Bifurcation diagram and global phase portraits of the quadratic vector fields of Class I

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Introduction

Ye Yanquian in (sec. 12 [Yanquian, 1984]) proved that all quadratic systems that may have a periodic orbit can be transformed by changes of variables to one of the following systems

$$\begin{array}{ll} \textit{Class I} & \begin{cases} \dot{x} = y \\ \dot{y} = -x + f\,y + l\,x^2 + m\,xy + n\,y^2 \end{cases} \\ \textit{Class II} & \begin{cases} \dot{x} = y(1 + a\,x) \\ \dot{y} = -x + f\,y + l\,x^2 + m\,xy + n\,y^2 \end{cases} \\ \textit{Class III} & \begin{cases} \dot{x} = y(1 + b\,x + c\,y) \\ \dot{y} = -x + f\,y + l\,x^2 + m\,xy + n\,y^2 \end{cases} \end{array} \tag{1}$$

Class I has four parameters, Class II five, and Class III six. Geometrically, Class III is generic, in Class II one finite singularity has escaped to infinity and in Class I, two finite singularities have escaped. Most works are thus done on Class I.

The paper [Perko, 1992] studies the case $n = 0$, and [Chen, 2020] the case $m = 0$. The PhD [Coll, 1987] tries to make a complete study but does not reach it even it gets very close.

Construction of the parameter space

We compactify our parameter space by considering that

$$\begin{aligned} \mathbb{R}^4 \backslash \{(0,\,0,\,0,\,0)\} &\longrightarrow \mathbb{RP}^3 = \mathbb{S}^3 / \sim \\ (f,\, l,\, m,\, n) &\longmapsto [f : l : m : n] \end{aligned}$$

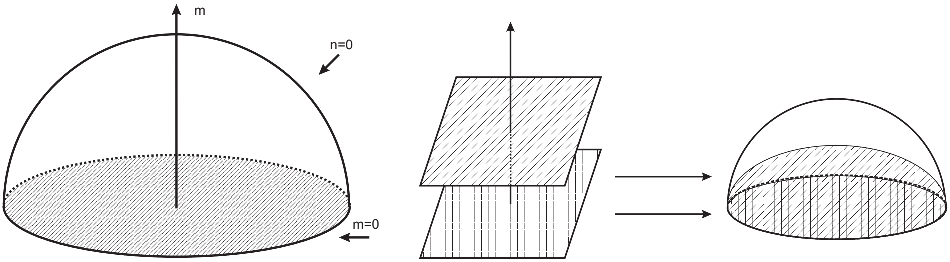
and given the symmetry $(x, y, t, f, l, m, n) \to (-x, y, -t, -f, l, -m, n)$ we only need to study $m \geq 0$.

When $n \neq 0$, we see the parameter space as the half-ball $\mathcal{B}_{1/2} = \{(f, l, m) \in \mathbb{R}^3 : f^2 + l^2 + m^2 < 1 \text{ i } n > 0\}$ with base $m = 0$. We have the bijection

$$\begin{aligned} \mathbb{RP}^3 \backslash \{n \neq 0\} &\longrightarrow \mathbb{R}^3 \\ [f : l : m : n] &\longrightarrow \left(\frac{f}{n}, \frac{l}{n}, \frac{m}{n} \right) \end{aligned}$$

Construction of the parameter space

When $n = 0$ we get the equation $l^2 + f^2 + m^2 = 1$, which corresponds to the border of the half-ball; i.e., $\mathbb{S}^2$. When $m = n = 0$ we are in the equator of the half-ball.



We have already used this compactification for reducing 4-parameter systems to 3-parameter ones in the past, and this has allowed us to study large families of codimension 2 quadratic systems.

Computing the bifurcation diagram

In the book [**?**] we present a set of invariants which allow to compute the algebraic bifurcation surfaces which have some geometrical meaning, in any normal form. This way we skip the classical way which need simple normal forms and computing singularities and eigenvalues. Every geometrical feature related with singularities, as well as invariant lines has its associated invariant.

Computing the bifurcation diagram

For this family the needed invariants are:

$\mathcal{S}1$: another finite singularity escapes to infinity (drawn in blue),

$$\mu_2 : \; l = 0;$$

$\mathcal{S}3$: there is a weak finite singularity, a focus or a saddle (drawn in yellow),

$$\mathcal{T}4 : \; 4f(fl + 2m)n(-m^2 + ln) = 0;$$

$\mathcal{S}4$: possible existence of an invariant straight line (drawn in magenta),

$$B_1 : -648(l^2 + 2flm + 4m^2 - 2ln + f^2 ln + 2fmn + n^2)$$
$$(l^2 + 2flm + 2ln + f^2 ln + 2fmn + n^2) = 0;$$

$\mathcal{S}5$: two infinite singularities coalesce (drawn in red),

$$\eta : \; 4n^2(m^2 - ln) = 0;$$

$\mathcal{S}6$: focus-node bifurcation (drawn in black),

$$W_4 : \; 16(-2 + f)(2 + f)(4l^2 + f^2 l^2 + 4flm + 4m^2)n^2(-m^2 + ln)^2 = 0.$$

Computing the bifurcation diagram

Now we need to compute all the singularities of these surfaces and all their intersections. This produces a list of curves and we need to compute if these curves intersect each other or not, and at which value of m. Those values of $m = m_i$ obtained are critical values which mean that the bifurcation diagram in slice $m = m_i$ will be possibly different from that on $m = m_i \pm \varepsilon$. These will become the singular slices. All other will be generic.
Also if some surface is tangent to the plane $m = m^*$, this slice will also be singular.
For doing that we use a Mathematica program prepared by us since the number of curves can be of several hundreds.

Computing the bifurcation diagram

Wolfram Mathematica 9.0 - [Chen-marcos.nb *]

File Edit Insert Format Cell Graphics Evaluation Palettes Window Help

overleaf.com/project/62eada064cef534350e538e7

Chen-marcos.nb *

```
km = 5; j = ListaSlices[[10]]; Print[j];
implicitplot[
 {WW4, 4 cn^2 (cm^2 - cl cn), 192 cl^2 (-cm^2 + cl cn), cl , 4 cf (cf cl + 2 cm) cn (-cm^2 + cl cn),
   -648 (cl^2 + 2 cf cl cm + 4 cm^2 - 2 cl cn + cf^2 cl cn + 2 cf cm cn + cn^2) (cl^2 + 2 cf cl cm + 2 cl cn + cf^2 cl cn + 2 cf cm cn + cn^2)} /. {cn → 1, cm → j}, cf, cl, -km, km, -7, km,
 1000]

2
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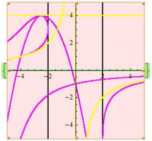



```
Solve[{4 cf (cf cl + 2 cm) cn (-cm^2 + cl cn) == 0, -648 (cl^2 + 2 cf cl cm + 4 cm^2 - 2 cl cn + cf^2 cl cn + 2 cf cm cn + cn^2) (cl^2 + 2 cf cl cm + 2 cl cn + cf^2 cl cn + 2 cf cm cn + cn^2) == 0} /. {cn → 1, cm → 2}]

{{cf → -4, cl → 1}, {cf → -5/2, cl → 4}, {cf → -4/3, cl → 3}, {cf → 0, cl → -1}, {cf → 0, cl → 1 - 4 i}, {cf → 0, cl → 1 + 4 i}, {cf → 4/5, cl → -5}}

Solve[{-648 (cl^2 + 2 cf cl cm + 4 cm^2 - 2 cl cn + cf^2 cl cn + 2 cf cm cn + cn^2) (cl^2 + 2 cf cl cm + 2 cl cn + cf^2 cl cn + 2 cf cm cn + cn^2) == 0} /. {cn → 1, cm → 2, cf → 2}]

{{cl → -5}, {cl → -5}, {cl → -7 - 2 √10}, {cl → -7 + 2 √10}}

N[%]

{{cl → -5.}, {cl → -5.}, {cl → -13.3246}, {cl → -0.675445}}

Solve[{4 cf (cf cl + 2 cm) cn (-cm^2 + cl cn) == 0} /. {cn → 1, cm → 2, cf → 3}]

{{cl → -4/3}, {cl → 4}}

N[((-19 - 7 * Sqrt[5]) / 2 + (-23 - 3 * Sqrt[53]) / 2) / 2]

-19.8732
```

Computing the bifurcation diagram

Apart from the algebraic surfaces, there may be other non-algebraic which are related with separatrix connections or double limit cycles. The only way to detect them is to look for coherence in the phase portraits in different points of a same region in a slice. If there is lack of coherence, then one or several non-algebraic surfaces must be added. It is not so difficult. Normally they are close to regions with limit cycles.

And these non-algebraic surfaces will for sure extend beyond the starting slice, so they must be restudied in every slice. If we detect some difference between two slices, another critical value $m = m^*$ must be added. This m^* must normally be computed numerically by approximation, even in this family we are lucky and the one we have is algebraic.

We will call $\mathcal{S}7$ to the union of all surfaces related to separatrix connections (which are not invariant lines) and will drawn it also in magenta.

In this family we can not have double limit cycles.

Bifurcation diagram

The slices we need are

1. $m_0 = 0$ singular (studied by [Chen, 2020]);
2. $m_1 = 1/8$ generic;
3. $m_2 = 1/4$ singular;
4. $m_3 = 2/7$ generic;
5. $m_4 = 1/3$ singular;
6. $m_5 = 2/5$ generic;
7. $m_6 = 1/2$ singular;
8. $m_7 = 2/3$ generic;
9. $m_8 = 1$ singular;
10. $m_9 = 3/2$ generic;
11. $m_{10} = 2$ singular (for non algebraic reasons);
12. $m_{11} = 3$ generic;
13. $m_{12} = 4$ singular;
14. $m_{13} = 5$ generic;
15. $m_{14} = \infty$ ($\equiv n = 0$) singular (studied by [Perko, 1992]).

Bifurcation diagram

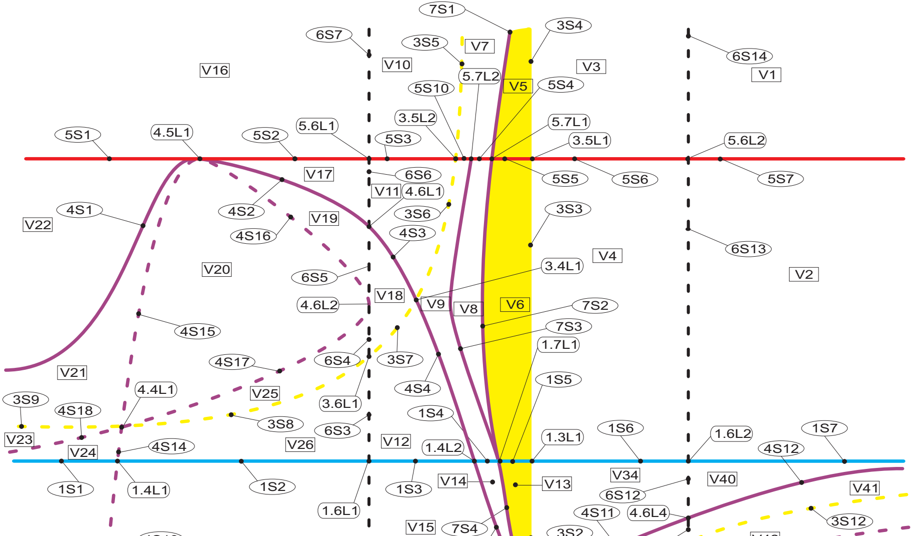
We start studying completely one generic slice, in our case $m = m_9 = 3/2$.
Remember de colours.

1. Blue: Finite $\to \infty$;
2. Yellow: Weak singularity;
3. Red: $\infty \leftrightarrow \infty$;
4. Magenta: separatrix connection;
5. Black: focus $\leftrightarrow$ node.

We draw in solid color when the surface implies a real topological bifurcation and with dashes when topology is not affected. This happen always with the Black curve and partially the Yellow and Magenta.
We call the regions with V_i, jS_i, $k.jL_i$ or P_i depending if they are Volumes, Surfaces, Lines or Points.

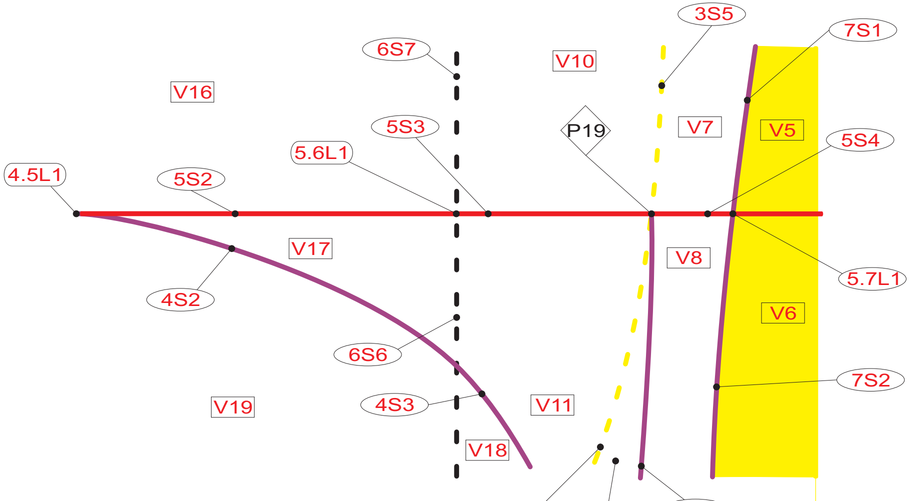
Bifurcation diagram

Main generic slice $m = m_9 = 3/2$.



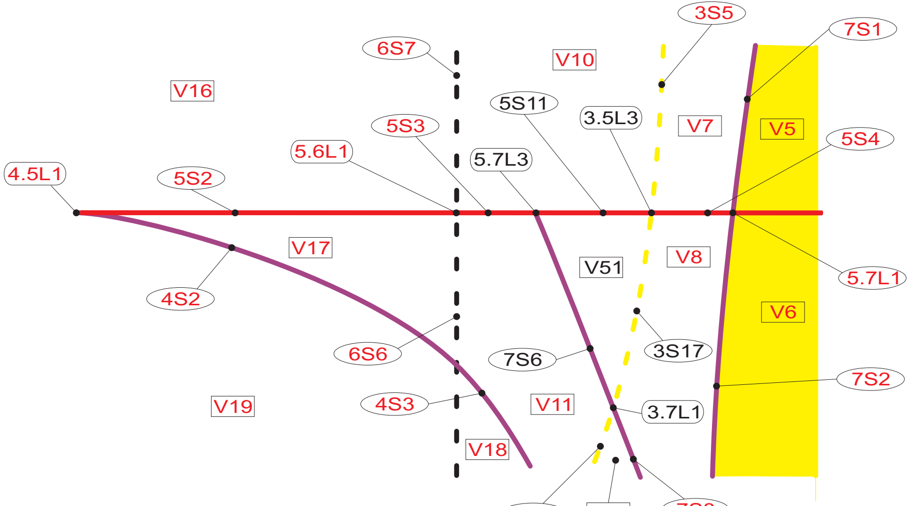
Bifurcation diagram

Now we move up to the next singular slice ($m = m_{10} = 2$) and make a zoom of the region that have changed.



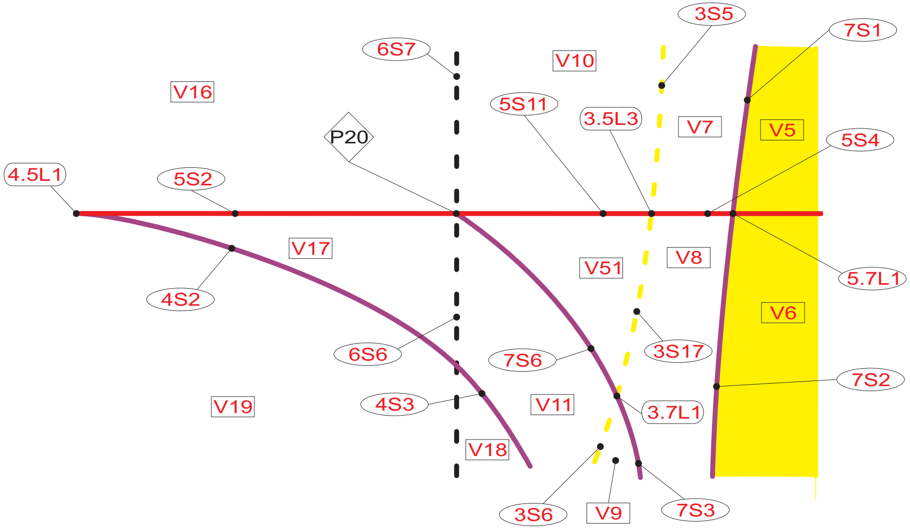
Bifurcation diagram

Next generic slice ($m = m_{11} = 3$). We use red labels for regions already known and black labels for new ones.



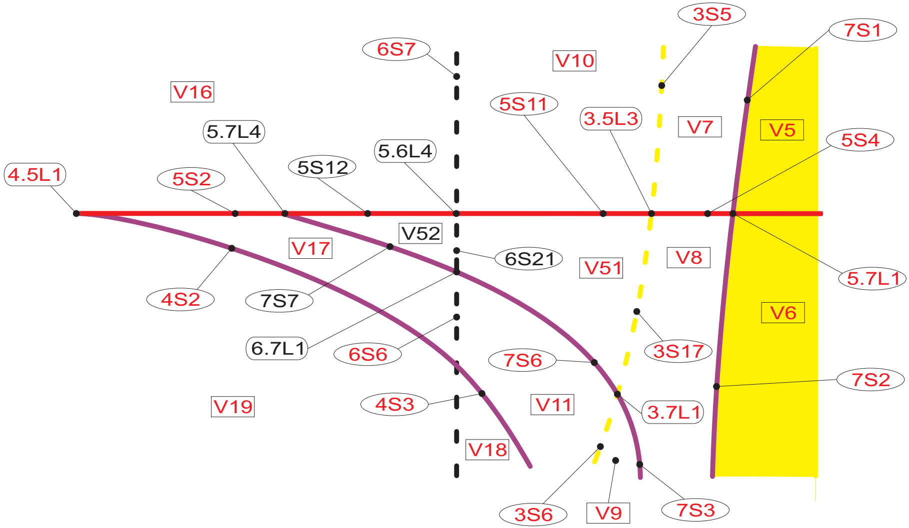
Bifurcation diagram

Next singular slice ($m = m_{12} = 4$).



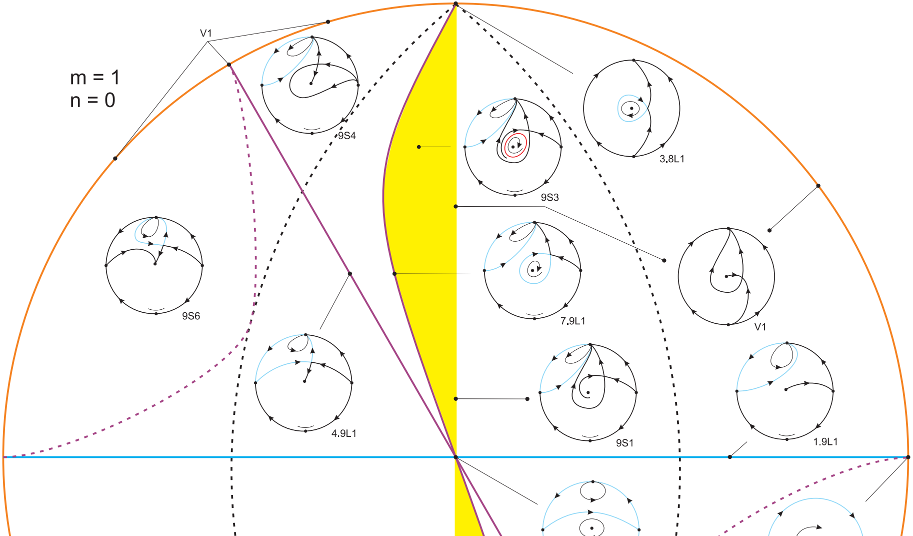
Bifurcation diagram

Next generic slice ($m = m_{13} = 5$).



Bifurcation diagram

Top slice ($m = m_{14} = \infty$) studied as $n = 0$ and $m = 1$ done by [Perko, 1992].

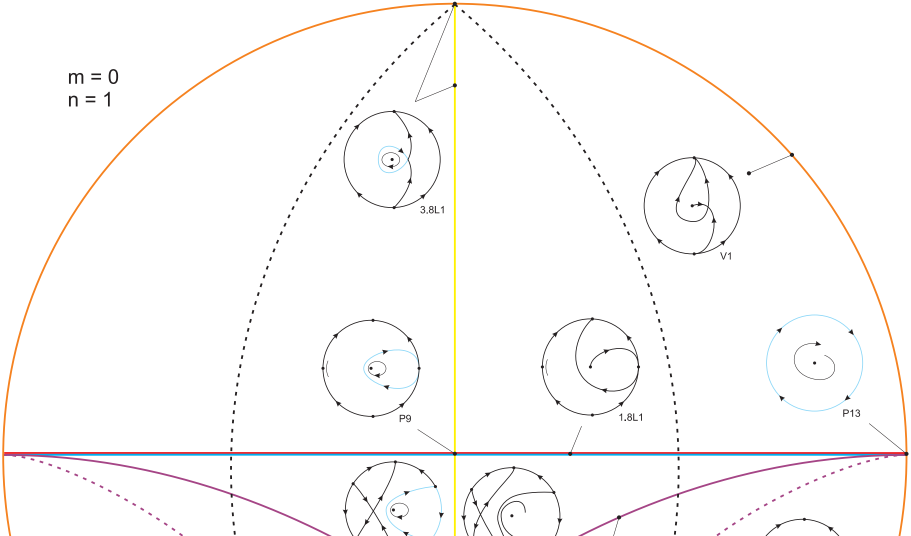


Bifurcation diagram

We must do a similar study for the slices below the reference onw $m = 3/2$. So we study what happens on singular slices $m = m_2 = 1/4$, $m_4 = 1/3$, $m_6 = 1/2$ and $m_8 = 1$ singular, and the intermidiate generic ones $m_1 = 1/8$, $m_3 = 2/7$, $m_5 = 2/5$ and $m_7 = 2/3$. And we end with the bottom slice $m = m_0 = 0$.

Bifurcation diagram

Bottom slice ($m = m_0 = 0$) done by [Chen, 2020].



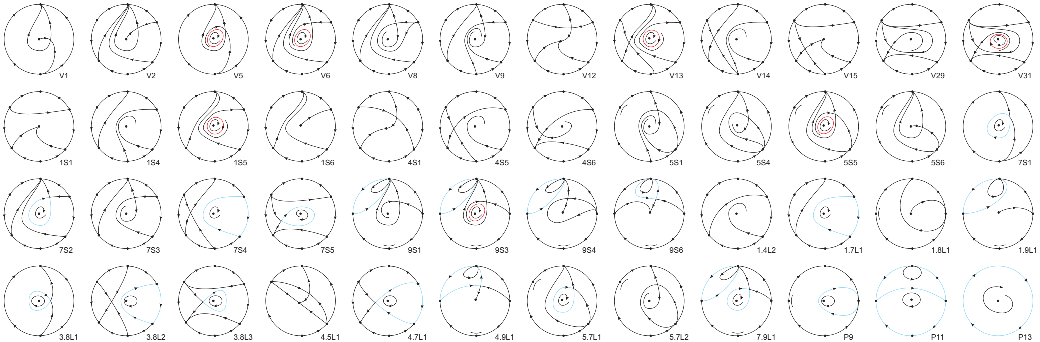
Bifurcation diagram

From this 3-dimensional bifurcation diagram it follows an interesting conclusion:
Even there are needed 15 slices (8 singular and 7 generic) to determine all the regions with some geometrical difference, from the topological point of view, one just needs the top and bottom slice plus any intermediate slice. In other words, if one had ever wished to complete the work of Class I after [Perko, 1992] and [Chen, 2020], one could have taken any slice and he would have obtained all the topologically different phase portraits.
However, and this is the value of this work, one could not be sure to have obtained all the phase portraits (modulo islands) until the full 4-dimensional study is done.
Moreover, this type of study must include always the top and bottom slice so to confirm that there is coherence between the bottom slice and the bottom-most generic slice, and same on the top.
We must point out that this type of study cannot confirm that no other phase portrait is possible. There is always the possibility of the existence of an "island" inside the parameter space where another phase portrait may live. We have never yet found such an "island" in any of our studies.

Main result

Resultat principal

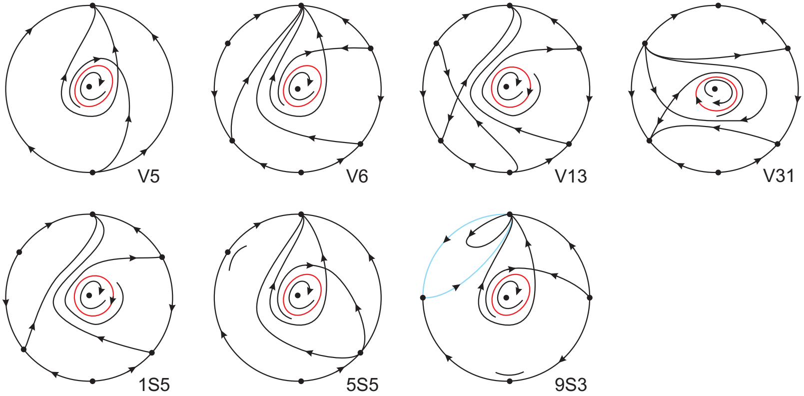
From the 263 regions given by the bifurcation diagram under study, 52 are 3-dimensional, 112 are 2-dimensional, 79 are 1-dimensional and 20 are 0-dimensionas. There are 48 topologically different phase portraits in the Class I.



Main result

Main result (cont.)

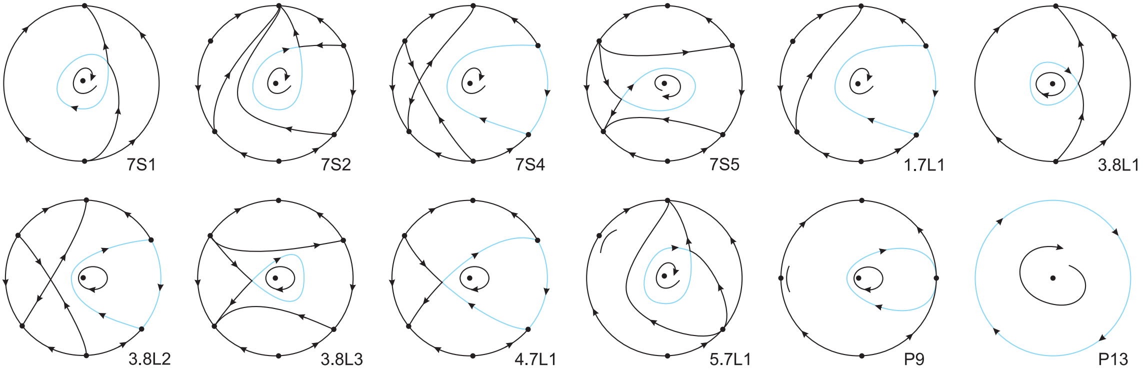
There are 7 phase portraits with a limit cycle. These are $V5$, $V6$, $V13$, $V31$, $1S5$, $5S5$ i $9S3$.



Main result (cont.)

Main result (cont.)

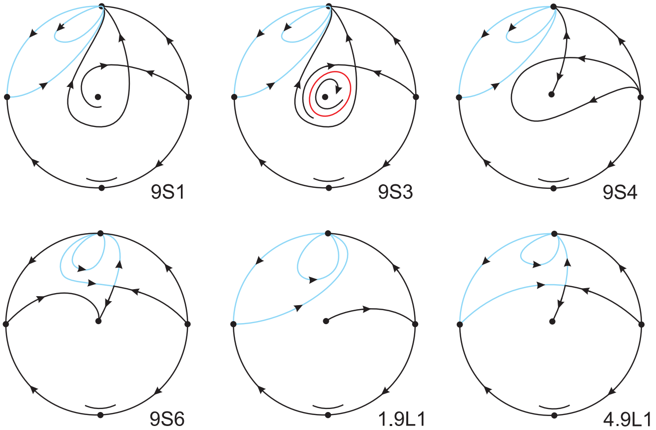
There are 12 phase portraits exactly with one graphic. These are $7S1$, $7S2$, $7S4$, $7S5$, $1.7L1$, $3.8L1$, $3.8L2$, $3.8L3$, $4.7L1$, $5.7L1$, $P9$ i $P13$.



Main result (cont.)

Main result (cont.)

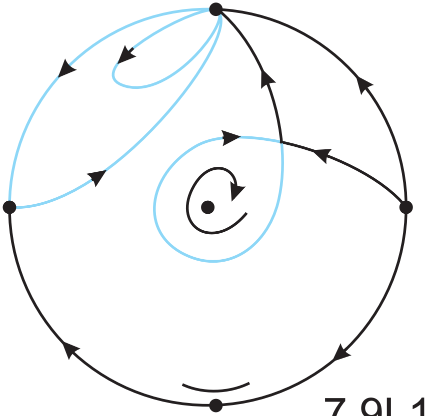
There are 6 phase portraits exactly one set of infinite number of graphics. These are $9S1$, $9S3$, $9S4$, $9S6$, $1.9L1$ and $4.9L1$.



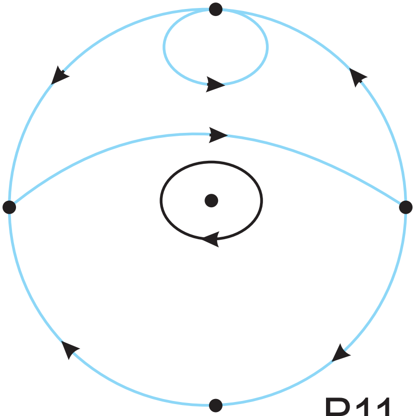
Main result (cont.)

Main result (cont.)

There are 2 phase portraits exactly with one set of infinite number of graphics plus an isolated graphic. These are $7.9L1$, $P11$.



7.9L1

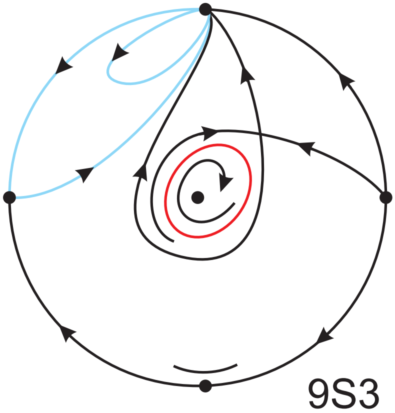


P11

Main result (cont.)

Main result (cont.)

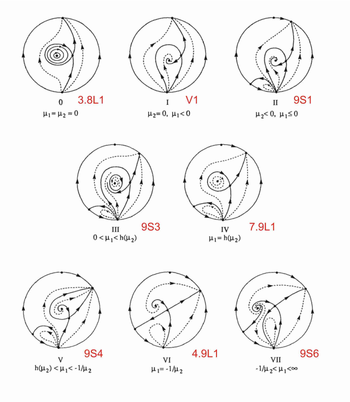
There is 1 phase portrait with a limit cycle and a set of infinite number of graphics. This is $9S3$.

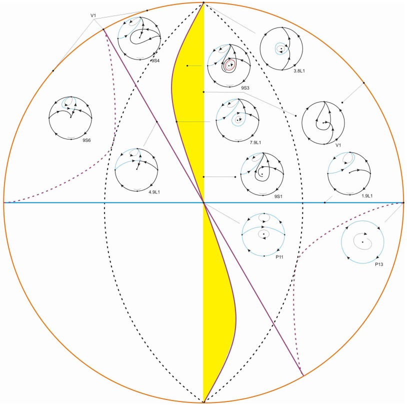


Main result (cont.)

Equivalences with: A global analysis of the Bogdanov-Takens system - L. Perko.

$$\begin{cases} \dot{x} = y \\ \dot{y} = -x + f\,y + x^2 + m\,xy \end{cases}$$

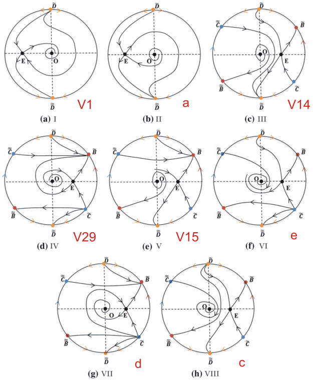


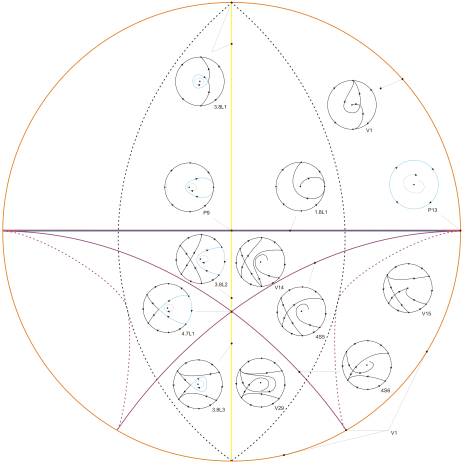


Main result (cont.)

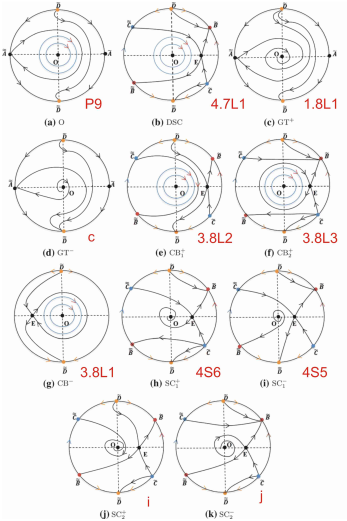
Equivalences with: Bifurcation Diagram and Global Phase Portraits of a Family of Quadratic Vector Fields in Class I - Man Jia, Haibo Chen i Hebai Chen.

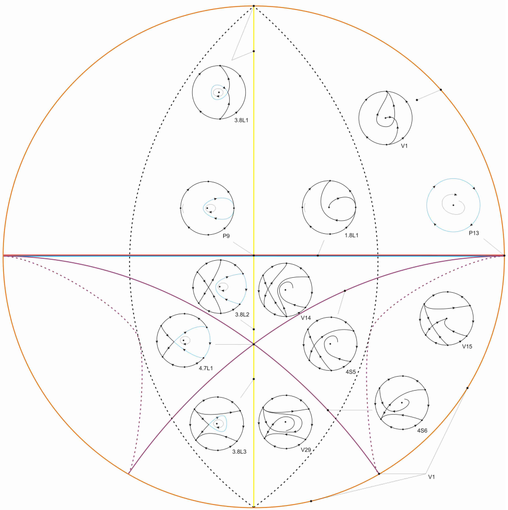
$$\begin{cases} \dot{x} = y \\ \dot{y} = -x - f\,y + l\,x^2 - y^2 \end{cases}$$





Main result (cont.)

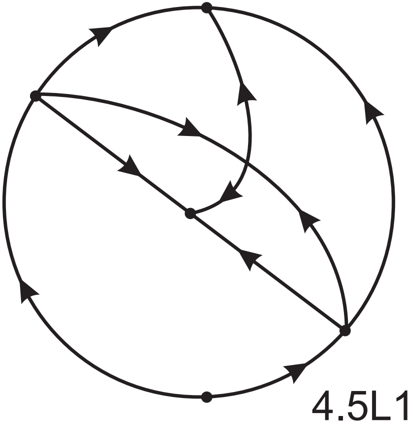


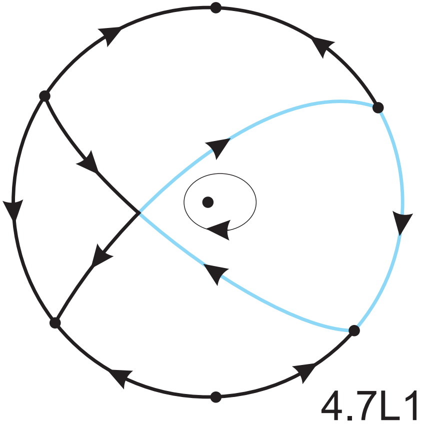


Main result (cont.)

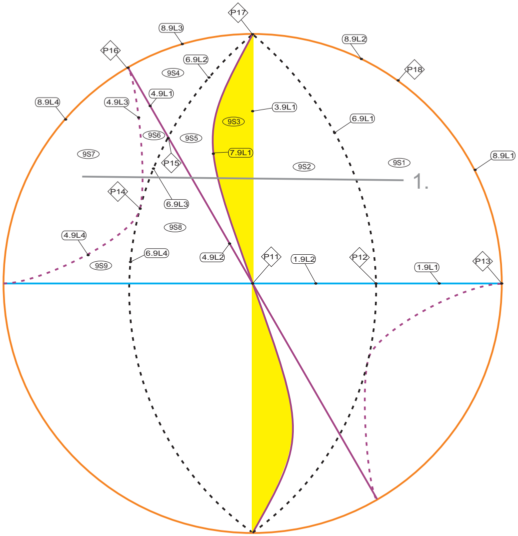
Main result (cont.)

The phase portraits 4.5$L1$ i 4.7$L1$ do not appear in the Ph. D. of B. Coll.

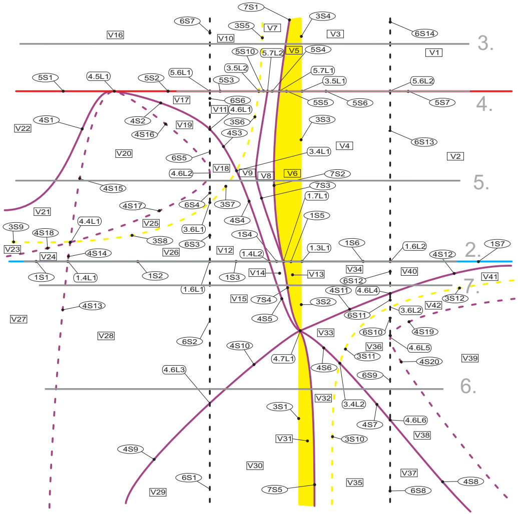




Comparing with the Ph. D. of Dr. Bartomeu Coll



Comparing with the Ph. D. of Dr. Bartomeu Coll



Conclussions

- This is a relatively quite easy family to be studied with the proper tools.
- Even it had been partially studied, it required a complete result as this.
- With the proper tools, now students of last year may afford problems that years ago were deserved for Ph. D.'s.
- The student was able to do this work in about half year while attending classes and working on his assignments during his last semester. This produced a time stress that even provoked an error in the memory which was discovered while preparing the exposition. And solved before it. Time stress is not good for research.
- Other two students of mine, did other 2 TFG's of 4-parametric quadratic systems, but these were much richer in slices and phase portraits and the TFG could contain only the complete study of some few slices. The complete work is planned to be done as TFM.

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Gràcies per la vostra atenció