1 Introduction to the PWL-ML model

The slow passage phenomenon appears in dynamical systems with multiple timescales, also referred to as slow-fast systems, and it marks the effect of the slow dynamics on bifurcations of the so-called fast subsystems, obtained when the dynamics of the slow variables are frozen and the slow variables are considered as parameters. Multiple timescales are ubiquitous in Neuroscience, therefore slow-fast systems are often used to capture dynamical behaviors of neural activity, for instance spiking or bursting oscillations. One example is the so-called Morris-Lecar model, which is a two-dimensional excitability biological model first introduced in the context of muscle fiber cells, and regularly used as a simple yet biophysical model of neural spike generation. The model’s equations take the form

\[
\begin{align*}
\dot{x} &= f(x) - y + I, \\
\dot{y} &= g(x) - y,
\end{align*}
\]  

(1)

where \( f \) and \( g \) are nonlinear functions whose graphs are cubic and sigmoid shaped, respectively; see Fig. 1 below. The ML model is quite used to reproduce some neural behaviors. In particular, it is widely studied from a mathematical point of view. Nevertheless, understanding the influence of the slow passage into this model requires a very extensive treatment.

2 Bifurcation sets of the PWL-ML

We consider the following piecewise linear caricature of the nonlinear functions of the Morris-Lecar model (PWL-ML) (1):

\[
\begin{align*}
f(x) &= \begin{cases} 
-\alpha & \text{if } x < -\sqrt{\epsilon}, \\
+\alpha & \text{if } x > \sqrt{\epsilon}, \\
0 & \text{if } -\sqrt{\epsilon} \leq x \leq \sqrt{\epsilon},
\end{cases} \\
g(x) &= \begin{cases} 
0 & \text{if } x < a, \\
2 & \text{if } x = a, \\
1 & \text{if } x > a.
\end{cases}
\]

(2)

Parameters \((a, \beta, \gamma)\) are given in terms of the constants \(k, \lambda\) and the parameters \((a, \varepsilon, I)\) in order to provide continuous nullclines. Moreover, the parameters are considered to be located in \(\mathbb{P} = \{a, \varepsilon, I \in \mathbb{R} : 0 < \varepsilon < k, 0 \leq a \leq 1\}\). We define the following subsets of \(\mathbb{P}\):

\[
\begin{align*}
SN_1 &= \{(a, \varepsilon, I) \in \mathbb{P} : \varepsilon < (a - 1)I < \sqrt{\varepsilon(1 + 2\delta)}\}, \\
SN_2 &= \{(a, \varepsilon, I) \in \mathbb{P} : \varepsilon(1 - (a - 1)\delta) < I < \varepsilon(1 + 2\delta)\}, \\
E_1 &= \{(a, \varepsilon, I) \in \mathbb{P} : \varepsilon(1 + 2\delta) = \varepsilon(1 - (a - 1)\delta)\}.
\end{align*}
\]

Theorem. The surfaces \(SN_1\) and \(SN_2\) are saddle-node bifurcation manifolds intersecting along the cusp-like bifurcation curve \(C\) and splitting the parameter set \(\mathbb{P}\) into two disconnected regions \(E_1\), where the PWL-ML system exhibits a unique equilibrium point, and \(E_2\), where it exhibits three homoclinic orbits. A sketch of these homoclinics is shown in Fig. 2.

3 Slow Passage through the homoclinic orbit

Since there exists a homoclinic orbit, we can study the slow passage phenomenon through a homoclinic loop by considering I as a slow drift. Hence, we study the following 3D extension of System (1):

\[
\begin{align*}
\dot{x} &= f(x) - y + I, \\
\dot{y} &= g(x) - y, \\
\dot{I} &= \sigma,
\end{align*}
\]

(3)

for \(|\varepsilon| \ll 1\). Hence, it is a slow-fast system, with \((x, y, I)\) the fast variables and \(I\) the slow one. For \(\sigma = 0\), the fast subsystem corresponds to System (1). We choose parameters in order to remain in region \(E_2\) where there exist: a saddle point; a saddle point, located on the repelling part of the critical manifold \(S_0^I\); with its stable \(W^s(S_0^I)\) and unstable \(W^u(S_0^I)\) branches; and an unstable one. Following Fenichel’s theorem, as we take \(\sigma \neq 0\), the family of equilibrium points perturbs into slow manifolds. In particular, a saddle-type slow manifold \(S^I_2\) appears at an \(\varepsilon\) distance from the saddle branch \(S^I_0\), having one stable manifold \(W^K(S^I_0)\) and one unstable manifold \(W^K(S^I_2)\).

The slow passage phenomenon through a homoclinic loop consists by the appearance of a finitely many number of paths \(W^K(S^I_0) \cap W^K(S^I_2)\) perturbed from the unique intersection \(W^K(S^I_0) \cap W^K(S^I_2)\), which is the homoclinic loop.

4 Conclusions

In this work, we study the effect of the slow passage through a homoclinic orbit. In this case, we apply a regulation technique of the vector field in order to get a homoclinic connection. The phenomenon transforms a homoclinic connection between two branches of a slow manifold into a finitely many maximal canard connections between them.

This behavior can help to understand the spike-adding phenomenon in bursting models. We apply a regulation technique of the vector field in order to get a homoclinic connection.

Therefore, starting over the discontinuous PWL-ML vector field, we find a homoclinic orbit and we extend it to the continuous one. This provides a surface of homoclinic orbits on the parameter space.