Hopf-Hopf and Hopf-Pitchfork bifurcations in coupled systems

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Motivation

- Coupled oscillatory systems: Isolated systems undergo a Hopf bifurcation
- Additional degeneracies may lead to Hopf-Hopf and Hopf-Pitchfork type bifurcations
- Likely, they become germs of complex bifurcation diagrams
- Particular case: Coupled neuron models

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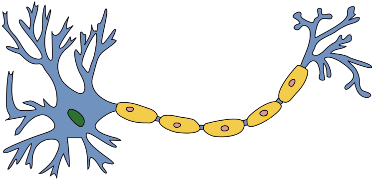

(Figure from **Quasar Jarosz at English Wikipedia (edited)**)

Coupled Oscillatory Systems

Two identical systems…

Isolated Oscillations

diffusively coupled

Dynamical Complexity?

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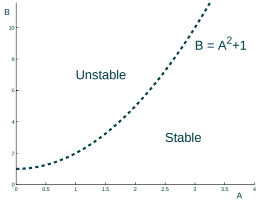
Coupled Oscillatory Systems

The Brusselator model:

$$\begin{cases} x' = A - (B+1)\,x + x^2\,y \\ y' = B\,x - x^2\,y \end{cases}$$

with A and B positive.

Supercritical Hopf bifurcation



The Coupled Brusselator System:

$$\begin{cases} x_1' = A - (B+1)x_1 + x_1^2 y_1 + \lambda_1(x_2 - x_1) \\ y_1' = Bx_1 - x_1^2 y_1 + \lambda_2(y_2 - y_1) \\ x_2' = A - (B+1)x_2 + x_2^2 y_2 + \lambda_1(x_1 - x_2) \\ y_2' = Bx_2 - x_2^2 y_2 + \lambda_2(y_1 - y_2) \end{cases}$$

where λ_1 and λ_2 are not negative.

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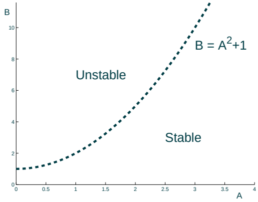

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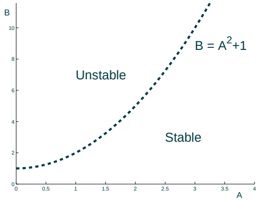

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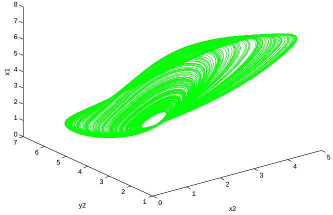
Projection of a strange attractor:



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Interaction between two systems with *simple dynamics* can develop more *dynamical complexity* (e.g. chaos).

[2] F. Drubi, S. Ibáñez, J. A. Rodríguez, *J. Differential Equations 239* (2007).

[3] F. Drubi, S. Ibáñez, J. A. Rodríguez, *Bull. Belg. Math. Soc. Simon Stevin 15* (2008)

[4] F. Drubi *et al.*, *Chaos, Solitons & Fractals* (2023).

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where λ_1 and λ_2 are not negative.

In any generic unfolding X_μ, with $\mu \in \mathbb{R}^n$, of an n-dimensional nilpotent singularity of codimension n, there exist two bifurcation curves of $(n-1)$-dimensional nilpotent singularities of codimension $n-1$ which are generically unfolded by X_μ.

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Coupled Oscillatory Systems

The dynamics of coupled system in the synchronization plane

$$S = \{x_1 = x_2,\ y_1 = y_2\}$$

are those of the isolated system, i.e., there is a Hopf bifurcation in S.

The simplest dynamics expected by a transversal plane S^* are:

- a zero eigenvalue in its linear part that will lead to a **Hopf-Pitchfork bifurcation** of codimension two or higher.
- a pair of imaginary eigenvalues in its linear part that will lead to a **Hopf-Hopf bifurcation** of codimension two or higher.

Both bifurcations are *partially* studied in the literature.

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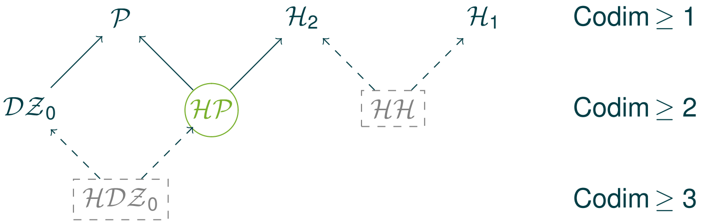

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A Coupled Neuron System

The Coupled Fitzhugh-Nagumo Oscillators

The FitzHugh-Nagumo (FN) model is raised from a traslation of Van der Pol's equation for a relaxation oscillator:

$$\begin{cases} x' = c\,\left(y + x - \frac{1}{3}x^3 + I\right) \\ y' = -\frac{1}{c}\,(x - a + b\,y) \end{cases}$$

with x the neuron **membrane potential** and y a **recovery variable**.

The **action potential** I corresponds to an external stimulus.

Assuming

$$0 < b < 1, \quad c > 0, \quad 1 - 2b/3 < a < 1, \quad \text{and} \quad b < c^2$$

the system has a unique **atractor** (a **resting state**) when $I = 0$ and a **Hopf bifurcation** on the parameter I.

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with $\alpha_1 \leq 0$.

The plane $S = \{x_1 = x_2,\ y_1 = y_2\}$ is invariant by the flow.

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with lineal diffusion parameteres α_1, α_2, ε_1, $\varepsilon_2 \in \mathbb{R}$.

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A change of coordinates:

$$u_1 = \frac{1}{2}(x_1 - x_2), \quad v_1 = \frac{1}{2}(y_1 - y_2), \quad u_2 = \frac{1}{2}(x_1 + x_2), \quad v_2 = \frac{1}{2}(y_1 + y_2)$$

provides the equivalent equations

$$\begin{cases} u_1' = c\,\left(v_1 + u_1 - \frac{1}{3}u_1^3 - u_1 u_2^2\right) - (2\alpha_1 + \varepsilon_1)u_1 \\ v_1' = -\frac{1}{c}(u_1 + bv_1) - (2\alpha_2 + \varepsilon_2)v_1, \\ u_2' = c\,\left(v_2 + u_2 - \frac{1}{3}u_2^3 - u_1^2 u_2\right) + \varepsilon_1 u_1, \\ v_2' = -\frac{1}{c}(u_2 - a + bv_2) + \varepsilon_2 v_1. \end{cases}$$

with the invariant (**synchronization**) plane $\overline{S} = \{u_1 = v_1 = 0\}$.
We will denote as $\overline{S}^*$ to a plane transverse to $\overline{S}$.

Linear Analysis at the equilibrium on the invariant plane

The equilibrium on the invariant plane $\overline{S}$ is of the form $(0, 0, p_2, q_2)$, where p_2 and q_2 fulfill the identities:

$$\frac{a}{b} + \left(1 - \frac{1}{b}\right) p_2 - \frac{1}{3} p_2^3 = 0 \quad \text{and} \quad q_2 = \frac{a - p_2}{b}.$$

The linear part at $(0, 0, p_2, q_2)$ is given by

$$\begin{pmatrix} c\,(1 - p_2^2) - (2\alpha_1 + \varepsilon_1) & c & 0 & 0 \\ -\frac{1}{c} & -\frac{b}{c} - (2\alpha_2 + \varepsilon_2) & 0 & 0 \\ \varepsilon_1 & 0 & c\,(1 - p_2^2) & c \\ 0 & \varepsilon_2 & -\frac{1}{c} & -\frac{b}{c} \end{pmatrix}.$$

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Elementary Local Bifurcations

$\mathcal{H}_{\overline{S}}$	$p_2^2 = 1 - b/c^2, \quad b < c$
$\mathcal{P}_{\overline{S}^*}$	$p_2^2 = 1 - 1/\left(b + c\left(2\alpha_2 + \varepsilon_2\right)\right) - \left(2\alpha_1 + \varepsilon_1\right)/c$
$\mathcal{H}_{\overline{S}^*}$	$p_2^2 = 1 - \frac{1}{c}\left(b/c + 2\alpha_1 + 2\alpha_2 + \varepsilon_1 + \varepsilon_2\right)$ $1 - \left(b/c - \left(2\alpha_1 + \varepsilon_1\right)\right)\left(b/c + 2\alpha_2 + \varepsilon_2\right) > 0$
$\mathcal{HP} = \mathcal{H}_{\overline{S}} \cap \mathcal{P}_{\overline{S}^*}$	$p_2^2 = 1 - b/c^2, \quad b < c$ $1/\big(b + c\left(2\alpha_2 + \varepsilon_2\right)\big) + \left(2\alpha_1 + \varepsilon_1\right)/c = b/c^2$ Hyperbolic eigenvalue: $-\left(2\alpha_1 + 2\alpha_2 + \varepsilon_1 + \varepsilon_2\right)$
$\mathcal{HH} = \mathcal{H}_{\overline{S}} \cap \mathcal{H}_{\overline{S}^*}$	$p_2^2 = 1 - b/c^2, \quad 2\alpha_1 + 2\alpha_2 + \varepsilon_1 + \varepsilon_2 = 0$ $\left(b/c + 2\alpha_2 + \varepsilon_2\right)^2 < 1, \quad b < c$

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Resonance phenomena

In the Hopf-Hopf case, the eigenvalues are:

$$\pm i\sqrt{1-\left(\frac{b}{c}-2\alpha_1-\varepsilon_1\right)^2} \quad \text{and} \quad \pm\, i\sqrt{1-\frac{b^2}{c^2}}.$$

If $2\alpha_1 + \epsilon_1 = 0$, the system will show a resonance 1:1. Hence, non-resonant bifurcations of codimension 2 can be unfolded.

As it occurs in **S. A. Campbell and M. Waite** (2001), where $\alpha_2 = \varepsilon_2 = 0$ and $\alpha_1 \leq 0$.

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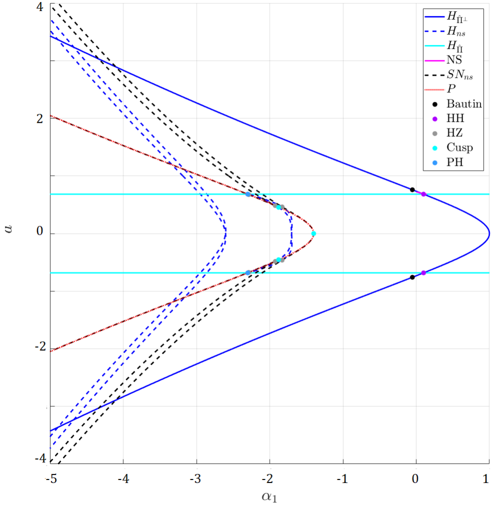
Bifurcation Diagram with AUTO: $b = 0.4$, $c = 2$, and $\alpha_2 = \varepsilon_2 = 0$



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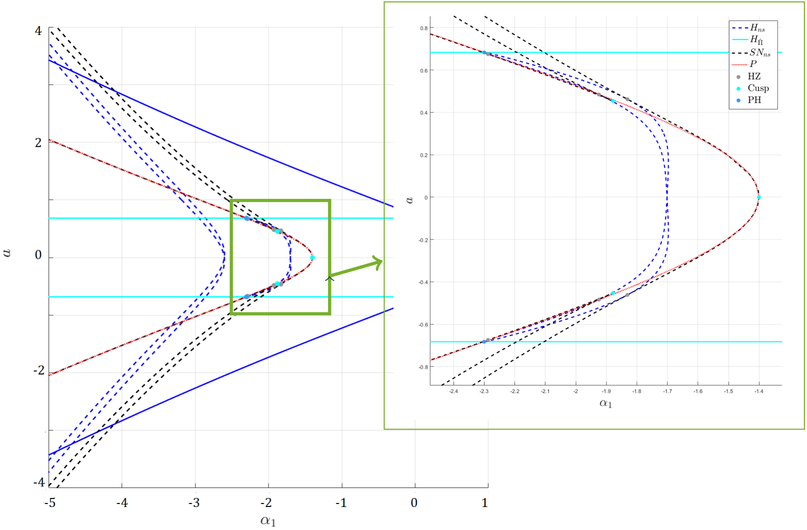

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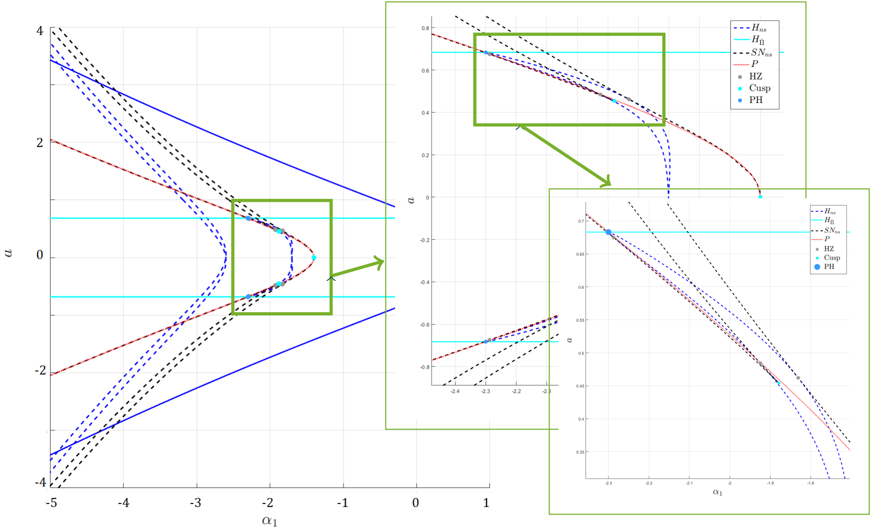

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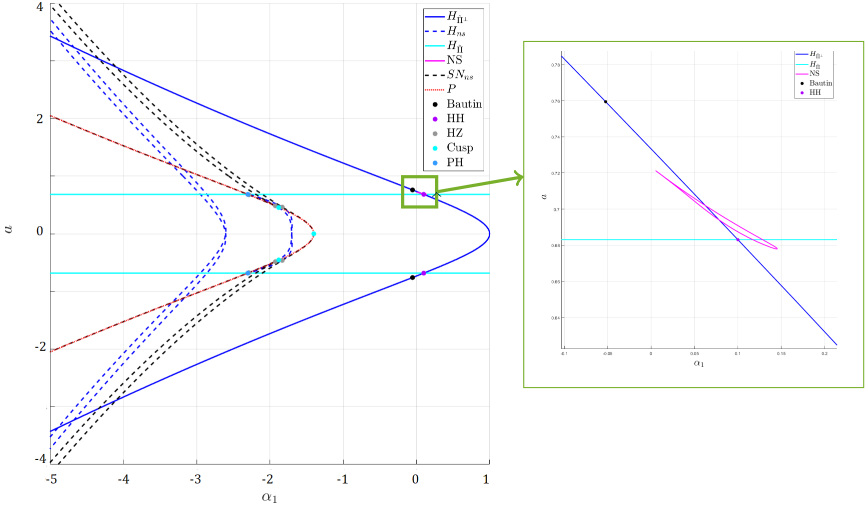

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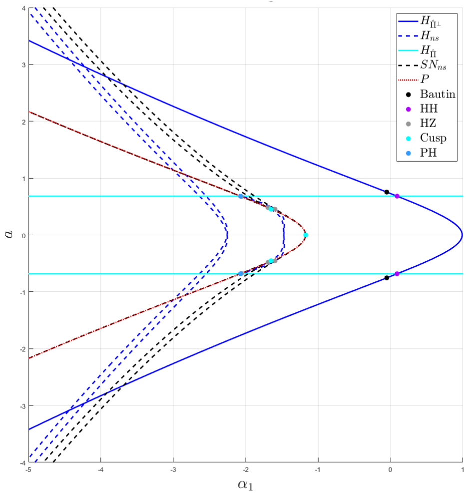
Bifurcation Diagram with AUTO: $b = 0.4$, $c = 2$, and $\alpha_2 = 0.01$



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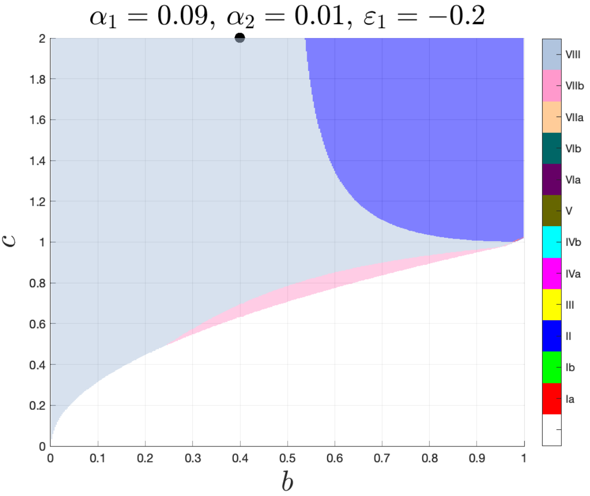
The Coupled Fitzhugh-Nagumo Oscillators

Classification of Hopf-Hopf singularities:



A Coupled Neuron System

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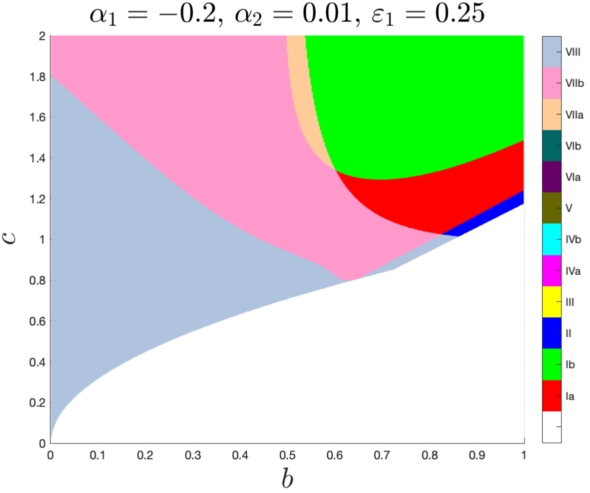

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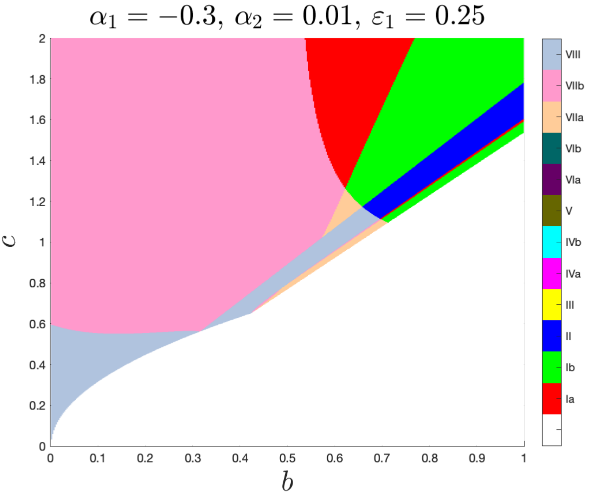

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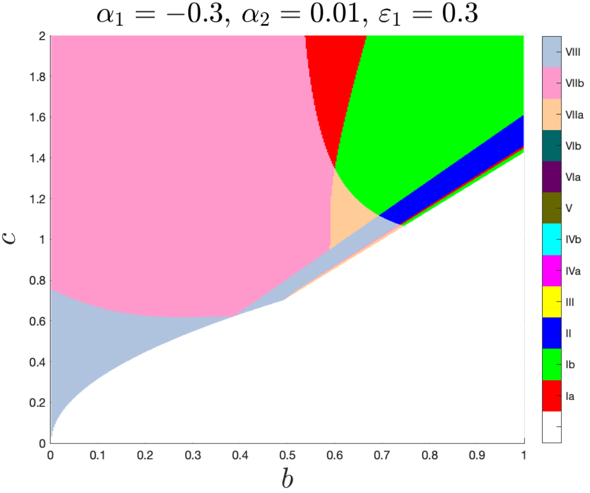

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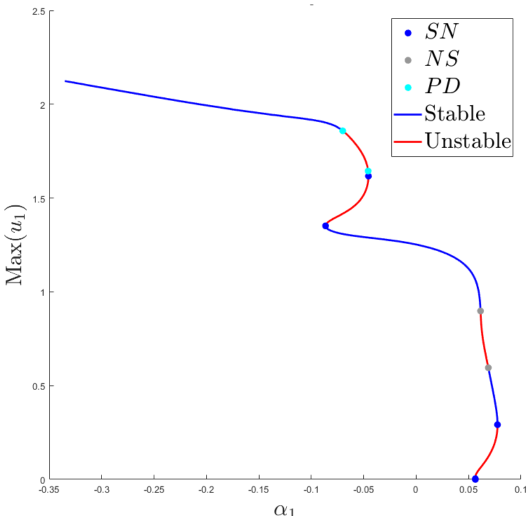
Continuation of a periodic orbit ($\alpha_2 = 0.01$):



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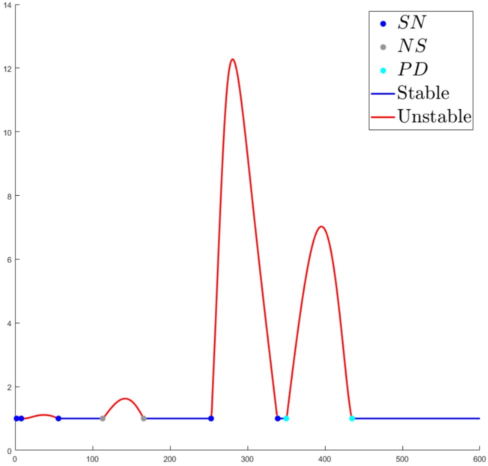
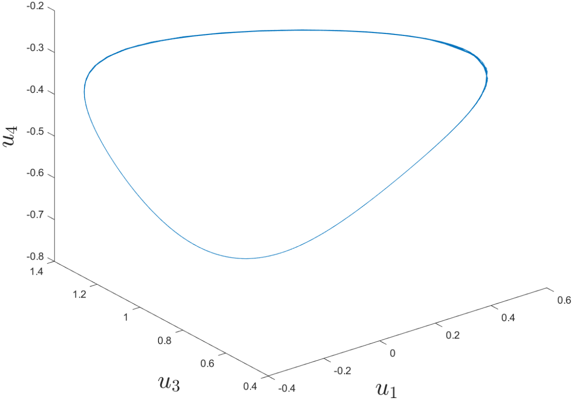
Modulus of eigenvalues (maximum) (α_2 = 0.01):



A Coupled Neuron System

The Coupled Fitzhugh-Nagumo Oscillators

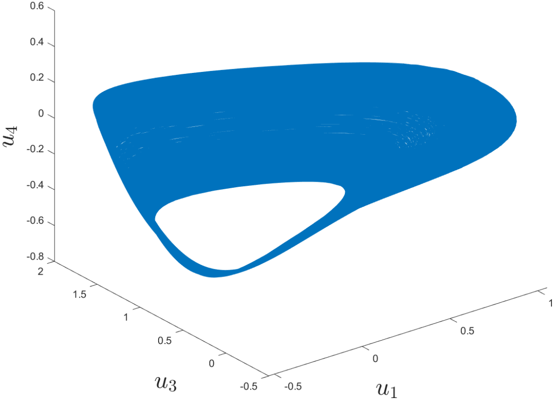
$$a = 0.7,\ \alpha_1 = 0.072926,\ \text{PO}$$



A Coupled Neuron System

The Coupled Fitzhugh-Nagumo Oscillators

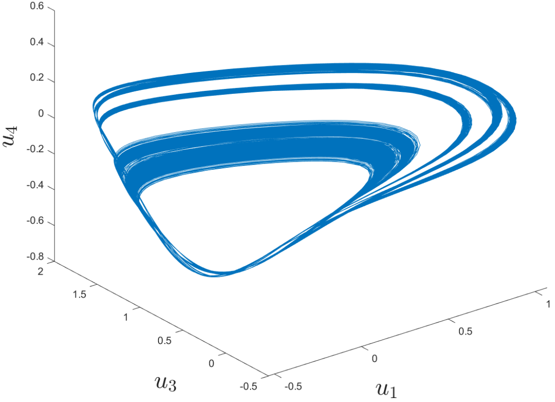
$a = 0.7$, $\alpha_1 = 0.065822$, 2-Torus



A Coupled Neuron System

The Coupled Fitzhugh-Nagumo Oscillators

$a = 0.7$, $\alpha_1 = 0.062702$, Chaotic Attractor?



Open Questions

- Is the case VIa associated with Hopf-Hopf singularities also feasible?
- What other cases can be obtained from resonance?
- Can all Hopf-Hopf types of codimension 2 be obtained?
- Can a classification of the Hopf-Pitchfork be provided?

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Thank you for your attention!