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Qualitative Theory of Planar Differential Systems

With 123 Figures and 10 Tables

 Springer

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Mathematics Subject Classification (2000): 34Cxx (34C05, 34C07, 34C08, 34C14, 34C20, 34C25, 34C37, 34C41), 37Cxx (37C10, 37C15, 37C20, 37C25, 37C27, 37C29)

Library of Congress Control Number: 2006924563

ISBN-10 3-540-32893-9 Springer Berlin Heidelberg New York
ISBN-13 3-540-32902-1 Springer Berlin Heidelberg New York

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Cover design: Erich Kirchner, Heidelberg
Typesetting by the authors and SPi using a Springer L^AT_EX macro package

Printed on acid-free paper SPIN: 11371328 41/3100/SPi 5 4 3 2 1 0

Preface

Our aim is to study *ordinary differential equations* or simply *differential systems* in two real variables

$$\begin{aligned}\dot{x} &= P(x, y), \\ \dot{y} &= Q(x, y),\end{aligned}\tag{0.1}$$

where P and Q are C^r functions defined on an open subset U of \mathbb{R}^2 , with $r = 1, 2, \dots, \infty, \omega$. As usual C^ω stands for analyticity. We put special emphasis onto *polynomial differential systems*, i.e., on systems (0.1) where P and Q are polynomials.

Instead of talking about the differential system (0.1), we frequently talk about its associated *vector field*

$$X = P(x, y) \frac{\partial}{\partial x} + Q(x, y) \frac{\partial}{\partial y}\tag{0.2}$$

on $U \subset \mathbb{R}^2$. This will enable a coordinate-free approach, which is typical in the theory of dynamical systems. Another way expressing the vector field is by writing it as $X = (P, Q)$. In fact, we do not distinguish between the differential system (0.1) and its vector field (0.2).

Almost all the notions and results that we present for two-dimensional differential systems can be generalized to higher dimensions and manifolds; but our goal is not to present them in general, we want to develop all these notions and results in dimension 2. We would like this book to be a nice introduction to the qualitative theory of differential equations in the plane, providing simultaneously the major part of concepts and ideas for developing a similar theory on more general surfaces and in higher dimensions. Except in very limited cases we do not deal with bifurcations, but focus on the study of individual systems.

Our goal is certainly not to look for an analytic expression of the global solutions of (0.1). Not only would it be an impossible task for most differential systems, but even in the few cases where a precise analytic expression can be found it is not always clear what it really represents. Numerical analysis of a

differential system (0.1) together with graphical representation are essential ingredients in the description of the phase portrait of a system (0.1) on U ; that is, the description of U as union of all the orbits of the system. Of course, we do not limit our study to mere numerical integration. In fact in trying to do this one often encounters serious problems; calculations can take an enormous amount of time or even lead to erroneous results. Based however on a priori knowledge of some essential results on differential systems (0.1), these problems can often be avoided.

Qualitative techniques are very appropriate to get such an overall understanding of a differential system (0.1). A clear picture is achieved by drawing a phase portrait in which the relevant qualitative features are represented; it often suffices to draw the “extended separatrix skeleton.” Of course, for practical reasons, the representation must not be too far from reality and has to respect some numerical accuracy. These are, in a nutshell, the main ingredients in our approach.

The basic results on differential systems and their qualitative theory are introduced in Chap. 1. There we present the fundamental theorems of existence, uniqueness, and continuity of the solutions of a differential system with respect the initial conditions, the notions of α - and ω -limit sets of an orbit, the Poincaré–Bendixson theorem characterizing these limit sets and the use of Lyapunov functions in studying stability and asymptotic stability. We analyze the local behavior of the orbits near singular points and periodic orbits. We introduce the notions of separatrix, separatrix skeleton, extended (and completed) separatrix skeleton, and canonical region that are basic ingredients for the characterization of a phase portrait.

The study of the singular points is the main objective of Chaps. 2, 3, 4, and 6, and partially of Chap. 5. In Chap. 2 we mainly study the elementary singular points, i.e., the hyperbolic and semi-hyperbolic singular points. We also provide the normal forms for such singularities providing complete proofs based on an appropriate two-dimensional approach and with full attention to the best regularity properties of the invariant curves. In Chap. 3, we provide the basic tool for studying all singularities of a differential system in the plane, this tool being based on convenient changes of variables called blow-ups. We use this technique for classifying the nilpotent singularities.

A serious problem consists in distinguishing between a focus and a center. This problem is unsolved in general, but in the case where the singular point is a linear center there are algorithms for solving it. In Chap. 4 we present the best of these algorithms currently available.

Polynomial differential systems are defined in the whole plane \mathbb{R}^2 . These systems can be extended to infinity, compactifying \mathbb{R}^2 by adding a circle, and extending analytically the flow to this boundary. This is done by the so-called “Poincaré compactification,” and also by the more general “Poincaré–Lyapunov compactification.” In both cases we get an extended analytic differential system on the closed disk. In this way, we can study the behavior of the orbits near infinity. The singular points that are on the circle at infinity are

called the infinite singular points of the initial polynomial differential system. Suitably gluing together two copies of the extended system, we get an analytic differential system on the two-dimensional sphere.

In Chap. 6 we associate an integer to every isolated singular point of a two-dimensional differential system, called its index. We prove the Poincaré–Hopf theorem for vector fields on the sphere that have finitely many singularities: the sum of the indices is 2. We also present the Poincaré formula for computing the index of an isolated singular point.

After singular points the main subjects of two-dimensional differential systems are limit cycles, i.e., periodic orbits that are isolated in the set of all periodic orbits of a differential system. In Chap. 7 we present the more basic results on limit cycles. In particular, we show that any topological configuration of limit cycles is realizable by a convenient polynomial differential system. We define the multiplicity of a limit cycle, and we study the bifurcations of limit cycles for rotated families of vector fields. We discuss structural stability, presenting a number of results and some open problems. We do not provide complete proofs but explain some steps in the exercises.

For a two-dimensional vector field the existence of a first integral completely determines its phase portrait. Since for such vector fields the notion of integrability is based on the existence of a first integral the following natural question arises: *Given a vector field on \mathbb{R}^2 , how can one determine if this vector field has a first integral?* The easiest planar vector fields having a first integral are the Hamiltonian ones. The integrable planar vector fields that are not Hamiltonian are, in general, very difficult to detect. In Chap. 8 we study the existence of first integrals for planar polynomial vector fields through the Darbouxian theory of integrability. This kind of integrability provides a link between the integrability of polynomial vector fields and the number of invariant algebraic curves that they have.

In Chap. 9 we present a computer program based on the tools introduced in the previous chapters. The program is an extension of previous work due to J. C. Artés and J. Llibre and strongly relies on ideas of F. Dumortier and the thesis of C. Herssens. Recently, P. De Maesschalck had made substantial adaptations. The program is called “Polynomial Planar Phase Portraits,” abbreviated as P4 [9]. This program is designed to draw the phase portrait of any polynomial differential system on the compactified plane obtained by Poincaré or Poincaré–Lyapunov compactification; local phase portraits, e.g., near singularities in the finite plane or at infinity, can also be obtained. Of course, there are always some computational limitations that are described in Chaps. 9 and 10. This last chapter is dedicated to illustrating the use of the program P4.

Almost all chapters end with a series of appropriate exercises and some bibliographic comments.

The program P4 is freeware and the reader may download it at will from <http://mat.uab.es/~artés/p4/p4.htm> at no cost. The program does not include either MAPLE or REDUCE, which are registered programs and must

VIII Preface

be acquired separately from P4. The authors have checked it to be bug free, but nevertheless the reader may eventually run into a problem that P4 (or the symbolic program) cannot deal with, not even by modifying the working parameters.

To end this preface we would like to thank Douglas Shafer from the University of North Carolina at Charlotte for improving the presentation, especially the use of the English language, in a previous version of the book.

Contents

1	Basic Results on the Qualitative Theory of Differential Equations	1
1.1	Vector Fields and Flows	1
1.2	Phase Portrait of a Vector Field	4
1.3	Topological Equivalence and Conjugacy	8
1.4	α - and ω -limits Sets of an Orbit	11
1.5	Local Structure of Singular Points	14
1.6	Local Structure Near Periodic Orbits	20
1.7	The Poincaré–Bendixson Theorem	24
1.8	Lyapunov Functions	30
1.9	Essential Ingredients of Phase Portraits	33
1.10	Exercises	35
1.11	Bibliographical Comments	41
2	Normal Forms and Elementary Singularities	43
2.1	Formal Normal Form Theorem	43
2.2	Attracting (Repelling) Hyperbolic Singularities	46
2.3	Hyperbolic Saddles	49
2.3.1	Analytic Results	49
2.3.2	Smooth Results	52
2.4	Topological Study of Hyperbolic Saddles	55
2.5	Semi-Hyperbolic Singularities	59
2.5.1	Analytic and Smooth Results	59
2.5.2	Topological Results	68
2.5.3	More About Center Manifolds	69
2.6	Summary on Elementary Singularities	71
2.7	Removal of Flat Terms	76
2.7.1	Generalities	76
2.7.2	Hyperbolic Case	79
2.7.3	Semi-Hyperbolic Case	81
2.8	Exercises	84
2.9	Bibliographical Comments	88

3	Desingularization of Nonelementary Singularities	91
3.1	Homogeneous Blow-Up	91
3.2	Desingularization and the Lojasiewicz Property	98
3.3	Quasihomogeneous blow-up	102
3.4	Nilpotent Singularities	107
3.4.1	Hamiltonian Like Case ($m < 2n + 1$)	109
3.4.2	Singular Like Case ($m > 2n + 1$)	110
3.4.3	Mixed Case ($m = 2n + 1$)	112
3.5	Summary on Nilpotent Singularities	116
3.6	Exercises	118
3.7	Bibliographical Comments	120
4	Centers and Lyapunov Constants	121
4.1	Introduction	121
4.2	Normal Form for Linear Centers	122
4.3	The Main Result	124
4.4	Basic Results	127
4.5	The Algorithm	133
4.5.1	A Theoretical Description	133
4.5.2	Practical Implementation	141
4.6	Applications	142
4.6.1	Known Examples	143
4.6.2	Kukles-Homogeneous Family	144
4.7	Bibliographical Comments	147
5	Poincaré and Poincaré–Lyapunov Compactification	149
5.1	Local Charts	149
5.2	Infinite Singular Points	154
5.3	Poincaré–Lyapunov Compactification	156
5.4	Bendixson Compactification	156
5.5	Global Flow of a Planar Polynomial Vector Field	157
5.6	Exercises	161
5.7	Bibliographical Comments	162
6	Indices of Planar Singular Points	165
6.1	Index of a Closed Path Around a Point	165
6.2	Deformations of Paths	168
6.3	Continuous Maps of the Closed Disk	170
6.4	Vector Fields Along the Unit Circle	170
6.5	Index of Singularities of a Vector Field	172
6.6	Vector Fields on the Sphere \mathbb{S}^2	176
6.7	Poincaré Index Formula	179
6.8	Relation Between Index and Multiplicity	181
6.9	Exercises	183
6.10	Bibliographical Comments	183

7	Limit Cycles and Structural Stability	185
7.1	Basic Results	185
7.2	Configuration of Limit Cycles and Algebraic Limit Cycles	192
7.3	Multiplicity and Stability of Limit Cycles	195
7.4	Rotated Vector Fields	196
7.5	Structural Stability	201
7.6	Exercises	204
7.7	Bibliographical Comments	210
8	Integrability and Algebraic Solutions in Polynomial Vector Fields	213
8.1	Introduction	213
8.2	First Integrals and Invariants	214
8.3	Integrating Factors	214
8.4	Invariant Algebraic Curves	215
8.5	Exponential Factors	217
8.6	The Method of Darboux	219
8.7	Some Applications of the Darboux Theory	223
8.8	Prelle–Singer and Singer Results	228
8.9	Exercises	229
8.10	Bibliographical Comments	230
9	Polynomial Planar Phase Portraits	233
9.1	The Program P4	233
9.2	Technical Overview	241
9.3	Attributes of Interface Windows	242
9.3.1	The Planar Polynomial Phase Portraits Window	242
9.3.2	The Phase Portrait Window	246
9.3.3	The Plot Orbits Window	251
9.3.4	The Parameters of Integration Window	252
9.3.5	The Greatest Common Factor Window	253
9.3.6	The Plot Separatrices Window	254
9.3.7	The Limit Cycles Window	255
9.3.8	The Print Window	256
10	Examples for Running P4	259
10.1	Some Basic Examples	259
10.2	Modifying Parameters	264
10.3	Systems with Weak Foci or Limit Cycles	273
10.4	Exercises	279
	Bibliography	285
	Index	295

List of Figures

1.1	An integral curve	2
1.2	Phase portrait of Example 1.7	7
1.3	Phase portraits of Example 1.8	7
1.4	The Flow Box Theorem	9
1.5	The arc $\{\varphi(t, p) : t \in [0, t_0]\}$	11
1.6	A limit cycle and some orbits spiralling to it	12
1.7	A subsequence converging to a point of a limit cycle	14
1.8	Sectors near a singular point	18
1.9	Curves surrounding a point p	19
1.10	Local behavior near a periodic orbit	21
1.11	Different classes of limit cycles and their Poincaré maps	22
1.12	Scheme of the section	25
1.13	Scheme of flow across the section	26
1.14	Definition of Jordan's curve	27
1.15	Impossible configurations	27
1.16	Possible configuration	27
1.17	Periodic orbit as ω -limit	28
1.18	Possible ω -limit sets	29
1.19	A singular point as ω -limit	29
1.20	A saddle-node loop	31
1.21	Figure for Exercise 1.5	36
1.22	Hint for Exercise 1.14	39
2.1	Transverse section around an attracting singular point	47
2.2	The flow on the boundary of V_0	53
2.3	A hyperbolic saddle	55
2.4	The transition close to a saddle	56
2.5	Modified transition close to a saddle	57
2.6	Comparing transitions close to two saddles	58
2.7	Flows of system (2.28)	61
2.8	Flow near a center manifold when $\lambda > 0$	64

XIV List of Figures

2.9	Flow near a center manifold when $\lambda < 0$	65
2.10	Saddle–nodes	68
2.11	Transition close to a semi-hyperbolic point	69
2.12	Phase portraits of non–degenerate singular points	72
2.13	Phase portraits of semi-hyperbolic singular points	74
3.1	Blow-up of Example 3.1	94
3.2	Local phase portrait of Example 3.1	95
3.3	Successive blowing up	97
3.4	Blowing up Example 3.2	98
3.5	Local phase portrait of Example 3.2	98
3.6	Some singularities of \bar{X} on ∂A_n	99
3.7	Samples of desingularizations of monodromic orbits	100
3.8	Blowing up a hyperbolic sector	101
3.9	Blowing up an elliptic sector	101
3.10	Blowing up of part of adjacent elliptic sectors	102
3.11	Quasihomogeneous blow-up of the cusp singularity	104
3.12	Calculating the Newton polygon	105
3.13	Desingularization of Hamiltonian like case when m is odd	110
3.14	Desingularization of Hamiltonian like case when m even	111
3.15	Blow-ups of the singular like case	111
3.16	Phase portraits of the singular like case	112
3.17	Blow-ups of the mixed case	113
3.18	Phase portraits of the mixed case	113
3.19	Phase portrait of (3.23)	115
3.20	Phase portrait of (3.24)	116
3.21	Phase portraits of nilpotent singular points	117
5.1	The local charts (U_k, ϕ_k) for $k = 1, 2, 3$ of the Poincaré sphere	151
5.2	A hyperbolic or semi-hyperbolic saddle on the equator of \mathbb{S}^2	155
5.3	Saddle-nodes of type SN1 and SN2 of $p(X)$ in the equator of \mathbb{S}^2	155
5.4	The phase portrait in the Poincaré disk of system (5.11)	158
5.5	The phase portrait in the Poincaré disk of system (5.12)	159
5.6	The phase portrait in the Poincaré disk of system (5.13)	160
5.7	Compactification of system (5.15)	161
5.8	Compactification of system (5.16)	161
5.9	Phase portraits of Exercise 5.2	162
5.10	Phase portraits of quadratic homogenous systems	163
6.1	Same image, different paths	166
6.2	Definition of $\varphi(t)$	166
6.3	Some examples of indices	167
6.4	Examples of homotopic closed paths	169
6.5	Closed path associated to a vector field	171
6.6	Point of index -1	174

6.7	Point of index 2	175
6.8	Piecing of \mathbb{D}^2	175
6.9	Stereographic projection for example 6.29	177
6.10	Phase portraits for example 6.29	177
6.11	The vector fields X' and X'' on \mathbb{D}^2	178
6.12	Index given by sectors	181
7.1	A limit cycle with a node inside	204
7.2	A limit cycle surrounding a saddle, two antisaddles, and two limit cycles in different nests	204
9.1	Representation of the Poincaré–Lyapunov disk of degree (α, β)	241
9.2	The <i>Planar Polynomial Phase Portraits</i> window	242
9.3	The <i>Main settings</i> window	243
9.4	The <i>Output</i> window	245
9.5	The <i>Poincaré Disc</i> window	247
9.6	The <i>Legend</i> window	248
9.7	The <i>View Parameters</i> window	249
9.8	A planar plot	250
9.9	The <i>Orbits</i> window	251
9.10	The <i>Parameter of Integration</i> window	252
9.11	The <i>Greatest Common Factor</i> window	253
9.12	The <i>Plot Separatrices</i> window	254
9.13	The <i>Limit Cycles</i> window	255
9.14	The <i>LC Progress</i> window	256
9.15	The <i>Print</i> window	257
10.1	The end of the calculations	260
10.2	Stable and unstable separatrices of system (10.1)	260
10.3	Some more orbits of system (10.1)	261
10.4	Stable and unstable separatrices of system (10.2)	262
10.5	Stable and unstable separatrices of system (10.3) for $a = 1$ and $l = -0.5$	263
10.6	Separatrix skeleton of the system (10.3) for $a = 1$ and $l = 0$	264
10.7	Stable and unstable separatrices of system (10.4)	265
10.8	Separatrix skeleton of the system (10.4)	266
10.9	<i>Epsilon</i> value too great	267
10.10	Good <i>epsilon</i> value	267
10.11	Phase portrait of system (10.5)	267
10.12	Stable and unstable separatrices of system (10.6) with $d = 0.1$ and $a = b = 0$	269
10.13	Stepping too fast	269
10.14	Separatrix skeleton of the system (10.6) with $d = 0.1$ and $a = b = 0$	270

XVI List of Figures

10.15 Stable and unstable separatrices of system (10.6) with $d = a = b = 0.1$	271
10.16 Stable and unstable separatrices of system (10.6) with $d = 0$...	271
10.17 Stable and unstable separatrices of system (10.7)	272
10.18 Portrait in the reduced mode of system (10.7)	273
10.19 Phase portrait of system (10.8) with $a = b = l = n = v = 1$	275
10.20 Phase portrait of system (10.9)	276
10.21 One orbit inside the limit cycle	276
10.22 The limit cycle	277
10.23 Phase portrait of system (10.10) with given conditions	278
10.24 Outer limit cycle	278
10.25 Looking for more limit cycles	279
10.26 When limit cycles are hard to find	279
10.27 Exercise 10.16	284