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Fractal analysis of planar nilpotent singularities

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Content

Fractal analysis of bifurcations

- Fractal analysis of bifurcation uses fractal invariants of an orbit to characterize a bifurcation.
- For us basic fractal invariants are box dimension (also called box counting, Minkowski, Bouligand dimension or limit capacity), and Minkowski content.
- We read these fractal invariants (and more!!) from ε-neighborhood of an orbit.

Definition of upper Minkowski content and upper box dimension

- *upper s-dimensional Minkowski content of the bounded set* $A \in \mathbb{R}^N$, $0 \le s \le N$: $\mathcal{M}^{*s}(A) = \limsup_{\varepsilon \to 0} \frac{|A_\varepsilon(A)|}{\varepsilon^{N-s}}$

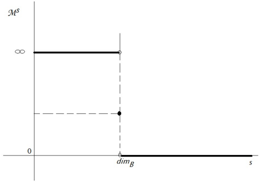


Figure: Minkowski content $\mathcal{M}^{*s}$ as function of $s \in [0, N]$

- *upper box dimension*: $\overline{\dim}_B A = \inf\{s \ge 0 \mid \mathcal{M}^{*s}(A) = 0\}$.

Definition of lower Minkowski content and lower box dimension

- analogously we define lower Minkowski content $\mathcal{M}_*^s$, lower box dimension $\underline{\dim}_B A$
- box dimension $s = \underline{\dim}_B A = \overline{\dim}_B A$
- $F(x)$ and $G(x)$, $F(x) \simeq G(x)$, as $x \to 0$, if exist $C_1, C_2, d > 0$ such that

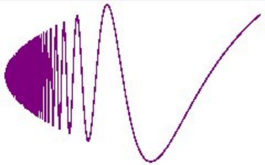
$$C_1 \leq F(x)/G(x) \leq C_2$$

  $x \in (0, d)$, such functions are comparable
- $\mathcal{M}^{*s}, \mathcal{M}_*^s \neq 0, \infty \Rightarrow |A_\varepsilon(A)| \simeq \varepsilon^{N-s}$, otherwise not comparable to any power of ε

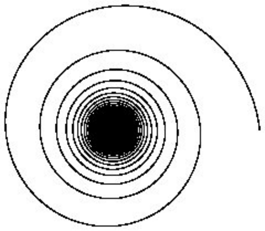
Examples appearing as orbits of dynamical systems

Chirp $f(x) = x^\alpha \sin x^{-\beta}$, $d = \dim_B \Gamma = 2 - \frac{1+\alpha}{1+\beta}$, $0 < \alpha < \beta \leq 1$

Spiral $r = \varphi^{-\alpha}$, $d = \dim_B \Gamma = \frac{2}{1+\alpha}$, $0 < \alpha \leq 1$ Claude Tricot formulas from the book *Curves and Fractal Dimension*, Springer-Verlag New York (1995)



$\dim_B \Gamma = \frac{7}{4}$, $\alpha = 0.5, \beta = 1$



$\dim_B \Gamma = \frac{4}{3}$, $\alpha = 0.5$

Sequence

$S \ldots (\frac{1}{n^\alpha})$ $d = \dim_B S = \frac{1}{1+\alpha}$, $\alpha > 0$

Motivation for studying bifurcations from fractal analysis

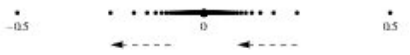
- A natural idea is that "density" of an orbit is related to the quantity and quality of objects which could be produced by perturbation of the system.
  Objects which could be produced in the bifurcation: fixed points, periodic orbits, limit cycles, polycycles.

- We are interested in connections between change of the box dimension and the bifurcation of dynamical systems.

- We are interested in connections between the value of the box dimension of an orbit and the multiplicity of the system near a fixed point or periodic orbit.
- How to read the normal form from ε -neighborhood of an orbit?

Which objects do we study?

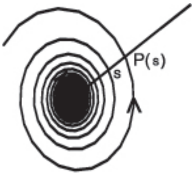
- We study discrete systems by the box dimension of an orbit near the fixed or periodic point



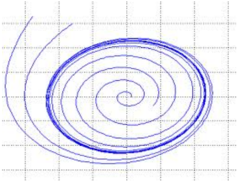
1-dim system generated by $x_{n+1} = x_n - x_n^2, \dim_B S_1 = \frac{1}{2}$

1-dim system generated by $x_{n+1} = x_n - x_n^5, \dim_B S_2 = \frac{4}{5}$

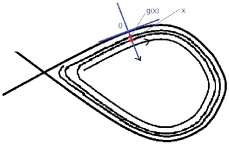
- We study continuous systems by
  -spiral trajectories near a focus, limit cycle and polycycle



focus



limit cycle



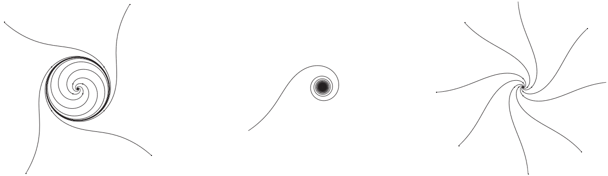
saddle-loop

-discrete systems generated by the Poincaré map
-discrete systems generated by the unit-time map

Weak focus in the normal form [Žubrinić, Ž]

$$\begin{cases} \dot{r} = r(r^{2l} + \sum_{i=0}^{l-1} a_i r^{2i}), \\ \dot{\varphi} = 1. \end{cases} \quad (1)$$

Hopf bifurcation occurs for $l = 1$ if $a_0 = 0$. Hopf-Takens bifurcation occurs for $l > 1$, producing l limit cycles in the system



(1) $a_0 < 0$ strong focus $r = 0$ and the limit cycle $r = \sqrt{-a_0}$. Spiral trajectories are exponential, box dimension equal to 1.
(2) $a_0 = 0$ the origin $r = 0$ is weak focus with $\dim_B \Gamma = 4/3$, where $r = (-2\varphi)^{-1/2}$, see figure in the middle.
(3) $a_0 > 0$ strong focus, exponential spirals, box dimension equal to 1.

Theorem

Γ a part of a trajectory of (1) near the origin.
(a) $a_0 \neq 0$, then the spiral Γ is of exponential type, that is, comparable with $r = e^{a_0 \varphi}$, and hence $\dim_B \Gamma = 1$.
(b) k is fixed, $1 \leq k \leq l$, $a_l = 1$ and $a_0 = \cdots = a_{k-1} = 0$, $a_k \neq 0$. Then Γ is comparable with the spiral $r = \varphi^{-1/2k}$, and

$$d := \dim_B \Gamma = \frac{4k}{2k+1}.$$

Theorem-Classification of analytic nilpotent singularities for planar vector fields, [Dumortier, Llibre, Artès]

Let $(0,0)$ be a singular point of the vector field given by

$$\begin{aligned} \dot{x} &= y + H_1(x,y), \\ \dot{y} &= H_2(x,y), \end{aligned} \tag{2}$$

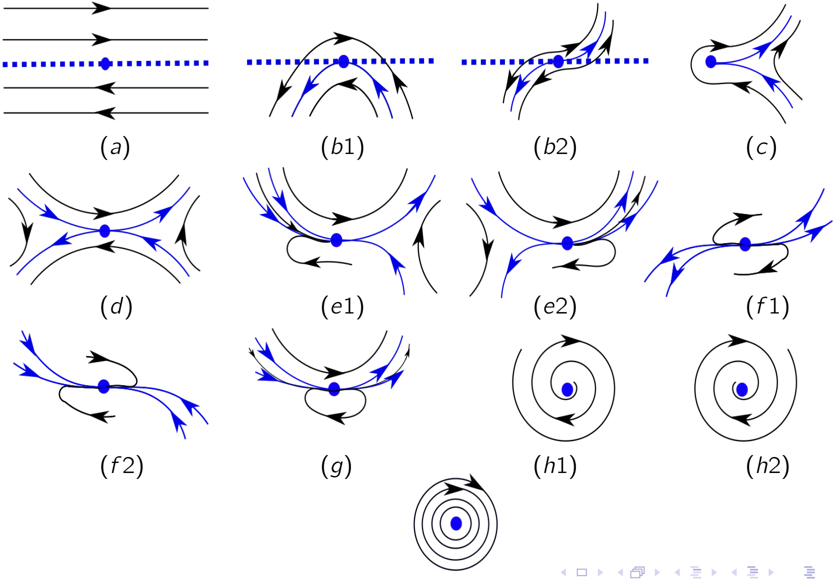
H_1 and H_2 are analytic functions near $(0,0)$, $j_1H_1(0,0) = j_1H_2(0,0) = 0$. Let $y = f(x)$ be the characteristic curve of (2), i.e. the solution of $y + H_1(x,y) = 0$ near $(0,0)$ and consider $F(x) = H_2(x, f(x))$ and $G(x) = (\frac{\partial H_1}{\partial x} + \frac{\partial H_2}{\partial y})(x, f(x))$. Then the following holds:

(1) If $F(x) \equiv G(x) \equiv 0$, then (2) has the curve of singularities $y = f(x)$ passing through the origin (see (a)).

(2) If $F(x) \equiv 0$, $G(x) = bx^n + o(x^n)$ for $n \in$, $n \geq 1$, $b \neq 0$, then (2) has curve of singularities $y = f(x)$ through origin (see (b1) or (b2)).

(3) If $G(x) \equiv 0$ and $F(x) = ax^m + o(x^m)$ for $m \in$, $m \geq 2$, $a \neq 0$, then
(i) If m is odd and $a > 0$, then the origin is a **saddle** (d); and if $a < 0$, then it is a **center or focus** ((h1)–(i));
(ii) If m is even, then the origin is a **cusp** (c).
(4) If $F(x) = ax^m + o(x^m)$ and $G(x) = bx^n + o(x^n)$, $m, n \in$, $m \geq 2$, $n \geq 1$, $a \neq 0$, $b \neq 0$, $\Delta = b^2 + 4a(n+1)$, then we have:
(i) If m is even, and
(i1) $m < 2n+1$, then the origin is a **cusp** (c);
(i2) $m > 2n+1$, then the origin is a **saddle-node** ((e1) or (e2)).
(ii) If m is odd and $a > 0$, then the origin is a **saddle** (d).
(iii) If m is odd, $a < 0$ and
(iii1) Either $m < 2n+1$, or $m = 2n+1$ and $\Delta < 0$, then the origin is a center or a focus ((h1)–(i));
(iii2) n is odd and either $m > 2n+1$, or $m = 2n+1$ and $\Delta \geq 0$, then the phase portrait has **one hyperbolic and one elliptic sector** (g);
(iii3) n is even and $m > 2n+1$ or $m = 2n+1$ and $\Delta \geq 0$, then the origin is a **node** (repelling if $b > 0$, attracting if $b < 0$). See (f1) or (f2).

Phase portraits of nilpotent singularities



The box dimension classification of nilpotent singularities in normal form along separatrices, [HDRVZ]

$m \geq 2, n \geq 1$		$a, b \neq 0$	type	separatrices	box dim.
Hamiltonian like case $m < 2n+1$	m odd	$a > 0$	saddle	$y = \pm\sqrt{\frac{2a}{m+1}x}^{\frac{m+1}{2}} + \dots$	$(\frac{m-1}{m+1}, \frac{m-1}{m+1}, \frac{m-1}{2m})$
		$a < 0$	center/focus	–	–
	m even	$a > 0$	cusp	$y = \pm\sqrt{\frac{2a}{m+1}x}^{\frac{m+1}{2}} + \dots, x > 0$	$(\frac{m-1}{m+1}, \frac{m-1}{m+1}, \frac{m-1}{2m})$
		$a < 0$	cusp	$y = \pm\sqrt{\frac{-2a}{m+1}(-x)}^{\frac{m+1}{2}} + \dots, x < 0$	$(\frac{m-1}{m+1}, \frac{m-1}{m+1}, \frac{m-1}{2m})$
Singular case $m > 2n+1$	m odd	$a > 0$	saddle	$y = \frac{b}{n+1}x^{n+1} + \dots$ $y = -\frac{a}{b}x^{m-n} + \dots$	$(\frac{n}{n+1}, \frac{n}{n+1}, \frac{n}{2n+1})$ $(\frac{m-n-1}{m-n}, \frac{m-n-1}{m-n}, \frac{m-n-1}{2m-2n-1})$
		$a < 0$ n odd	elliptic		
		$a < 0$ n even	node		
	m even	–	saddle-node		
	$m = \infty$	–	curve of sing.	$y = \frac{b}{n+1}x^{n+1} + \dots$	$(\frac{n}{n+1}, \frac{n}{n+1}, \frac{n}{2n+1})$
Mixed case $m = 2n+1$	–	$a > 0$	saddle	$y = \frac{b \pm \sqrt{\Delta}}{2(n+1)}x^{n+1} + \dots$	$(\frac{n}{n+1}, \frac{n}{n+1}, \frac{n}{2n+1})$
	n odd	$a < 0, \Delta \geq 0$	elliptic		
	n even	$a < 0, \Delta \geq 0$	node		
	–	$a < 0, \Delta < 0$	center/focus	–	–
$m = n = \infty$	–	–	curve of sing.	–	–

Table: Box dimension of 2-dim orbit S, the second (resp. third) component is the box dimension of the 1-dim orbit A (resp. B) consisting of the x-components (resp. y-components) of S

Nilpotent focus

We study limit cycles near the origin in an analytic δ-family of planar systems

$$\begin{aligned} \dot{x} &= y + X(x, y, \delta) \\ \dot{y} &= Y(x, y, \delta) \end{aligned} \tag{3}$$

with $\delta = (\delta_1, \delta_2, \dots, \delta_l) \in D \subset \mathbb{R}^l$ where D is a simply connected domain and $X, Y = \mathcal{O}(|x, y|^2)$ for all $\delta \in D$.

- Let $y = f(x, \delta)$ be the characteristic curve of (3)
- define $F(x, \delta) = -Y(x, f(x, \delta), \delta)$
- define $G(x, \delta) = -(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y})(x, f(x, \delta), \delta)$.

Poincaré map on the characteristic curve

Assumptions for F and G for all $\delta \in D$:

$$\begin{aligned} F(x,\delta) &= \sum_{j\geq 2n-1} a_j(\delta)x^j, \;\; n\geq 2, \;\; a_{2n-1}(\delta)>0, \text{ and} \\ G(x,\delta) &= \sum_{j\geq n-1} b_j(\delta)x^j, \;\; b_{n-1}^2(\delta)-4n\, a_{2n-1}(\delta)<0. \end{aligned} \tag{4}$$

- Theorem 1 implies that (3) under these assumptions has a center or a focus at the origin $\forall\delta \in D$.
- By Theorem Han-Romanovski there is an analytic function $P(x_0,\delta)$ in $x_0 = 0$ s.t. partial derivative of P w.r.t. x_0 in $(0,\delta)$ is positive,

$$P(x_0,\delta) = x_0 + \sum_{j\geq 1} v_j(\delta)x_0^j$$

for $|x_0|$ sufficiently small.

Cyclicity of nilpotent focus

Theorem (Han, Romanovski)

Let system (3) satisfy the conditions (4) for all $\delta \in D$. Write $p_n = (1 + (-1)^n)/2$.

(1) If there is an **integer** $k \geq 1$ such that

$$\sum_{j=1}^{k+1} \left| v_{2j-1+p_n} \right| > 0, \quad \forall \delta \in D,$$

then there exists a neighborhood $\mathcal{U}$ of the origin such that the system (3) has **at most** k **limit cycles** in $\mathcal{U}$ for all $\delta \in \bar{D}$, where $\bar{D}$ is any compact subset of D.

(2) If there is $\delta_0 \in D$ such that $v_{2k+1+p_n}(\delta_0) \neq 0$, then for all $\delta \in D$ near δ_0 system (3) has **at most** k **limit cycles** in a neighborhood of the origin.

Cyclicity of nilpotent focus, [HDRVZ]

Theorem

Cyclicity of nilpotent focus and box dimension

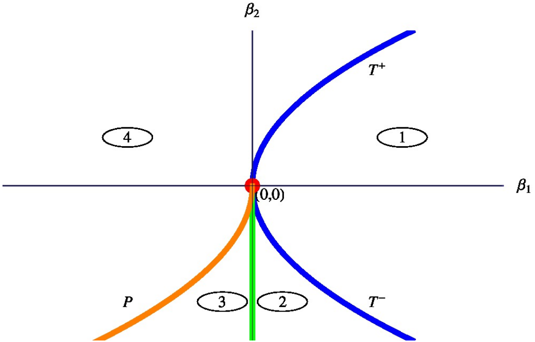
- Let system (3) satisfy the conditions (4) for all $\delta \in D$.
- Let $\Gamma(\delta_0)$ be a spiral trajectory of (3) near the origin for some $\delta_0 \in D$.
- Let $P(x, \delta_0)$ be the Poincaré map of (3) with $\delta = \delta_0$ on the characteristic curve $y = f(x, \delta_0)$.
- Suppose that the sequence $S(x_1) = (x_i)_{i \geq 1}$ defined by $x_{i+1} = P(x_i, \delta_0)$ (a stable focus) or $x_{i+1} = P^{-1}(x_i, \delta_0)$ (an unstable focus), with $x_1 > 0$ small and fixed, has the box dimension equal to $1 - \frac{1}{2k+1}$ **with** n **odd or** $1 - \frac{1}{2k+2}$ **with** n **even** where $k \geq 1$ is an integer.
- Then for all $\delta \in D$ near δ_0 the system (3) has **at most** k **limit cycles** in a neighborhood of the origin.

Bogdanov-Takens bifurcation

Let us consider the normal form for the Bogdanov-Takens bifurcation

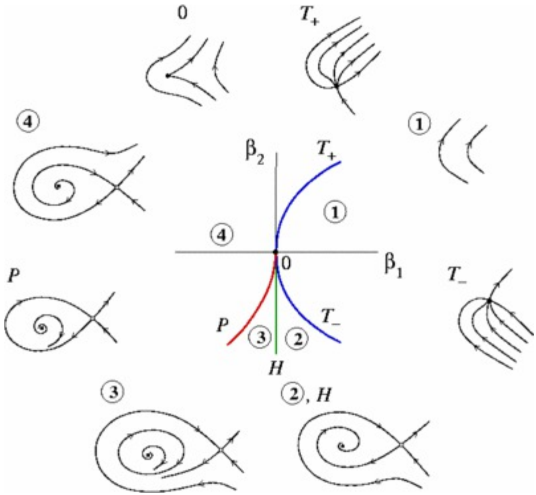
$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= \beta_1 + \beta_2 x + x^2 - xy, \end{aligned} \tag{5}$$

where $\beta_{1,2} \in \mathbb{R}$ are parameters.

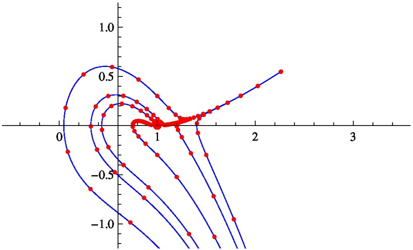


Bifurcation diagram for Bogdanov-Takens bifurcation

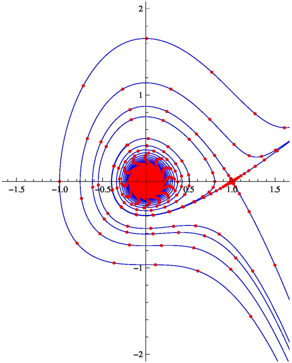
Fractal analysis of Bogdanov-Takens bifurcation



- For $\beta_1 = \beta_2 = 0$ we have a cusp and we get $\dim_B S = \dim_B S_x = 1/3$, $\dim_B S_y = 1/4$ on the separatrices
- At region 1 there are no singularities
- By passing through the curve T^- a saddle and a node appear, it is a saddle-node bifurcation curve. On the center manifold we have $\dim_B S = \dim_B S_x = 1/2$ and $\dim_B S_y = 0$



curve T^-**,** $\dim_B S = 1/2$

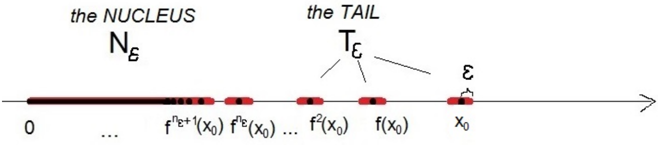


curve H, $\dim_B S = 4/3$

- Somewhere in region 2 the node becomes the focus, at curve H a limit cycle is born- the Hopf bifurcation, the box dimension of a spiral trajectory is 4/3, the box dimension of the Poincaré map is 2/3, the box dimension of a discrete spiral 4/3 (LHD)
- Passing through the curve P saddle homoclinic bifurcation occurs, a saddle-loop appears (box dimension of spiral near saddle-loop, MR, VC)
- In region 4 the saddle-loop is broken and there are two singularities, a saddle and a node
- If we continue the journey clockwise and finally return to region 1, once more the saddle-node bifurcation occurs (curve T^+)
- The box dimension is nontrivial when some local bifurcation occurs
- All hyperbolic cases inside the regions have trivial box dimensions.

Numerical verification of box dimensions from Table 1

To numerically estimate $\dim_B S$ of a planar 2-dimensional orbit S from Theorem 2, we use the decomposition of S into tail and nucleus.



Define $\epsilon_k = \frac{\sqrt{(x_k - x_{k+1})^2 + (y_k - y_{k+1})^2}}{2}$, here $\epsilon_k \to 0$ monotonically. The lower bound on $\underline{\dim}_B S$ and the upper bound on $\overline{\dim}_B S$,

$$\underline{\dim}_B S \geq \lim_{k\to\infty} \left(2 - \frac{\ln\left((k+1)\epsilon_k^2 \pi\right)}{\ln \epsilon_k} \right), \tag{6}$$

$$\overline{\dim}_B S \leq \lim_{k\to\infty} \left(2 - \frac{\ln\left(2\epsilon_k (x_{k+1} + y_{k+1}) + (k+1)\epsilon_k^2 \pi\right)}{\ln \epsilon_k} \right). \tag{7}$$

Numerical verification for cusp (Hamiltonian case)

Setting $\beta_1 = \beta_2 = 0$ in Bogdanov-Takens normal form

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= x^2 - xy. \end{aligned} \tag{8}$$

The first step is to numerically solve the system for the initial condition (x_0, y_0) taken on the separatrix

$$y = -\sqrt{\frac{2}{3}} x^{\frac{3}{2}} + \cdots, x > 0,$$

tending to the origin and given in Table 1. By numerically solving the system (8) we mean computing a finite number K of elements in the orbit $S = \{(x_k, y_k)\}_{k \in \mathbb{N}}$ generated by the unit-time map of (8), which we denote by $S' = \{(x_k, y_k)\}_{1 \le k \le K}$.

Numerical verification for saddle-node (singular case) and saddle

We take the system

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= x^4 + xy, \end{aligned} \tag{9}$$

which is solved numerically in the same sense as the previous system. The initial condition is taken on a separatrix $y = \frac{1}{2}x^2 + \cdots$ or $y = -x^3 + \cdots$ Similarly we solve numerically the system

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= x^3 + xy, \end{aligned} \tag{10}$$

where the initial condition is on the separatrix $y = x^2 + \cdots$

Table

			Num. est. using (6) and (7)	
System and separatrix	Box dim.	Theory value	Lower estimate	Upper estimate
cusp (8), $y = -\sqrt{\frac{2}{3}}x^{\frac{3}{2}} + \cdots$	$\dim_B S$	1/3	0.337205	0.338300
saddle-node (9), $y = \frac{1}{2}x^2 + \cdots$	$\dim_B S$	1/2	0.503807	0.506537
saddle-node (9), $y = -x^3 + \cdots$	$\dim_B S$	2/3	0.666374	0.702302
saddle (10), $y = x^2 + \cdots$	$\dim_B S$	1/2	0.502648	0.505373

Table: Numerically computed estimates for box dimension $\dim_B S$.

All numerical calculations are implemented using Wolfram Mathematica 12.0. At https://github.com /FRABDYN /NilpotentSingularities code is available for download.

Main reference

L. Horvat Dmitrović, R. Huzak, D. Vlah, V. Županović, Fractal analysis of planar nilpotent singularities and numerical applications, Journal of Differential Equations 293(2021) 122

Thank you!