A bifurcation analysis of a contact-based epidemic spreading

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Joint work with A. Arenas, A. Garijo and S. Gómez

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gsd UAB grup de sistemes dinàmics

Setting of the problem

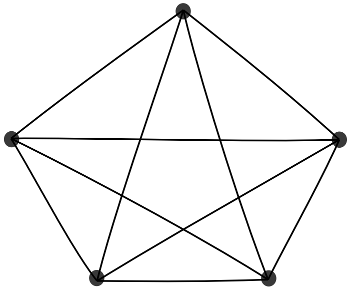
We study a discrete dynamical system that is a mathematical model for the well-known susceptible-infected-susceptible (SIS) epidemic spreading model.

The space:

- We consider a connected undirected network $\mathcal{N}_n$ made up of n nodes, whose weights $r_{ij} \in [0, 1]$ represent the contact probability between the nodes i and j.
- The $n \times n$ contacts matrix $R = (r_{ij})$ is symmetric and irreductible. We also assume the absence of self-loops, thus $r_{ii} = 0$.
- The nodes may stand for persons, cities, countries, airports, train stations, ...

Setting of the problem

First example

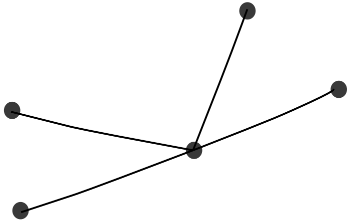


Clique network $\mathcal{C}_5$

$$R = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

Setting of the problem

Second example



Star network $\mathcal{S}_5$

$$R = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Setting of the problem

We define a discrete dynamical system based on the infection process on the network. In the SIS model on networks each node may be in one of two different states: susceptible (healthy) or infected.

The discrete-time dynamic of the SIS makes that, at each time step,

- susceptible nodes may get infected with probability β by contacts with their infected neighbours, while
- infected nodes may recover with probability μ.

We consider that, at each time step, all nodes contact to all their neighbours, known as *reactive* process.

We also add the possibility of one-step reinfections, which means that an infected node that has recovered may become infected by its neighbours within the same time step.

Setting of the problem

Denoting by p_i^k the probability that the node i is infected at the time step k, its evolution is given by the equation

$$p_i^{k+1} = (1 - q_i^k)(1 - p_i^k) + (1 - \mu)p_i^k + \mu(1 - q_i^k)p_i^k \qquad (1)$$

where $q_i^k := \prod_{j=1}^{k}(1 - \beta r_{ij} p_j^k)$ is the probability that the node i is not infected by any neighbour at time step k

The summands account for the three different ways that a node may be infected at time $k+1$:

1. Being susceptible at time k and getting infected at time $k+1$.
2. Being infected and not recovering.
3. Being infected, recover and becoming infected again (one-step reinfection).

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Setting of the problem

Setting $\mathbf{p} = (p_1, p_2, \ldots, p_n) \in \mathbb{R}^n$, this discrete dynamical system is governed by the iteration of the map

$$\begin{aligned} F: \mathbb{R}^n &\to \mathbb{R}^n \\ \mathbf{p} &\mapsto F(\mathbf{p}) \end{aligned}$$

where the i-th component of F is given by

$$F_i(\mathbf{p}) = 1 - \big(1 - (1-\mu)p_i\big)q_i(\mathbf{p})$$

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In other words, if $(p_1^0, \dots, p_n^0)$ is the vector of initial conditions then $(p_1^k, \dots . p_n^k) = F^k(p_1^0, \dots, p_n^0)$ where $F^k = F \circ F^{k-1}$.

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F maps $[0,1]^n$ to $[0,1]^n$ and we restrict the study of the dynamical system generated by F on the compact set $\Omega = [0,1]^n$.

Setting of the problem

Numerical simulations show that these kind of systems, governed by the map F, converge to an asymptotic distribution

$$\lim_{k\to\infty} F^k(\mathbf{p}) = \mathbf{p}^\infty = (p_1^\infty, \ldots, p_n^\infty)$$

independently on the initial condition $\mathbf{p} \in \Omega$. Hence it seems that there exists a fixed point that is a global attractor for the discrete dynamical system under consideration.

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The numerical simulations also show that the location of this global attractor $\mathbf{p}^\infty$ undergoes a bifurcation at $\beta_0 := \frac{\mu}{\rho(R)}$, where $\rho(R)$ is the spectral radius of the matrix R.

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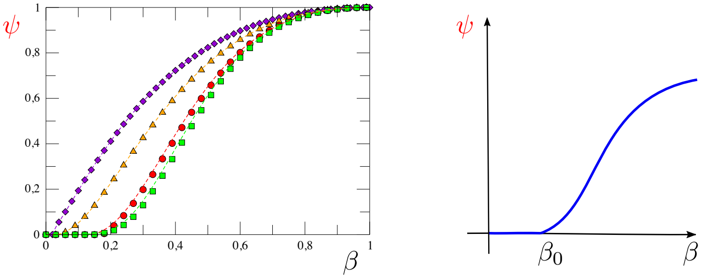
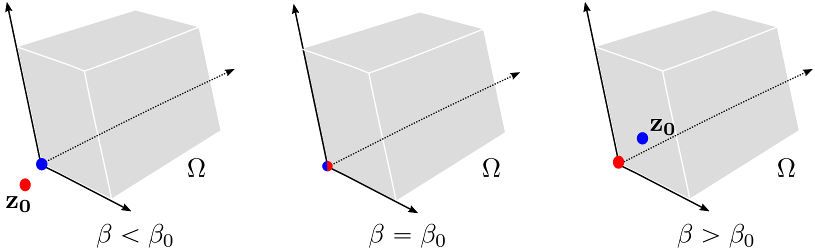


Expected fraction of infected nodes, $\psi := \frac{1}{n}\sum_{i=1}^{n} p_i^{\infty}$, as a function of the infection probability β.

On the left, numerical results by using two different simulation methods and several contact matrices. On the right, sketch showing the epidemic threshold β_0

Main result

The origin $\mathbf{0} = (0, \ldots, 0)$ is a fixed point of F for any $\beta, \mu \in [0,1]$. We prove that for each $\mu \in (0,1)$ this fixed point undergoes a transcritical bifurcation at the epidemic threshold $\beta_0 := \frac{\mu}{\rho(R)}$. Indeed, the origin is a stable fixed point for $\beta < \beta_0$ and, as β tends to β_0, it collides with an unstable fixed point $\mathbf{z_0}$ coming from outside Ω. Then, for $\beta > \beta_0$, the origin is unstable while $\mathbf{z_0}$ is stable and inside Ω.



For $\beta \approx \beta_0$ the fixed point in red is unstable and the one in blue stable.

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Main Theorem

Let us consider a connected undirected network $\mathcal{N}_n$ with associated matrix R and parameters $\beta, \mu \in (0,1)$. Then the following holds:

(a) The origin $\mathbf{0}$ is a fixed point of F for all parameter value and, for each μ, it undergoes a transcritical bifurcation as the β varies through the bifurcation value $\beta_0 := \frac{\mu}{\rho(R)}$.

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(b) If $\beta < \beta_0$ then $\mathbf{0}$ is a stable hyperbolic fixed point of F and $\lim_{k\to\infty} F^k(\mathbf{x}) = \mathbf{0}$ for all $\mathbf{x} \in [0, 1]^n$.

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(c) If $\beta > \beta_0$ then there exists a fixed point $\mathbf{z}_0$ of F in the interior of $[0,1]^n$ that is stable and verifying $\lim_{k\to\infty} F^k(\mathbf{x}) = \mathbf{z}_0$ for all $\mathbf{x} \in [0,1]^n \backslash \{\mathbf{0}\}$. Moreover the map $\beta \mapsto \|\mathbf{z}_0\|_2$ is monotonous increasing.

Sketch of the proof of (a)

Theorem (Sotomayor) (1/2)

Let $f\colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ be a $\mathscr{C}^2$ map verifying the following:

(a) $\mathbf{x}_0$ is a fixed point for all ν, i.e., $f(\mathbf{x}_0;\nu) = \mathbf{x}_0$ for all ν.

(b) The Jacobian matrix of $f(\,\cdot\,;\nu_0)$ evaluated at $\mathbf{x} = \mathbf{x}_0$, that is $D_{\mathbf{x}} f(\mathbf{x}_0;\nu_0)$, has a simple eigenvalue $\lambda = 1$ and all the other eigenvalues have modulus strictly smaller than one.

(c)

C. Robinson, "Dynamical systems. Stability, symbolic dynamics, and chaos", Second edition. Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1999.

Sketch of the proof of (a)

Theorem (Sotomayor) (2/2)

(c) The derivatives

$$\mathbf{w}\left[D_{\mathbf{x}\mathbf{x}} f(\mathbf{x}_0;\nu_0)(\mathbf{v},\mathbf{v})\right] = \sum_{i,j,k=1}^{n} w_k v_i v_j \frac{\partial^2 f_k(\mathbf{x}_0;\nu_0)}{\partial x_i \partial x_j}$$

$$\mathbf{w}\left[D_{\mathbf{x}\nu} f(\mathbf{x}_0;\nu_0)\mathbf{v}\right] = \sum_{i,k=1}^{n} w_k v_i \frac{\partial^2 f_k(\mathbf{x}_0;\nu_0)}{\partial x_i \partial \nu}$$

are different from zero, where $\mathbf{v}$ and $\mathbf{w}$ are respectively the right (column) and left (row) eigenvectors for $\lambda = 1$ of $D_{\mathbf{x}} f(\mathbf{x}_0;\nu_0)$.

Then the discrete dynamical system that yields the iteration of the map $\mathbf{x} \mapsto f(\mathbf{x};\nu)$ undergoes a transcritical bifurcation at the fixed point $\mathbf{x}_0$ as ν varies through the bifurcation value $\nu = \nu_0$.

Sketch of the proof of (b)

Lemma 1

Let D be a convex subset of $\mathbb{R}^n$ and consider a $\mathscr{C}^1$ mapping $G\colon D \to \mathbb{R}^n$ such that $\|DG_x\|_p \leqslant \kappa$ for all $x \in D$. Then $\|G(x) - G(y)\|_p \leqslant \kappa \|x - y\|_p$ for all $x, y \in D$.

Lemma 2

(a) If A and B are nonnegative square matrices with $A \preceq B$ then $\rho(A) \leqslant \rho(B)$ and $\|A\|_2 \leqslant \|B\|_2$.

(b) If A is a nonnegative square matrix then $\rho(Id + A) = 1 + \rho(A)$.

(c) If A is a symmetric matrix then $\|A\|_2 = \rho(A)$.

(d) If A is a nonnegative square matrix then $\rho(A)$ is an eigenvalue of A and there is a nonnegative vector $\mathbf{u} \neq \mathbf{0}$ such that $A\mathbf{u} = \rho(A)\mathbf{u}$. Moreover the algebraic multiplicity of the eigenvalue $\rho(A)$ is 1 in case that A is an irreductible matrix.

Sketch of the proof of (b)

We have

$$\frac{\partial F_i(\mathbf{x})}{\partial x_i} = (1-\mu)q_i(\mathbf{x})$$

$$\frac{\partial F_i(\mathbf{x})}{\partial x_j} = \beta r_{ij}\big(1-(1-\mu)x_i\big)\prod_{\substack{k=1\\k\neq j}}^{n}(1-\beta r_{ik}x_k).$$

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Thus, by applying Lemma 1, F is a contraction on $[0,1]^n$ provided that $\beta < \beta_0 := \frac{\mu}{\rho(R)}$. Then (b) follows by the Contraction Mapping Theorem.

Sketch of the proof of (c)

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- $\{F^k(\mathbf{x}_\varepsilon)\}_{k \in \mathbb{N}}$ converges to a fixed point $\mathbf{z}_0$ of F because each one of the entries is a monotonous increasing sequence of real numbers smaller than 1

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- $\{F^k(\mathbf{x}_\varepsilon)\}_{k\in\mathbb{N}}$ converges to a fixed point $\mathbf{z}_0$ of F because each one of the entries is a monotonous increasing sequence of real numbers smaller than 1
- $\{F^k(\mathbf{1})\}_{k\in\mathbb{N}}$ converges to a fixed point $\mathbf{z}_1$ of F

Sketch of the proof of (c)

- Given $\mathbf{a}, \mathbf{b} \in [0,1]^n$ with $\mathbf{a} \preceq \mathbf{b}$ we define the hypercube

$$\Omega(\mathbf{a}, \mathbf{b}) := \{\mathbf{z} \in [0,1]^n : \mathbf{a} \preceq \mathbf{z} \preceq \mathbf{b}\}$$

- Then $F^k\Big(\Omega(\mathbf{x}_\varepsilon, \mathbf{1})\Big) \subset \Omega\left(F^k(\mathbf{x}_\varepsilon), F^k(\mathbf{1})\right)$ for all $k \in \mathbb{N}$
- $\{F^k(\mathbf{x}_\varepsilon)\}_{k \in \mathbb{N}}$ converges to a fixed point $\mathbf{z}_0$ of F because each one of the entries is a monotonous increasing sequence of real numbers smaller than 1
- $\{F^k(\mathbf{1})\}_{k \in \mathbb{N}}$ converges to a fixed point $\mathbf{z}_1$ of F
- $\bigcap_{k \geqslant 1} F^k\Big(\Omega(\mathbf{x}_\varepsilon, \mathbf{1})\Big) \subset \Omega\left(\mathbf{z}_0, \mathbf{z}_1\right)$

Sketch of the proof of (c)

- Given $\mathbf{a}, \mathbf{b} \in [0,1]^n$ with $\mathbf{a} \preceq \mathbf{b}$ we define the hypercube

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- $\{F^k(\mathbf{x}_\varepsilon)\}_{k\in\mathbb{N}}$ converges to a fixed point $\mathbf{z}_0$ of F because each one of the entries is a monotonous increasing sequence of real numbers smaller than 1
- $\{F^k(\mathbf{1})\}_{k\in\mathbb{N}}$ converges to a fixed point $\mathbf{z}_1$ of F
- $\bigcap_{k\geqslant 1} F^k\Big(\Omega(\mathbf{x}_\varepsilon, \mathbf{1})\Big) \subset \Omega\left(\mathbf{z}_0, \mathbf{z}_1\right)$
- $\varepsilon > 0$ arbitrary $\Rightarrow \bigcap_{k\geqslant 1} F^k\Big([0,1]^n \backslash \{\mathbf{0}\}\Big) \subset \Omega\left(\mathbf{z}_0, \mathbf{z}_1\right)$

Sketch of the proof of (c)

Part (c) will follow if we show that $\mathbf{z}_0 = \mathbf{z}_1$. To this end we appeal to

H. Amann, *Fixed points equations and nonlinear eigenvalue problems in ordered Banach spaces*, SIAM Review (1976)

Theorem 24.3

If E is an ordered Banach space whose positive cone P has nonempty interior, D is a convex subset of E and $f : D \to E$ is a strongly increasing and strongly order concave map with a fixed point $x_0 \in D$, then f has at most one fixed point $\bar{x}$ with $\bar{x} > x_0$.

Thanks very much!!

Have a safe trip back home