Slow passage through a homoclinic loop in a piecewise linear version of the Morris-Lecar model

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1 Introduction to the PWL-ML model

The **slow passage phenomenon** appears in dynamical systems with multiple timesacles, also referred to as *slow-fast systems*, and it marks the effect of the slow dynamics on bifurcations of the so-called *fast subsystem*, obtained when the dynamics of the slow variables are frozen and the slow variables considered as parameters. Multiple timescales are ubiquitous in Neuroscience, therefore slow-fast systems are often used to capture dynamical behaviors of neural activity, for instance spiking or bursting oscillations. One example is the so-called *Morris-Lecar model* (ML), which is a two-dimensional excitable biophysical model first introduced in the context of muscle fiber cells, and regularly used as a simple yet biophysical model of neural spike generation[1]. The model's equations take the form

$$\begin{cases} \varepsilon\dot{x} = f(x) - y + I, \\ \dot{y} = g(x) - y. \end{cases} \tag{1}$$

where f and g are nonlinear functions whose graphs are cubic and sigmoid shaped, respectively; see **Fig. 1** below. The ML model is quite used to reproduce some neural behaviors. In particular, it is widely studied from a mathematical point of view. Nevertheless, understanding the influence of the slow passage into this model requires a very extensive treatment.

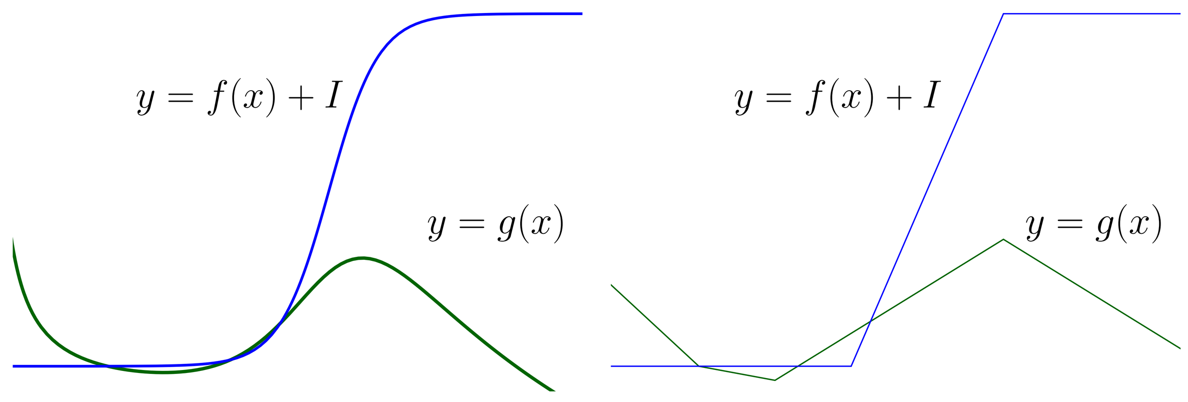

Fig. 1: Nullclines of the ML Model. On the left, the original ML model; on the right, the proposed PWL version, which captures the essentials of the geometry of the original model.

We propose a *Piecewise Linear* (PWL) simplification of the Morris-Lecar model. PWL models are well known to reproduce qualitative and quantitative aspects of smooth systems, while being more amenable to analysis. Hence, we study the slow passage phenomenon in the context of this PWL model; in particular, via a homoclinic bifurcation similar to the one that appears in the ML model.

2 Bifurcation sets of the PWL-ML

We consider the following piecewise linear caricature of the nonlinear functions of the Morris-Lecar model (PWL-ML) (1):

$$f(x) = \begin{cases} -x & \text{if } x < -\sqrt{\varepsilon}, \\ \delta x + \lambda & \text{if } |x| \leq \sqrt{\varepsilon}, \\ kx + \beta & \text{if } \sqrt{\varepsilon} < x < 1, \\ -x + \gamma & \text{if } x \geq 1, \end{cases} \quad g(x) = \begin{cases} 0 & \text{if } x < a, \\ lx + n & \text{if } a \leq x < 1, \\ 1 & \text{if } x \geq 1. \end{cases} \tag{2}$$

Parameters $\{\lambda, \beta, \gamma, l, n\}$ are given in terms of the constants δ, k and the parameters (a, ε, I) in order to provide continuous nullclines. Moreover, the parameters are considered to be located in $\mathcal{P} = \{(a, \varepsilon, I) \in \mathbb{R}^3 : 0 < \varepsilon < k, \sqrt{\varepsilon} \leq a \leq 1\}$. We define the following subsets of $\mathcal{P}$:

$$\begin{aligned} \mathcal{SN}_1 &= \{(a, \varepsilon, I) \in \mathcal{P} : \sqrt{\varepsilon} < a < 1, I = -\sqrt{\varepsilon}(1 + 2\delta)\}, \\ \mathcal{SN}_2 &= \{(a, \varepsilon, I) \in \mathcal{P} : \sqrt{\varepsilon} < a < 1, I = -(ak + \beta)\}, \\ \mathcal{C} &= \{(a, \varepsilon, I) \in \mathcal{P} : a = \sqrt{\varepsilon}, I = -(ak + \beta)\}, \\ \mathcal{E}_3 &= \{(a, \varepsilon, I) \in \mathcal{P} : \sqrt{\varepsilon} < a < 1, -(ak + \beta) < I < -\sqrt{\varepsilon}(1 + 2\delta)\}, \\ \mathcal{E}_1 &= \mathcal{P} \setminus (\mathcal{SN}_1 \cup \mathcal{E}_3 \cup \mathcal{SN}_2 \cup \mathcal{C}). \end{aligned}$$

Theorem. *The surfaces $\mathcal{SN}_1$ and $\mathcal{SN}_2$ are saddle-node bifurcation manifolds intersecting along the cusp-like bifurcation curve $\mathcal{C}$ and splitting the parameter set $\mathcal{P}$ into two disconnected regions $\mathcal{E}_1$, where the PWL-ML system exhibits a unique equilibrium point, and $\mathcal{E}_3$ where it exhibits three hyperbolic equilibrium points: one attracting, one saddle and one repelling.*

We look for conditions to the existence of a *homoclinic bifurcation* to a saddle point in $\mathcal{P}$.

Regularization Technique. We notice that, moving the parameter a, it is possible to obtain:

- A continuous vector field, when $a < 1$ or;
- A discontinuous vector field, when $a = 1$.

This process can be understood as a regulation of the discontinuous vector field, allowing to extend the existence of global dynamical objects from the discontinuous context to the continuous one.

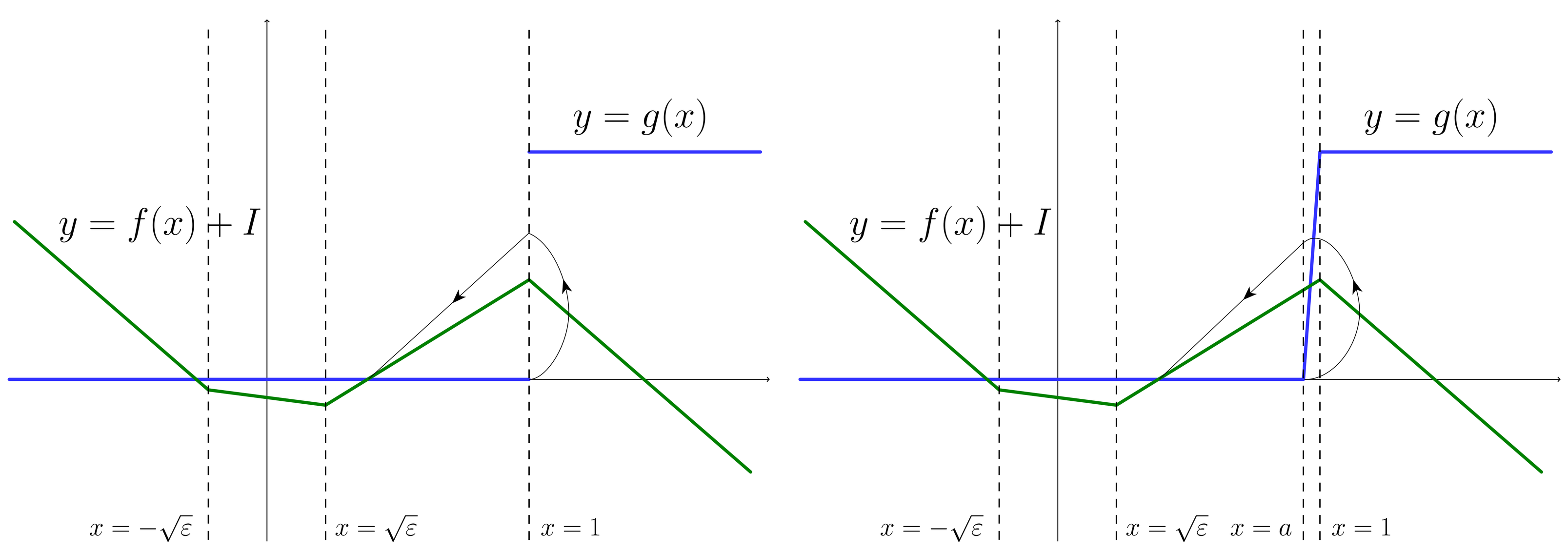

Fig. 2: Representation of the nullclines. On the left, we depict the discontinuous case ($a = 1$); on the right, the continuous case ($a < 1$). We plot a sketch of a homoclinic orbit and its regulation.

Therefore, starting over the discontinuous PWL vector field, we find a homoclinic orbit and we extend it to the continuous one. This provides a surface of homoclinic orbits on the parameter space.

Theorem. *There exist values $\varepsilon_0 \in (0, k)$, $a_0 \in (\sqrt{\varepsilon_0}, 1)$ and a function $H \in \mathcal{C}^1$ defined over $W := [\sqrt{\varepsilon}, 1] \times [0, \varepsilon_0)$ such that, for $(a, \varepsilon) \in W$, System (1) exhibits a homoclinic orbit to a saddle point at $I = H(a, \varepsilon)$, providing a homoclinic orbit manifold $\mathcal{H}_0 = \{(a, \varepsilon, H(a, \varepsilon)) : (a, \varepsilon) \in W\}$.*

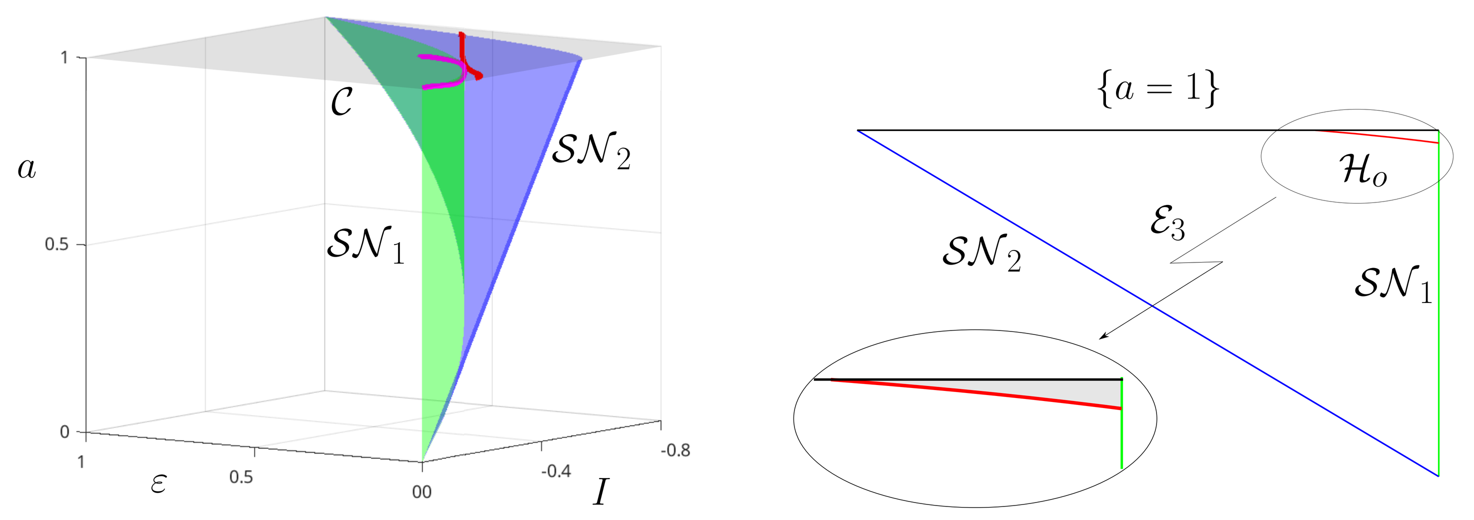

Figure 3: Bifurcation diagram in Region $\mathcal{P}$ and a section given by $\{\varepsilon = 1/3\}$.

3 Slow Passage through the homoclinic orbit

Since there exists a homoclinic orbit, we can study the slow passage phenomenon through a homoclinic loop by considering I as a *slow drift*. Hence, we study the following 3D extension of System (1):

$$\begin{cases} \varepsilon\dot{x} = f(x) - y + I, \\ \dot{y} = g(x) - y, \\ \dot{I} = \sigma, \end{cases} \tag{3}$$

for $|\sigma| \ll 1$. Hence, it is a slow-fast system, with (x, y) the fast variables and I the slow one. For $\sigma = 0$, the fast subsystem corresponds to System (1). We choose parameters in order to remain in region $\mathcal{E}_3$, where there coexist: a stable point; a saddle point, located on the repelling part of the *critical manifold* $\mathcal{S}_0^{sd}$, with its *stable* $W^s(\mathcal{S}_0^{sd})$ and *unstable* $W^u(\mathcal{S}_0^{sd})$ branches; and an unstable one.

Following Fenichel's theorems[2], as we take $\sigma \neq 0$, the family of equilibrium points perturbs into slow manifolds. In particular, a saddle-type *slow manifold* $\mathcal{S}_\sigma^{sd}$ appears at an $O(\sigma)$ distance from the saddle branch $\mathcal{S}_0^{sd}$, having one stable manifold $W^s(\mathcal{S}_\sigma^{sd})$ and one unstable manifold $W^u(\mathcal{S}_\sigma^{sd})$.

The slow passage phenomenon through a homoclinic loop consists by the appearence of a finitely many number of paths $W^u(\mathcal{S}_\sigma^{sd}) \cap W^s(\mathcal{S}_\sigma^{sd})$ perturbed from the unique intersection $W^u(\mathcal{S}_0^{sd}) \cap W^u(\mathcal{S}_0^{sd})$, which is the homoclinic loop.

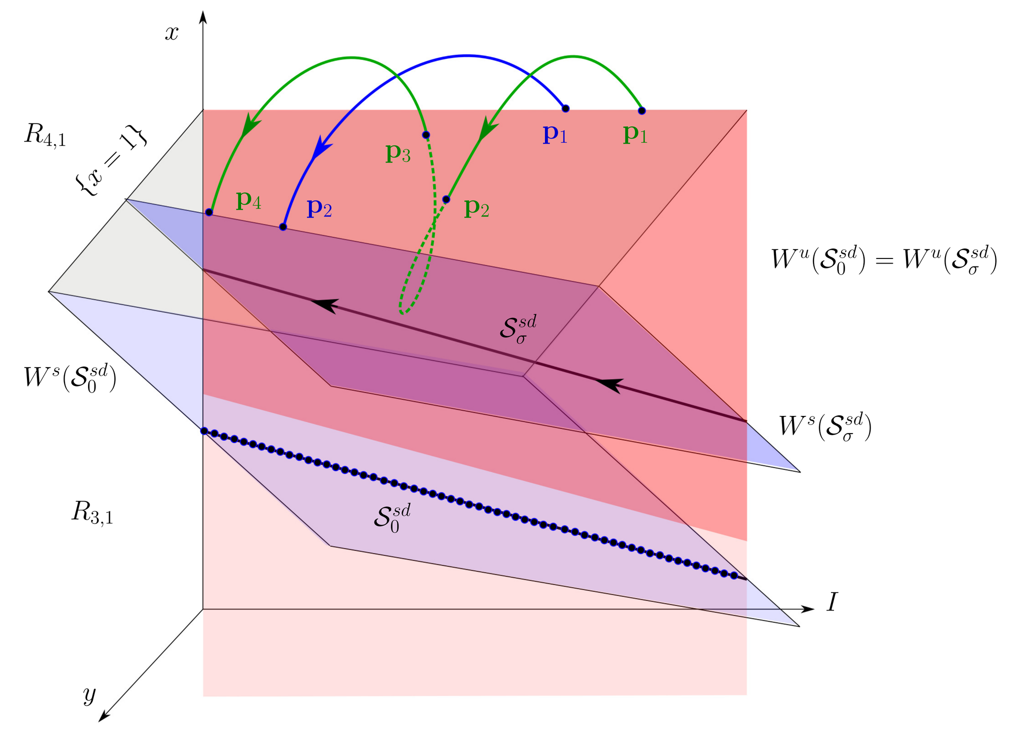

Figure 4. Sketch of the invariant planes and maximal canards.

These intersections provide a finite number of maximal canard orbits $N(\sigma)$, all of them perturbing from the homoclinic orbit. The existence of these orbits is quite difficult to prove, so we only provide the existence of maximal canard orbits related to 1 or 2 spikes, and give complementary numerical evidences.

4 Conclusions

In this work, we study the effect of the slow passage through a homoclinic orbit. In this case,

- We apply a regulation technique of the vector field in order to get a homoclinic connection.
- The phenomenon transforms a homoclinic connection between two branches of a slow manifold into a finitely many maximal canard connections between them.
- This behavior can help to understand the spike-adding phenomenon in bursting models.
- It remains to prove the existence of maximal canard orbits generally in a neighborhood of $\sigma = 0$.

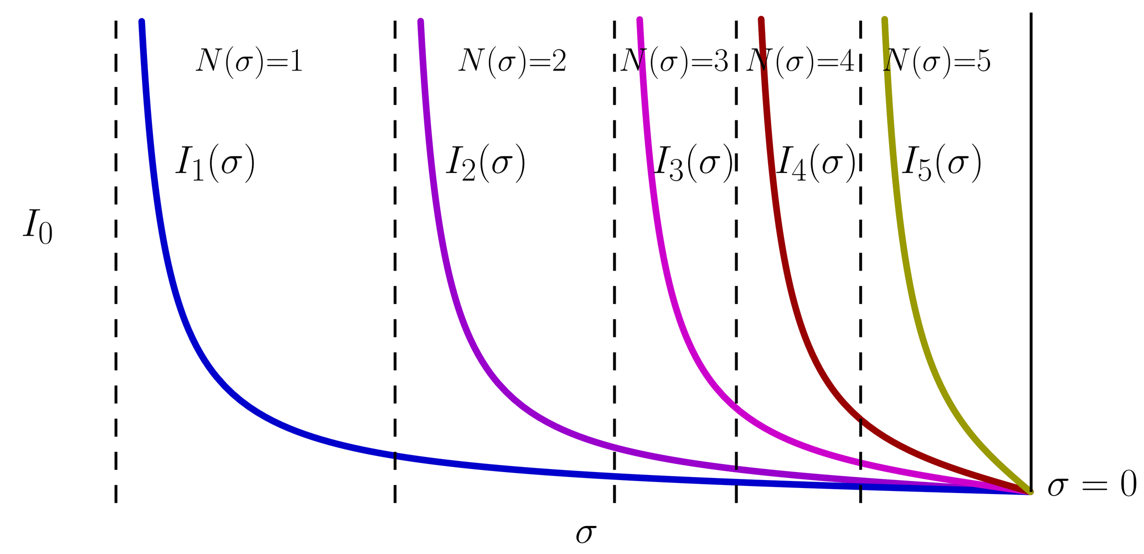

Figure 5: Sketch of the function $N(\sigma)$.

[1] C. Morris and H. Lecar, Biophysical Journal **35**(1): 193-213, 1981.
[2] N. Fenichel, Journal of Differential Equations **31**(1): 53-98, 1979.