Combinatorial dynamics and entropy in dimension one

Lluís Alsedà

Departament de Matemàtiques,
Universitat Autònoma de Barcelona,
08193–Bellaterra,
Barcelona
SPAIN

Jaume Llibre

Departament de Matemàtiques,
Universitat Autònoma de Barcelona,
08193–Bellaterra,
Barcelona
SPAIN

Michał Misiurewicz

Institute of Mathematics,
Warsaw University,
Banacha 2,
02–097 Warszawa
POLAND

The wind goeth toward the south, and turneth about unto the north; it whirleth about continually, and the wind returneth again according to his circuits.

[Ecclesiastes 1.6]

Preface

In middle 1970's, many scientists started to investigate one dimensional dynamical systems. They were mathematicians interested in a pure theory and physicists, biologists, etc. who sought models applicable for their problems. The reasons of popularity of these systems were complex. The theory of dynamical systems came to a point where most of problems considered at that time as important were either solved or turned out to be very difficult. The study of the simplest systems with complex behavior promised new problems, easier to solve and shedding light on similar problems for more general (many dimensional or even infinite dimensional) systems. Such simplest systems with "sufficiently complex" behavior are flows (differential equations) in three dimensions, invertible maps (homeomorphisms, diffeomorphisms) in two dimensions and non-invertible maps in one dimension. Various notions of chaotic behavior started to form and interval maps proved to be sufficiently complex to display most kinds of chaos. The paper of Li and Yorke [218] "Period three implies chaos" helped very much to make those ideas popular. The simplicity of tools necessary to study these systems attracted mathematicians (and physicists) from many areas.

On the other hand, in the natural sciences ideas of reducing dimension of mathematical models were developed. Although the full model may be a dynamical system in many dimensions, even infinite, its "most interesting" part can have much less dimension. Moreover, if we pass from a flow to a non-invertible map, we can reduce this dimension even more, in many cases to one. This procedure was used successfully in order to reduce a model of atmospheric behavior to the Lorenz family of flows in three dimensions and then to a class of maps in one dimension. Another application of this procedure resulted in an explanation of the period doubling phenomena for higher dimensional systems via the Feigenbaum universality theory for one dimensional systems.

Investigating one dimensional models with computers has two advantages compared to many dimensional models. The first advantage is that it is quicker. The second advantage is its visual image: if you want to draw a two dimensional picture, you can use one axis for the space and save another one for a parameter.

The theory of one dimensional dynamical systems has grown out in many directions. One of them has its roots in the Sharkovskiĭ Theorem. This beautiful theorem describes the possible sets of periods of all cycles (periodic orbits) of a continuous map of an interval (or the real line) into itself. They are given by a certain linear order in the natural numbers; to get such a set one "cuts" this order in an appropriate place and takes the right-hand part. All proofs of this theorem inevitably lead to an idea of a "type" of a cycle. When ordering the points of a cycle $p_1 < p_2 < \ldots < p_n$, one gets a cyclic permutation σ corresponding to it, such that p_i is mapped to $p_{\sigma(i)}$. Immediately a problem arises: what happens if we try to consider cyclic permutations instead of periods? Then the usual thing happens, the more problems you solve, the more new ones arise. The whole theory which was developed basing on these ideas, deals mainly with combinatorial objects; permutations, graphs etc. We decided to call it *combinatorial dynamics*.

It is important to be able to measure the complexity of a system, or the degree of "chaos" present in it. To some extent this can be done by counting cycles of various periods or various types. A better way to do it is to compute the topological entropy of a system. Topological entropy is considered in almost all papers on combinatorial dynamics. There are also other specific problems in the entropy theory.

We decided that the development of combinatorial dynamics and entropy theory for one dimensional dynamical systems had gone far enough to try to write a book on them. The subject was never treated in a unified way. We found many gaps in the theory. We found results belonging to so called "folk knowledge": everybody interested in the subject knows that they are true, but nobody can point out where to find them in the literature. In many cases the techniques existing now allow one to prove theorems in a simpler way than they were proved originally.

Of course, we could not try to include in the book everything that has been done in the subject. The theory is being developed all the time and we could have never stopped writing. Also we had to exclude some relevant and interesting subjects, again in order to be able to finish the book in a finite time. However, we would like the reader to be aware of many connections of the material presented in the book with other problems. We describe briefly these connections using

> ♣ small print (we also refer to it as to small print) which we mark like this. ♣

We treat each small print as a unit. If there are several small prints in a row, we nevertheless mark them separately. We also mark the ends of the proofs by ■ and the ends of examples and remarks by □ .

As yet, there is no standard terminology in combinatorial dynamics. Even such fundamental object as a *cycle* (= *periodic orbit*) has two different names (if we do not count the name *circuit* used by Ecclesiastes). Sometimes we had to choose between existing names, and sometimes we had to introduce new ones consistent with the rest of our terminology. This has to be taken into account when reading this book and relevant papers.

We tried to find all literature on the subject covered by the book (this does not include small prints; we only chose a few positions for each small print). We are aware that probably we had overlooked some papers; even though we tried, finding them all seemed to be impossible. In certain cases we also omitted some papers consciously. This applies to the short versions of papers whose full versions can be found as easily as the short ones. We also had to stop adding new references at some point; therefore the very recent ones are missing. An even bigger problem was to give credit to various authors. At the end of every section we include a part called "Historical remarks", where we try to indicate who proved first the results of this section and whose ideas we follow in it. We want to apologize to every author to whom we did not give enough credit. We know that we are not good historians. We also want to apologize that our names appear more often than they should, but after all, when writing the book we had to follow mainly our own ideas and we were writing the things we knew the best.

We address the book to the general mathematical audience. We tried to keep the book on the elementary level. We hope that it is available even to students after some basic courses in mathematics.

In many proofs it is very important to keep track of relative position of the points under consideration and to see where they are mapped. Sometimes we provided appropriate (we hope) figures. If there are no such figures, we advise the reader to make his/her own ones when reading the proofs.

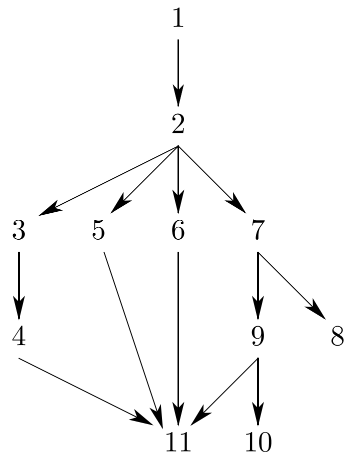
We would like to explain the main interdependences between various sections of this book. The book is divided into four chapters: Preliminaries, Interval maps, Circle maps and Entropy. The material from the Preliminaries Chapter is extensively used in the Interval and Circle Chapters.

The chapter on interval maps is organized essentially in a linear way: each section uses the previous ones.

The situation for the chapter on circle maps is more complicated. The easiest way to explain the relations between Sections 1 – 11 of this chapter is by means of the diagram shown in the figure on the next page. Moreover, the results of Section 2.1 are used in Sections 6, 7 and 9 and the results of Section 2.2 in Sections 6, 7 and 10 of the Circle chapter. This chapter contains also an appendix, which should be treated as a large small print. To understand it, one needs the whole theory developed in the Interval and Circle chapters.

The chapter on entropy is organized again essentially in a linear way, except that Sections 4 and 5 are independent on each other. The whole Interval chapter intervenes in Section 4 and the whole Circle chapter in Sections 7 and 8.

A reader interested in books presenting different approaches to one-



The relations between Sections 1 – 11 of the Circle chapter.

dimensional dynamics may wish to read for instance [60], [107], [118], [269], [274], [275], [289], [290] or [314].

We want to express our gratitude to Zbigniew Nitecki and Karen Brucks for careful reading of the manuscript and many useful remarks.

We feel indebted to Universitat Autònoma de Barcelona, Centre de Recerca Matemàtica, Warsaw University, Northwestern University and to the organizers of European Conferences on Iteration Theory '87 and '89 and of the Third Czechoslovak Summer School on Dynamical Systems for creating opportunities for us to meet and work together.

We were partially supported by the DGICYT grants numbered PB86–0351 and PB90–0695, and by the European Communities grant number 86400432DEPUJU1.

Contents