Monodromic singular points in switching curves of planar piecewise analytical differential systems

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We deal with piecewise analytical vector fields X (in short, PWAVF) defined by a pair (X^+, X^-), where X^+ and X^- are analytical vector fields on regions of the plane separated by an analytical curve Σ. The curve Σ is called *switching curve* which separates the plane in two regions Σ^+ and Σ^- having defined on each region the analytical vector fields X^+ and X^-, respectively.

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The few works that deal with this topic study the so-called "pseudo focus" or "sewed focus"

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(i) *Focus-Focus type*: both systems X^+ and X^- have a singular point at p with eigenvalues $\lambda \in \mathbb{C} \setminus \mathbb{R}$ and their solutions turn around p counterclockwise or clockwise sense.

(ii) *Focus-Parabolic* (resp. *Parabolic-Focus*) : X^+ (resp. X^-) has a singular point of focus type at p (i.e. X^+ has a singular point at p with eigenvalues $\lambda \in \mathbb{C} \setminus \mathbb{R}$) while the solutions of X^- (resp. X^+) have a parabolic contact (i.e. a second order contact point) with Σ at p, the solution by p is locally contained in Σ^+ (resp. Σ^-).

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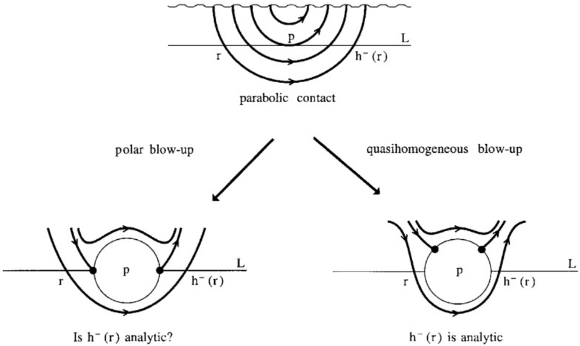
(iii) *Parabolic-Parabolic*: the solutions of both systems have a parabolic contact at p with Σ in such a way that the flow of X turns around p.

In [1] the authors introduce techniques that make it possible to study the stability of the monodromic singular points described in the definitions (i), (ii), and (iii). Moreover, they obtain general expressions for the first three Lyapunov constants for these points and generate limit cycles for some concrete examples.

[1] B. Coll, A. Gasull, and R. Prohens, *Degenerate Hopf bifurcations in discontinuous planar systems*, J. Math. Anal. Appl., 253 (2001), pp. 671–690.



Analyticity of the return map for parabolic contacts.

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The second one is Dulac's problem restricted to piecewise analytical planar vector fields, i.e., the existence of a neighborhood of the Σ-monodromic singular point (whenever possible) "free of limit cycles".

Σ-Monodromic singular points

Let $Y : \mathbb{R}^2 \to \mathbb{R}^2$ be an analytical vector field and p be a singular point of Y and $\gamma = \{\gamma(t) : t \in \mathbb{R}\}$ be an orbit of Y that tends to p when t tends to $\pm\infty$.

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In this case, γ is a *characteristic orbit* if there exists $\lim_{t\to\pm\infty}\gamma(t)/|\gamma(t)|$, where $|\cdot|$ is the Euclidean norm.

We have that p is a *monodromic singular point* of Y if there is no characteristic orbit associated with it.

Consider $f : \mathbb{R}^2 \to \mathbb{R}$ an analytical function having 0 as a regular value and denote $\Sigma = f^{-1}(0)$ and $\Sigma^{\pm} = \{p \in \mathbb{R}^2 : \pm f(p) > 0\}$.

Consider $f:\mathbb{R}^2\to\mathbb{R}$ an analytical function having 0 as a regular value and denote $\Sigma=f^{-1}(0)$ and $\Sigma^{\pm}=\{p\in\mathbb{R}^2:\pm f(p)>0\}$. Let $X=(X^+,X^-)$ be a piecewise analytical vector field defined by

$$X(q)=\begin{cases} X^+(q) & \text{if} \quad f(q)\geq 0,\\ X^-(q) & \text{if} \quad f(q)\leq 0,\end{cases}$$

where $X^{\pm}$ are analytical planar vector fields. We note that X can be bi-valued at the points of Σ.

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As usual, here and in what follows, Yf will denote the derivative of the function f in the direction of the vector Y, i.e. $Yf = \langle \nabla f, Y \rangle$. Moreover, $Y^n f = Y(Y^{n-1}f)$.

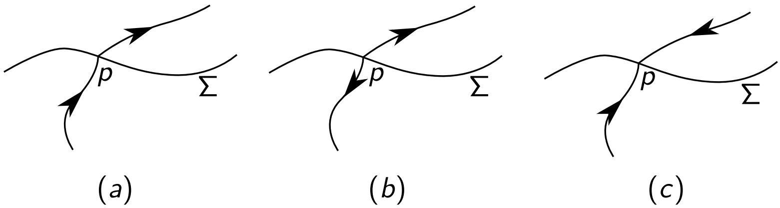
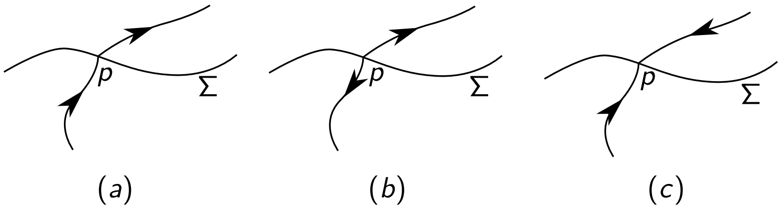


Figure: Illustrating different types of $p \in \Sigma$: (a) Sewing; (b) Escaping; (c) Sliding.



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On the arcs ES and SL we define the *Filippov vector field* F_X associated to $X = (X^+, X^-)$, as follows: if $p \in SL$ or ES, then $F_X(p)$ denotes the vector tangent to Σ in the cone spanned by $X^+(p)$ and $X^-(p)$.

A point $p \in \Sigma$ is called a Σ-*regular point* of X if $p \in SW$ or if $p \in SL$ or $p \in ES$ then $F_X(p) \neq 0$. Otherwise, p is a Σ-*singular point* of X.

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A continuous closed curve γ consisting of two regular trajectories, one of X^+ and another of X^-, and two points $\{p_1, p_2\} = \Sigma \cap \gamma$ is called a Σ-*closed crossing orbit* (or simply Σ-*closed orbit*) of X, if $\{p_1, p_2\}$ are sewing points and γ meets Σ transversally in $\{p_1, p_2\}$.

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ii) There exists a regular trajectory γ of X^+ (resp. X^-) with $\lim_{t\to\pm\infty} \gamma(t) = p$, and there exist a neighborhood V of p such that $\gamma \cap V \subset V \cap \Sigma^+$ (resp. $\gamma \cap V \subset V \cap \Sigma^-$);

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iii) For all neighborhood V of p there exists $q \in V \cap \Sigma$ such that $Xf(q) \cdot Yf(q) < 0$.

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A Σ-singular point p is a $\mathit{fold}_{n^\pm}$ of $X^\pm$ if it is a fold point of order $n^\pm$, i.e., If p is an isolated Σ-singular point of X and X does not have Σ-characteristic orbits associated with p we call it Σ-*monodromic singular point*.

A Σ-singular point p is a $fold_{n^\pm}$ of $X^\pm$ if it is a fold point of order $n^\pm$, i.e.,

$$X^\pm(p) \neq 0, X^\pm f(p) = \cdots = (X^\pm)^{n^\pm - 1} f(p) = 0 \text{ and } (X^\pm)^{n^\pm} f(p) \neq 0,$$

where $n^\pm$ are even. In this case we have that $X^\pm$ has a contact of order $n^\pm$ with Σ at p (see Figure 2).

We say that p is a *visible fold*$_{n^\pm}$ for $X^\pm$ if $\pm(X^\pm)^{n^\pm} f(p) > 0$. If $\pm(X^\pm)^{n^\pm} f(p) < 0$ then we say that p is an *invisible fold*$_{n^\pm}$ for $X^\pm$.

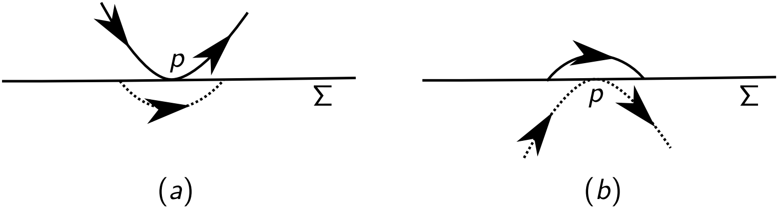

Figure: Illustrating different types of folds: (a) Visible fold$_{n^+}$ for X^+; (b) Invisible fold$_{n^+}$ for X^+.

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We say that Y has a *characteristic orbit in K* associated with p if there are a characteristic orbit γ of Y associated with p and a neighborhood V of p such that $\gamma \cap V \subset V \cap K$. Otherwise, we say that Y has no characteristic orbit in K.

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Let $p \in \overline{K} \subset \mathbb{R}^2$ an isolated singular point of an analytical vector field Y.

We say that Y has a *characteristic orbit in* K associated with p if there are a characteristic orbit γ of Y associated with p and a neighborhood V of p such that $\gamma \cap V \subset V \cap K$. Otherwise, we say that Y has no characteristic orbit in K.

The following theorem classifies the Σ-monodromic singular points of X.

Theorem 2

Let $X = (X^+, X^-)$ be a piecewise analytical vector field. We suppose that p is a Σ-singular point of X such that $X^+f(p) = X^-f(p) = 0$ and $X^+f \cdot X^-f\ |_V > 0$ in a V-neighborhood of $p \in \Sigma$ with the unique exception of p.

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i) *If $X^+(p) \neq 0$ and $X^-(p) \neq 0$, then p is a Σ-monodromic singular point of X if and only if it is a fold of order n^+ of X^+ with $(X^+)^{n^+} f(p) < 0$ and it is a fold of order n^- of X^- with $(X^-)^{n^-} f(p) > 0$.*

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ii) *If $X^+(p) = 0$ (resp. $X^-(p) = 0$) and $X^-(p) \neq 0$ (resp. $X^+(p) \neq 0$), then p is a Σ-monodromic singular point of X if and only if X^+ has no characteristic orbit in Σ^+ (resp. X^- has no characteristic orbit in Σ^-) associated with p and p is a fold of order n of X^- (resp. X^+) with $(X^-)^n f(p) > 0$ (resp. with $(X^+)^n f(p) < 0$).*

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ii) *If $X^+(p) = 0$ (resp. $X^-(p) = 0$) and $X^-(p) \neq 0$ (resp. $X^+(p) \neq 0$), then p is a Σ-monodromic singular point of X if and only if X^+ has no characteristic orbit in Σ^+ (resp. X^- has no characteristic orbit in Σ^-) associated with p and p is a fold of order n of X^- (resp. X^+) with $(X^-)^n f(p) > 0$ (resp. with $(X^+)^n f(p) < 0$).*

iii) *If $X^+(p) = X^-(p) = 0$, then p is a Σ-monodromic singular point of X if and only if X^+ has no characteristic orbit in Σ^+ and X^- has no characteristic orbit in Σ^- associated with p, respectively.*

Theorem 3

Let p be an isolated Σ-singular point of a piecewise analytical vector field $X = (X^+, X^-)$. If p is a Σ-monodromic singular point, then the Poincaré return map is well defined.

Analyticity of the displacement function

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Hypothesis H. *Let* $X = (X^+, X^-)$ *be a piecewise analytical vector field having an isolated* Σ*-monodromic singular point* $p_0 \in \Sigma$*. Without loss of generality we assume that* p_0 *is the origin and the orbits of* X *turn around the origin in counterclockwise sense. We say that* X *satisfies the Hypothesis* **H** *if*

(i) there exists a continuous change of coordinates such that the switching curve Σ, in these new coordinates, is $\{y = 0\}$; and

(i) there exists a continuous change of coordinates such that the switching curve Σ, in these new coordinates, is $\{y = 0\}$; and

(ii) there are integers p, q, r and s such that in the weight polar coordinates $(x, y) = (\rho^p \cos\theta, \rho^r \sin\theta)$ in $\Sigma \cup \Sigma^+$ and $(x, y) = (\rho^q \cos\theta, \rho^s \sin\theta)$ in $\Sigma \cup \Sigma^-$ the systems associated with vector fields X^+ and X^- are equivalent to differential equations

$$\frac{d\rho}{d\theta} = \frac{F^{\pm}(\theta,\rho)}{G^{\pm}(\theta,\rho)}, \tag{1}$$

where $F^{\pm}$ and $G^{\pm}$ are analytical functions with $F^{\pm}(\theta, 0) = 0$ for all $\theta \in \mathbb{R}$, $G^+(\theta, 0) \neq 0$ for all $\theta \in [0, \pi]$ and $G^-(\theta, 0) \neq 0$ for all $\theta \in [-\pi, 0]$.

Theorem 4

If a piecewise analytical vector field $X = (X^+, X^-)$ has a Σ-monodromic singular point and satisfies the Hypothesis **H**, *then there is a choice of coordinates in Σ such that in these coordinates the displacement function is analytic.*

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Corollary 5

With the hypothesis of Theorem 4, we have that p is free of limit cycles.

Proposition 3.1

Let $X = (X^+, X^-)$ be a piecewise analytical vector field and $\Sigma = f^{-1}(0)$ a curve, where $f : \mathbb{R}^2 \to \mathbb{R}$ is an analytical function having 0 as its regular value. Assume that p is a fold point of order $n^\pm$ of $X^\pm$ with $(X^+)^{n^+} f(p) < 0$, $(X^-)^{n^-} f(p) > 0$ and $X^+ f \cdot X^- f \mid_{V \setminus \{p\}} > 0$ for some neighborhood V of p, then X satisfies the Hypothesis **H**.

Proposition 3.2

Let $X = (X^+, X^-)$ be a piecewise analytical vector field and $\Sigma = f^{-1}(0)$ a curve, where $f : \mathbb{R}^2 \to \mathbb{R}$ is analytic having 0 as a regular value. Assume that p is a fold point of order $n^\pm$ of $X^\pm$ with $\pm(X^\pm)^{n^\pm} f(p) < 0$ or p is a singular point of $X^\pm$ having linear part with eigenvalues $\alpha \pm \beta i$, with $\beta \neq 0$. Then, if $X^+ f \cdot X^- f \,|_{V \setminus \{p\}} > 0$ for some neighborhood V of p, X satisfies the Hypothesis **H**.

Proposition 3.3

Let $X = (X^+, X^-)$ be a piecewise analytical vector field and $\Sigma = f^{-1}(0)$ a curve, where $f : \mathbb{R}^2 \to \mathbb{R}$ is analytic having 0 as a regular value. Assume that $p \in \Sigma$ is either a monodromic nilpotent singular point or a nilpotent cusp of X^+ (resp. X^-) such that Σ is transversal to the eigenspace of $DX^+(p)$ (resp. $DX^-(p)$), locally the characteristic orbits of the cusp p are contained in Σ^- (resp. Σ^+) in the cusp case, and p is a fold point of order n^- of X^- with $(X^-)^{n^-}f(p) > 0$ (resp. n^+ of X^+ with $(X^+)^{n^+}f(p) < 0$) or p is a singular point of X^- (resp. X^+) having linear part with eigenvalues $\alpha \pm \beta i$, with $\beta \neq 0$. Then, if $X^+f \cdot X^-f \mid_{V\setminus\{p\}} > 0$ for some neighborhood V of p, X satisfies the Hypothesis **H**.

From now on, we call the Σ-monodromic singular points that are nilpotent singular points of at least one of the analytical vector fields $X^{\pm}$ of Σ-*monodromic nilpotent singular points*.

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Our best result for the Σ-monodromic nilpotent singular points is Proposition 3.3. However, if more restrictive hypotheses are assumed, we can find other types of Σ-monodromic nilpotent singular points, not contemplated in this proposition, which also satisfy Hypothesis **H**.

For example, if we assume from the start that Σ is the x axis, X^+ and X^- are both in nilpotent normal form

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= g(x) + yh(x) + y^2 B(x,y), \end{aligned} \tag{2}$$

(where $g(x) = ax^m + o(x^m)$, for some $m \geq 2$ with $a \neq 0$, and the function h is either identically null or there exist $b \neq 0$ and $n \geq 1$ such that $h(x) = bx^n + o(x^n)$) and the origin is a Σ-monodromic nilpotent singular point obtained by a nilpotent cusp and a monodromic nilpotent singular point.

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Proposition 3.4

Let $X = (X^+, X^-)$ be a piecewise analytical vector field with switching curve $\Sigma = \{y = 0\}$, with X^- or X^+ given by $Y : \mathbb{R}^2 \to \mathbb{R}^2$ where

$$Y : \begin{cases} \dot{x} = y, \\ \dot{y} = g(x) + yh(x) + y^2 B(x, y), \end{cases} \tag{3}$$

$g(x) = ax^m + o(x^m)$, $h(x) = bx^n + o(x^n)$ and B analytical. Assume $a < 0$, n odd, $m = 2n + 1$ and $b^2 + 4a(n + 1) \geq 0$. If $b > 0$ (resp. $b < 0$) then Y does not have characteristic orbit in $\Sigma^- \cup \Sigma$ (resp. $\Sigma^+ \cup \Sigma$) at the origin. Moreover, if $b > 0$ and $Y = X^-$ (resp. $b < 0$ and $Y = X^+$) then the half-Poincaré return map Π^- (resp. Π^+) is well defined and it is analytic.

Applications

Proposition 4.1

Let $X = (X^+, X^-)$ be a piecewise analytical vector field having an isolated Σ-monodromic singular point and satisfying the Hypothesis **H**. *Then the half-Poincaré return maps $\Pi^\pm$ have the following expansions in power series*

$$
\begin{aligned}
\Pi^+(x_0) = &-(u_1^+(\pi))^p x_0 - p(u_1^+(\pi))^{p-1}u_2^+(\pi)x_0^{\frac{p+1}{p}} \\
&- \left(\frac{1}{2}(p-1)p(u_1^+(\pi))^{p-2}(u_2^+(\pi))^2 + p(u_1^+(\pi))^{p-1}u_3^+(\pi)\right)x_0^{\frac{p+2}{p}} - \dots
\end{aligned}
$$

$$
\begin{aligned}
\Pi^-(x_0) = &-(u_1^-(-\pi))^q x_0 - q(u_1^-(-\pi))^{q-1}u_2^-(-\pi)x_0^{\frac{q+1}{q}} \\
&- \left(\frac{1}{2}(q-1)q(u_1^-(-\pi))^{q-2}(u_2^-(-\pi))^2 + q(u_1^-(-\pi))^{q-1}u_3^-(-\pi)\right)x_0^{\frac{q+2}{q}} - \cdots
\end{aligned}
$$

where $u_i^\pm(\theta)$ is the coefficient of order i in the expansion of solution $\rho = \rho^\pm(\theta, \rho_0)$ (with $\rho^\pm(0, \rho_0) = \rho_0$) of equation (1) in powers of ρ_0.

Corollary 6

With the hypothesis of the above proposition, if $(p,q)=1$ *then the origin is a* Σ*-center if and only if* $\Pi^{+}(x_0) = -(u_1^{+}(\pi))^{p}x_0$, $\Pi^{-}(x_0) = -(u_1^{-}(-\pi))^{q}x_0$ *and* $(u_1^{+}(\pi))^{p} - (u_1^{-}(-\pi))^{q} = 0$.

In what follows we consider piecewise analytical vector fields $X = (X^+, X^-)$ having the origin as a Σ-monodromic singular point and the switching curve $\Sigma = \{y = 0\}$.

Cusp-Fold$_2$:

In the half-plane Σ^+ we consider the vector field X^+ associated with the system

$$X^+ : \begin{cases} \dot{x} = -y^2 + bxy, \\ \dot{y} = x, \end{cases} \tag{4}$$

which has a cusp singular point at origin.

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In the half-plane Σ^+ we consider the vector field X^+ associated with the system

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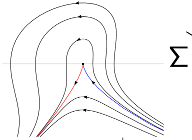
which has a cusp singular point at origin.

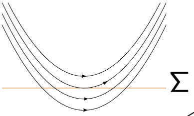
In the half-plane Σ^- we consider the vector field X^- associated with the system

$$X^- : \begin{cases} \dot{x} = 1, \\ \dot{y} = x, \end{cases} \qquad (5)$$

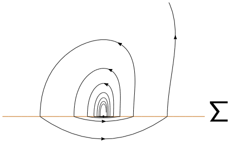
which has a fold point of order 2 (because $X^-(0,0) \neq (0,0)$, $X^- f(0,0) = 0$ and $(X^-)^2 f(0,0) = 1$). Therefore, by Theorem 2 (ii), the origin is a Σ-monodromic singular point.



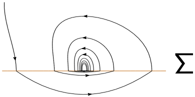
A cusp for X^+ with $b > 0$

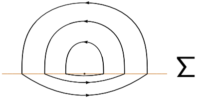


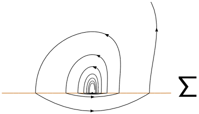
A fold_2 for X^-



The Cusp-Fold_2 case for X with $b > 0$







A stable Σ-focus for X with $b < 0$ A Σ-center for X with $b = 0$ An unstable Σ-focus for X with $b > 0$

Cusp-(HE-singular point):

Consider a Σ-monodromic singular point formed by a cusp singular point for X^+ and a degenerate singular point with an elliptic sector and a hyperbolic sector for X^-.

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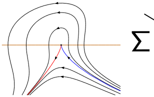
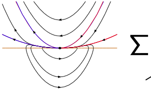
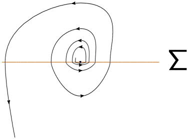
For X^- the origin as a nilpotent singular point with one elliptic sector and one hyperbolic sector, called *HE-singular point*.



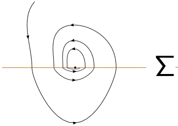
A cusp for X^+,with $b > 0$

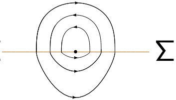


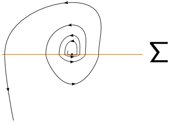
A HE-singular point for X^-



The Cusp-(HE-singular point) case for X with $b > 0$







A stable Σ-focus for X with $b < 0$ A Σ-center for X with $b = 0$ An unstable Σ-focus for X with $b > 0$

Fold$_2$-Fold$_4$:

Consider a Σ-monodromic singular point where both X^+ and X^- have fold points at the origin.

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and in the half-plane Σ^- we have

$$X^- : \begin{cases} \dot{x} = y + 1, \\ \dot{y} = x^3 + cx^2y. \end{cases} \tag{8}$$

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As $X^+(0,0) \neq (0,0)$, $X^+ f(0,0) = 0$ and $(X^+)^2 f(0,0) = -1$, the origin is a fold point of order 2 of X^+, called fold$_2$.

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As $X^+(0,0) \neq (0,0)$, $X^+f(0,0) = 0$ and $(X^+)^2f(0,0) = -1$, the origin is a fold point of order 2 of X^+, called fold$_2$. The origin is a fold point of order 4 of X^- (called fold$_4$), provided that $X^-(0,0) \neq (0,0)$, $X^-f(0,0) = (X^-)^2f(0,0) = (X^+)^3f(0,0) = 0$ and $(X^+)^4f(0,0) = 6$.

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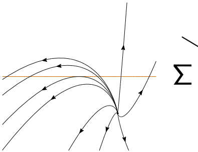
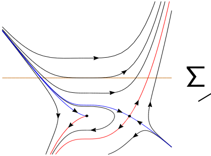
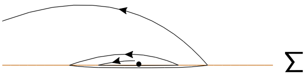


A fold_2 for X^+ with $a = b = 1$



A fold_4 for X^- with $c = 1$



The Fold_2-Fold_4 case for X, with $a = b = c = 1$.

(i) If $a + b < 0$ the origin is a stable Σ-focus of X;

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(ii) If $a + b > 0$ the origin is an unstable Σ-focus of X;

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(ii) If $a + b > 0$ the origin is an unstable Σ-focus of X;

(iii) If $a + b = 0$ and $c > 0$ the origin is a stable Σ-focus of X;

(i) If $a+b<0$ the origin is a stable Σ-focus of X;
(ii) If $a+b>0$ the origin is an unstable Σ-focus of X;
(iii) If $a+b=0$ and $c>0$ the origin is a stable Σ-focus of X;
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(ii) If $a+b>0$ the origin is an unstable Σ-focus of X;
(iii) If $a+b=0$ and $c>0$ the origin is a stable Σ-focus of X;
(iv) If $a+b=0$ and $c<0$ the origin is an unstable Σ-focus of X.
(v) If $a+b=c=0$ the origin is a Σ-center of X.

Elementary-Degenerate:

Consider a Σ-monodromic singular point where the origin is an elementary singular point for X^+ and it is a degenerate one for X^-.

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$$X^+ : \begin{cases} \dot{x} = ax - by + cx^2, \\ \dot{y} = bx + ay, \end{cases} \tag{9}$$

with $b > 0$,

Elementary-Degenerate:

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with $b > 0$, and on the half-plane Σ^- we have

$$X^- : \begin{cases} \dot{x} = -y^3, \\ \dot{y} = x^3 + dx^4 y. \end{cases} \tag{10}$$

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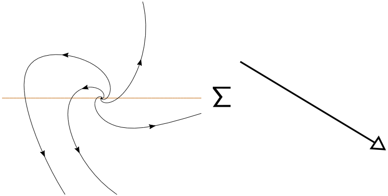
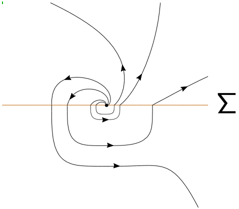
with $b > 0$, and on the half-plane Σ^- we have

$$X^- : \begin{cases} \dot{x} = -y^3, \\ \dot{y} = x^3 + dx^4 y. \end{cases} \tag{10}$$

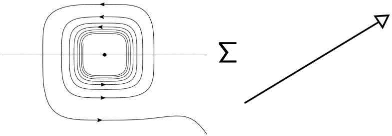
Note that the origin is a monodromic singular point of X^+ and X^-, as the origin has no characteristics directions. By Theorem 2 (iii), the origin is a Σ-monodromic singular point of X, and the trajectories of X are oriented counterclockwise sense.



An unstable focus for X^+ with $a = b = c = 1$



The Elementary-Degenerate case for X with $a = b = c = d = 1$



An unstable weak focus for X^- with $d = 1$

(i) If $a < 0$ the origin is a stable Σ-focus of X;

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(ii) If $a > 0$ the origin is an unstable Σ-focus of X;
(iii) If $a = 0$ and $d > 0$ the origin is a stable Σ-focus of X;

(i) If $a < 0$ the origin is a stable Σ-focus of X;
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(ii) If $a > 0$ the origin is an unstable Σ-focus of X;
(iii) If $a = 0$ and $d > 0$ the origin is a stable Σ-focus of X;
(iv) If $a = 0$ and $d < 0$ the origin is an unstable Σ-focus of X.
(v) If $a = d = 0$ the origin is a Σ-center of X.

References

K. S. Andrade, M. R. Jeffrey, R. M. Martins, and M. A. Teixeira, *On the Dulac's problem for piecewise analytic vector fields*, J. Differential Equations, 266 (2019), pp. 2259–2273.

C. Buzzi, J. C. Medrado and C. Pessoa, *Monodromy and Dulac's Problem for Piecewise analytical planar vector fields*, Preprint, arXiv:2106.09827.

B. Coll, A. Gasull, and R. Prohens, *Degenerate Hopf bifurcations in discontinuous planar systems*, J. Math. Anal. Appl., 253 (2001), pp. 671–690.

M. Esteban, E. Freire, E. Ponce and F. Torres, *On normal forms and return maps for pseudo-focus points*, J. Math. Anal. Appl., 507 (2022), pp. 125774.

D. D. Novaes and L. A. Silva, *Lyapunov coefficients for monodromic tangential singularities in Filippov vector fields*, J. Differential Equations, 300 (2021), pp. 565–596.

Thank you for your attention!