What is the asymptotic behaviour of solutions, $\{x_n\}_{n\in\mathbb{N}}$, in terms of n, when $\lim_{n\to\infty} x_n = 0$?

Some examples

The difference equation

$$x_{n+1} = \frac{x_{n-k}}{1 + x_n + \cdots + x_{n-k+1}}, \quad \text{for } k = 1$$

- Berg and Stevi'c in 2002, gives the first five terms in the asymptotic expansions of the solutions

$$\frac{2}{n} + \frac{2}{n^2} \ln n + \frac{a}{n^3} \ln^2 n \leq x_n \leq \frac{2}{n} + \frac{2}{n^2} \ln n + \frac{b}{n^3} \ln^2 n,$$

  where $a < 2 < b$, and proves existence of $\{x_n\}_n \to 0$

- Stevi'c in 2006, for all k, proved the existence of a positive solution, $\{x_n\}_n$, converging to zero, by assuming that the first five terms in the asymptotic expansion of x_n have the following form:

$$x_n \sim \frac{a}{n} + \frac{b \ln n + c}{n^2} + \frac{d \ln^2 n + e \ln n}{n^3}.$$

Some examples

The second example concerns the difference equation, Berg in 2008 and by Berg-Stević in 2011,

$$x_n = \frac{x_{n-k}}{1 + x_{n-1} \dots x_{n-k+1}}, \tag{2}$$

- existence of a solution s.t. $\{x_n\}_n \to 0$ is proved for $k = 3$ (2008) and for all k (2011), by assuming that the asymptotic expansion of x_n

$$x_n = \frac{1}{\sqrt{n}} \left(a + \frac{b}{n} + \frac{1}{n^2} (c \ln^2 n + d \ln n + e) \right),$$

where the coefficients are fixed s.t. x_n is proved to be a solution.

- Idea: For $k = 3$, the authors approximate the discrete equation by the differential equation

$$x(1 + x^2) = x - 3x'$$

and the approximate solution is given by $x(t) = \sqrt{\frac{3}{2t}} \left(1 + \frac{3}{8t} \ln t\right)$

Recalling the complex case $0 \leq |\lambda| < 1$

In the complex case,

$$F(z) = \lambda z + \sum_{i \geq 1} a_i z^{k+i}, \quad \lambda, z, a_i \in \mathbb{C}, \quad a_1 \neq 0$$

by using conformally conjugate (c.c.) functions to F, it is possible reduce this study to more simpler cases (canonical forms):

- $0 < |\lambda| < 1$, the linearization Theorem (**Koenigs**) applies
  $\Rightarrow F(z)$ is c.c. to λz
- $\lambda = 0$, the conjugation theorem for superattracting fixed points (**Boettcher**) can be used
  $\Rightarrow F(z)$ is c.c. to z^{k+1}

A first Goal:

when $0 \leq |\lambda| < 1$, give the asymptotic behaviour of solutions, $\{x_n\}_{n \in \mathbb{N}}$, in terms of n, for the dynamics in the real case

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Main results. Asymptotic dynamical properties in $1d$

Asymptotic behaviour in the real case, when $0 \leq |\lambda| < 1$.

$$x_{n+1} = F(x_n) = \lambda x_n + \sum_{i=1}^{\infty} a_i x_n^{k+i}, \quad k \geq 1, \quad a_1 \neq 0, \quad 0 \leq |\lambda| < 1.$$

1. if $0 < |\lambda| < 1$, then
$$x_n = \overline{x}_0 \lambda^n + O(\lambda^{2n}), \quad \overline{x}_0 = \overline{x}_0(x_0)$$
2. if $\lambda = 0$, then
$$x_n = a\beta^{(k+1)^n} + O\left(\beta^{(k+1)^n+1}\right), \quad \beta = a_1^{\frac{2}{k}} x_0, \quad |a| = |a_1|^{-\frac{1}{k}}$$

Idea of the proof: We use the conformally conjugate theory, as in the complex case.

The complex case $\lambda = 1$

Concerning the asymptotic behaviour of complex solutions, $\{z_n\}_{n\in\mathbb{N}}$, in terms of n, for parabolic diffeomorphisms $z_{n+1} = F(z_n)$

- **Resman 2013**, proved that the asymptotic development of z_n is given by

$$z_n = \sum_{i=1}^{k} g_i n^{-i/k} + g_{k+1} n^{-\frac{k+1}{k}} \log n + o(n^{-\frac{k+1}{k}} \log n),$$

where the coefficients $g_i = g_i(k, A, a_2, \ldots, a_i) \in \mathbb{C}$, $i = 1, \ldots, k+1$, are complex-valued functions, and $A = (-ka_1)^{-\frac{1}{k}}$.

A second goal:

when $\lambda = 1$, give the complete asymptotic development of x_n in the real case.

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Recalling the real case $\lambda = 1$. Parabolic case

When assuming a parabolic fixed point at the origin for

$$x_{n+1} = F(x_n) = x_n + \sum_{i=1}^{\infty} a_i x_n^{k+i}, \quad x \in \mathbb{R}^m.$$

- The linear part of the map F at the fixed point is the identity and a whole neighborhood of the origin is a center manifold
- However there may exist invariant submanifolds of points which go to the origin by the iteration of the map (stable manifolds)

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Main results. Asymptotic dynamical properties in 1d

Asymptotic behaviour in the parabolic type fixed point case, $\lambda = 1$.

$$F(x) = x + a_1 x^{k+1} + a_2 x^{k+2} + \ldots, \quad a_1 \neq 0, \quad k \geq 1, \quad x, a_i \in \mathbb{R}$$

Theorem 1

For each x_0 initial condition belonging to an attracting domain of the origin, i.e. such that $\lim_{n\to\infty} x_n = 0$,

$$x_n = \sum_{p=0}^{\infty} \sum_{i=1}^{k} \frac{1}{n^{p+i/k}} \left(\sum_{j=0}^{p} g_p^{i,j} \ln^j n \right),$$

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Main results. Asymptotic dynamical properties in $1d$

A similar result on the asymptotic behaviour in the case, $\lambda = -1$.

$$F(x) = -x + a_1 x^{k+1} + a_2 x^{k+2} + \dots, \quad a_1 \neq 0, \quad k \geq 1, \quad x, a_i \in \mathbb{R}$$

Corollary 2

$$x_n = \sum_{p=0}^{\infty} \sum_{i=1}^{\overline{k}} \frac{1}{n^{p+i/\overline{k}}} \left(\sum_{j=0}^{p} g_p^{i,j} \ln^j n \right),$$

for some integer number $\overline{k}$ such that: $\overline{k} = k$ if k is even, and $\overline{k} > k$ if k is odd; and the coefficients $g_p^{i,j} = g_p^{i,j}(k, a_1, a_2, \dots)$ are real valued functions.

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Idea of the proof of Theorem 1

Lemma 3 (Technical result)

Given $n_0 \in \mathbb{N}$, let us consider the recurrence relation

$$u_{n+1} - u_n = f(n) + o(f(n)), \quad n \geq n_0, \quad n \in \mathbb{N},$$

where f is a real and continuous non-negative function, defined on $\mathbb{R}$. Suppose that $u_{n_0} > 0$ and that f is a monotonous function on $[n_0, +\infty)$. Consider a function F such that

$$F'(x) = f(x), \quad x \geq n_0,$$

and suppose that one of the following hypotheses holds:

1. $\lim_{x\to+\infty} F(x) = \infty$ *and* $\lim_{n\to+\infty} f(n)/f(n+1) = 1$,
2. $\lim_{x\to+\infty} F(x) \in \mathbb{R}$

Then, for some $F(x)$, $x \geq n_0$,

$$u_n = F(n) + o(F(n)).$$

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$$x_{n+1} = F(x_n) = x_n + a_1 x_n^{k+1} + a_2 x_n^{k+2} + \dots, \quad a_1 \neq 0, \quad \lim_{n\to\infty} x_n = 0.$$

- Step1. (c.o.v. $x \to \omega$) $x = A\omega^{-\frac{1}{k}}$, where $A = (-ka_1)^{-\frac{1}{k}}$,

$$\omega_{n+1} = \omega_n + 1 + \sum_{i=1}^{\infty} \frac{c_i}{\omega_n^{i/k}} \implies \omega_{n+1} - \omega_n = 1 + o(1)$$

- Step 2. $\omega_{n+1} - \omega_n = 1 + o(1) \Rightarrow$ (Lemma 4) $\omega_n = n + o(n)$
- Step 3. We define $p_n = \omega_n - n$
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- Step 5. (Lemma 4)
  $p_n = \sum_{i=1}^{k-1} \frac{e_i}{n^{(i-k)/k}} + e_k \ln(n) + \sum_{i=k+1}^{k(q+1)} \frac{e_i}{n^{(i-k)/k}} + o(n^{-q})$

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- Step 6.

$$\omega_n = n + p_n = n + \sum_{i=1}^{k-1} \frac{e_i}{n^{(i-k)/k}} + e_k \ln n + \sum_{i=k+1}^{k(q+1)} \frac{e_i}{n^{(i-k)/k}} + o(n^{-q})$$

- Step 7. If we undo the c.o.v.

$$x_n = A\omega_n^{-\frac{1}{k}} = \frac{A}{n^{1/k}} \left(\frac{1}{1+M}\right)^{1/k},$$

where $M = \sum_{\substack{i=1 \\ i \neq k}}^{k(q+1)} \frac{e_i}{n^{(k-i)/k}} + e_k \frac{1}{n} \ln n + o(n^{-q-1})$, using Taylor development of $(1/(1+M))^{1/k}$, on the variable M, in a neighbourhood of the origin, up to order p, then

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Idea of the proof of Theorem 1

- $$x_{n,p} = \sum_{m=0}^{p} \sum_{i=1}^{k} \frac{1}{n^{m+i/k}} \left(\sum_{j=0}^{m} \overline{g}_m^{i,j} \ln^j n \right) + \frac{1}{n^{p+1+1/k}} \overline{g}_{p+1}^{1,p+1} \ln^{p+1} n + R_p(n).$$

- Step 9. formula

$$x_n = \sum_{p=0}^{\infty} \sum_{i=1}^{\overline{k}} \frac{1}{n^{p+i/\overline{k}}} \left(\sum_{j=0}^{p} g_p^{i,j} \ln^j n \right),$$

follows by using mathematical induction on p. □

Remark

We are just involved into the control how the functions $(\ln^i n)/n^{j/k}$ emerge, not into fix the coefficients $g_p^{i,j}$

An application to 2d difference equations

The difference equation

$$x_{n+1} = \frac{x_{n-k}}{1 + x_n + \cdots + x_{n-k+1}}$$

case $k = 1$:

- Berg 2002, gives the first five terms in the asymptotic expansions of the solutions and proves existence of $\{x_n\}_n \to 0$
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Defined through the shift function $G(x, y)$,

$$(x_{n+1}, y_{n+1}) = G(x_n, y_n) := \left(y_n, \frac{x_n}{1 + y_n}\right)$$

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An application to 2d difference equations

Consider

$$(x_{n+1}, y_{n+1}) = F(x_n, y_n)$$

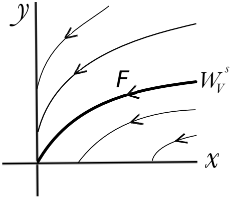
where

$$F = G^2 := G \circ G \quad \Rightarrow \quad F(x,y) = \left(\frac{x}{1+y}, \frac{y(1+y)}{1+x+y} \right).$$

If we define the stable invariant manifold of the origin, as

$$W_V^s = \{(x,y) \in U : \pi^1 F^m(x,y) \in V, \pi^2 F^m(x,y) > 0, m \geq 0, \\ F^m(x,y) \to 0, \text{as } m \to \infty\},$$

where $V = (0, r)$, then



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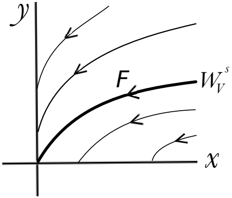
$$W_V^s = \{(x,y) \in U : \pi^1 F^m(x,y) \in V, \pi^2 F^m(x,y) > 0, m \geq 0, \\ F^m(x,y) \to 0, \text{as } m \to \infty\},$$

where $V = (0, r)$, then



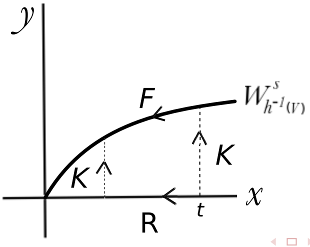
Parameterization method

Consider $F : U \subset \mathbb{R}^{1+n} \to \mathbb{R}^{1+n}, F(0,0) = 0, DF(0,0) = Id, F \in C^r$

The Aim: $\exists$ 1-D invariant manifolds passing through the origin.

- Parameterization method: One tries to look at the same time for the parameterization of the invariant manifold and for a version of the dynamics on it (**Ref: Cabré, Fontich et al, 2003,2004, 2007**).
- It consists in looking simultaneously for a parameterization of a curve, $K : I_0 \subset \mathbb{R} \to \mathbb{R}^{1+n}$, and a representation of the dynamics on the curve, $R : I_0 \subset \mathbb{R} \to \mathbb{R}$ such that

$$F \circ K = K \circ R, \qquad \text{(invariance equation)}$$



An application to 2d difference equations

Consider $F(x,y) = \left(\frac{x}{1+y}, \frac{y(1+y)}{1+x+y}\right)$.

Theorem 4

1. *There exists an open region in the first quadrant on which*
2. *the invariant manifold of the origin, W^s_V, is the graph of an analytic function, K, where*

$$K(x) = x - \frac{1}{2}x^2 + \frac{1}{2}x^3 - \frac{9}{16}x^4 + \frac{5}{8}x^5 - \frac{41}{64}x^6 + O(x^7),$$

3. *the dynamics on W^s_V is given by the analytic function R,*

$$R(t) = t - t^2 + \frac{3}{2}t^3 - \frac{5}{2}t^4 + \frac{69}{16}t^5 - \frac{15}{2}t^6 + O(t^7),$$

An application to $2d$ difference equations

4. on W_V^s, the first terms of the asymptotic development of the solutions corresponding to the difference equation

$$t_{n+1} = R(t_n)$$

are given by

$$t_n = \frac{1}{n} + \frac{1}{n^2}\left(-c + \frac{1}{2}\ln n\right) + \frac{1}{n^3}\left(c^2 - \frac{1+4c}{4}\ln n + \frac{1}{4}\ln^2 n\right)$$
$$+ \frac{1}{n^4}\left(g_3^{1,0} + g_3^{1,1}\ln n + g_3^{1,2}\ln^2 n + g_3^{1,3}\ln^3 n\right)$$
$$+ \frac{1}{n^5}\left(g_4^{1,0} + g_4^{1,1}\ln n + g_4^{1,2}\ln^2 n + g_4^{1,3}\ln^3 n + g_4^{1,3}\ln^4 n\right) + o\left(\frac{\ln^4 n}{n^5}\right)$$

where $g_p^{i,j} = g_p^{i,j}(c)$, being c a constant parameter that is fixed from the intitial condition of the orbit.

An application to 2d difference equations

Idea of the proof (some facts):

- The origin, $(0,0)$, is a parabolic fixed point
- Points on the coordinate axes are fixed points
- $V(x,y) = x^2 + y^2$ is a Lyapunov function
- In a neighbourhood of the origin, we observe that F is an analytic function given by

$$F(x,y) = (x - xy + xy^2 + o(\|(x,y)\|^3), y - xy + xy^2 + x^2y + o(\|(x,y)\|^3))$$

To prove the existence and analyticity of one-dimensional stable manifold associated to $(0,0)$, we apply Theorem 4.1 of Baldomá, Fontich 2004, [BF2004].

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An application to 2d difference equations

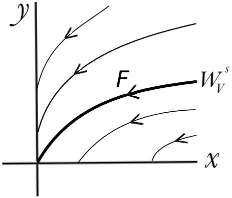
To obtain the first terms of the asymptotic development of t_n given by

$$t_{n+1} = R(t_n)$$

it is enough to apply **Theorem 1** to the obtained expression of $R(t)$.

Remark

Since we can identify $x_n = t_n$, the obtained expression of t_n gives the asymptotic expansion of x_n, in terms of n. This expansion includes the one proved by Berg 2002.



Taking a look in higher dimensions. Examples

For higher dimensions, concerning the asymptotic dynamic behaviour of solutions tending to the origin, we wonder if

Question

can previous results on 1d difference equations be extrapolated to higher dimensions?

i.e. for $x_n \in W_V^s$, is

$$x_n = \sum_{p=0}^{\infty} \sum_{i=1}^{k} \frac{1}{n^{p+i/k}} \left(\sum_{j=0}^{p} g_p^{i,j} \ln^j n \right) ?$$

Taking a look in higher dimensions. Examples

- Let us take a look at the asymptotic dynamic behaviour of solutions tending to the origin
- through the study of two families of difference equations defined in a neighbourhood of the origin in $3d$

We proceed by:

- assuming, if necessary, that the invariant manifold of the origin W_V^s is the graph of an analytic function K.
- using the parameterization method

$$F \circ K = K \circ R, \qquad \text{(invariance equation)}.$$

where R is the representation of the dynamics on the curve.

- approach analytically K and R.

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Taking a look in higher dimensions. Examples

Both examples agree with the following scheme.

- Consider the "shift" function

$$(x_{n+1}, y_{n+1}, z_{n+1}) = G(x_n, y_n, z_n) = (y_n, z_n, g(x_n, y_n, z_n)),$$

  where g is given either by

$$g(x, y, z) = \frac{x}{1 + y + z} \qquad \text{or} \qquad g(x, y, z) = \frac{x}{1 + yz}.$$

- We take the three-dimensional scheme iteration, F, given by

$$F = G^3 := G \circ G \circ G$$

i.e. by

$$F(x, y, z) = (g(x, y, z), g(y, z, g(x, y, z)), g(z, g(x, y, z), g(y, z, g(x, y, z))))$$

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- Consider the "shift" function

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i.e. by

$$F(x,y,z) = (g(x,y,z), g(y,z,g(x,y,z)), g(z,g(x,y,z),g(y,z,g(x,y,z))))$$

Taking a look in higher dimensions. Example 1.

Example 1: $g(x,y,z) = \frac{x}{1+y+z}$

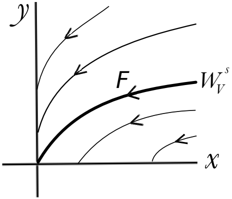
$$(x_{n+1}, y_{n+1}, z_{n+1}) = F(x_n, y_n, z_n)$$

- Each component of F agrees with the difference equation

$$x_{n+1} = \frac{x_{n-k}}{1 + x_n + \cdots + x_{n-k+1}}, \qquad \text{when} \quad k = 2.$$

  Stević 2006 proves existence of $\{x_n\}_n \to 0$, for all k.

- On each coordinate plane, the dynamical behaviour of F coincides with the one given in case $k = 1$, v.g.



Taking a look in higher dimensions. Example 1.

By using the parameterization method,

$$F \circ K = K \circ R, \quad \text{invariance equation}$$

-K is assumed to be the analytic graph of W^s_V

-R the representation of the dynamics on the curve,

$$\Downarrow$$

1. There exists an invariant curve, K, through the origin, analytically approached by

$$K(t) = (t, t - \frac{2}{3}t^2 + \frac{10}{9}t^3 - \frac{58}{27}t^4 + o(t^4), t - \frac{4}{3}t^2 + \frac{28}{9}t^3 - \frac{224}{27}t^4 + o(t^4)).$$

2. The dynamics on K is approached by the analytic function R

$$R(t) = t - 2t^2 + 6t^3 - \frac{182}{9}t^4 + \frac{214}{3}t^5 - \frac{20762}{81}t^6 + o(t^6).$$

Taking a look in higher dimensions. Example 1.

Proposition 1

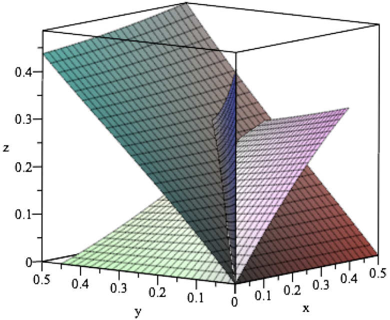
There exist two positive solutions in W_V^s with different asymptotic speeds developments in n when approaching the origin, given by

$$x_n = \frac{1}{n} + \frac{1}{n^2}\left(-c + \frac{1}{2}\ln n\right) + \frac{1}{n^3}\left(c^2 - \frac{1+4c}{4}\ln n + \frac{1}{4}\ln^2 n\right) + \frac{1}{n^4}\left(g_3^{1,0} + g_3^{1,1}\ln n + g_3^{1,2}\ln^2 n + g_3^{1,3}\ln^3 n\right) + o\left(\frac{\ln^3 n}{n^4}\right)$$

while the other is given by

$$x_n = \frac{1}{2n} + \frac{1}{4n^2}\left(-2c + \ln n\right) + \frac{1}{8n^3}\left(\frac{1}{9} + 4c^2 - (1+4c)\ln n + \ln^2 n\right) + \frac{1}{16n^4}\left(g_3^{1,0} + g_3^{1,1}\ln n + g_3^{1,2}\ln^2 n + \ln^3 n\right) + o\left(\frac{\ln^3 n}{n^4}\right)$$

Taking a look in higher dimensions. Example 1.



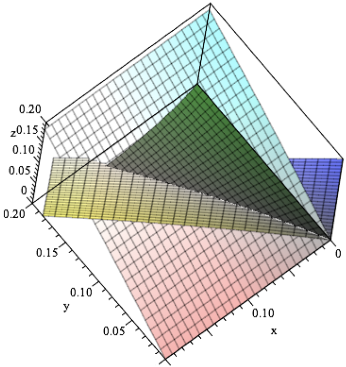


Figure: Stable invariant manifold of the origin: arrowhead. Different perspectives. Two different asymptotic speeds developments in n when approaching the origin

Taking a look in higher dimensions. Example 2.

Example 2: $g(x,y,z) = \frac{x}{1+yz}$

$$(x_{n+1}, y_{n+1}, z_{n+1}) = F(x_n, y_n, z_n)$$

- Each component of F agrees with the difference equation

$$x_n = \frac{y_{n-k}}{1 + y_{n-1} \dots x_{n-k+1}}, \qquad \text{when} \quad k = 3.$$

  Berg 2008 and Berg-Stević 2011 (for all k) proved the existence of $\{x_n\}_n \to 0$.
- Making a guess, they also fix the first five terms in the asymptotic expansion of x_n.
- All points in the coordinate planes are fixed points.

Taking a look in higher dimensions. Example 2.

By using the parameterization method,

$$F \circ K = K \circ R, \quad \text{invariance equation}$$

-K is assumed to be the analytic graph of W^s_V

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1. There exists an invariant curve, K, through the origin, analytically approached by

$$K(t) = (t, t - \frac{1}{3}t^3 + \frac{1}{3}t^5 - \frac{1}{3}t^7 + o(t^7), t - \frac{2}{3}t^3 + t^5 - \frac{5}{3}t^7 + o(t^7)).$$

2. The dynamics on K is approached by the analytic function R

$$R(t) = t - t^3 + 2t^5 - \frac{41}{9}t^7 + \frac{32}{3}t^9 - \frac{2014}{81}t^{11} + \mathrm{o}(t^{11}).$$

Taking a look in higher dimensions. Example 2.

Proposition 2

There exists a positive solution in W_V^s whose asymptotic behaviour is

$$x_n = \frac{1}{\sqrt{2n}}\left(1+\frac{1}{n}\left(-\frac{c}{2}+\frac{1}{8}\ln n\right)-\frac{1}{32n^2}\left(\frac{7}{9}+\ln n\right)\right.$$
$$+\frac{1}{256n^3}\left(\frac{55}{18}+\frac{31}{9}\ln n-\ln^2 n\right)$$
$$\left.+\frac{1}{n^4}\left(g_4^{1,0}+g_4^{1,1}\ln n+g_4^{1,2}\ln^2 n+g_4^{1,3}\ln^3 n\right)\right)+o\left(\frac{\ln^3 n}{n^{9/2}}\right),$$

being c a constant parameter that is fixed from the initital condition.

Taking a look in higher dimensions. Example 2 $g(x,y,z)=\frac{x}{1+yz}$

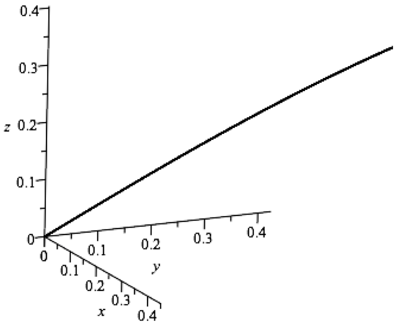

Figure: Stable invariant manifold of the origin: curve.

Per Molts d'Anys Armengol!!