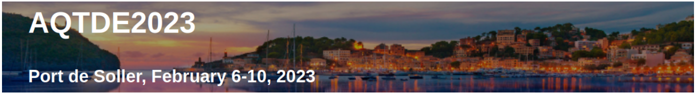
Slow passage through bifurcations by PWL systems

A.E. Teruel



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Slow passage phenomenon

- Given a 1-parameter ODE,

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \mu)$$

consider slow dynamics for the parameter μ with $0 < \varepsilon \ll 1$.

- Fast subsystem ($\varepsilon = 0$), hence μ is constant.
- It is expected the understanding of the full flow, from that of the fast subsystem.

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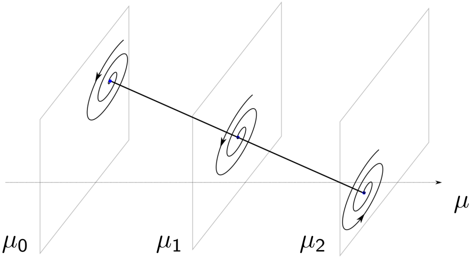

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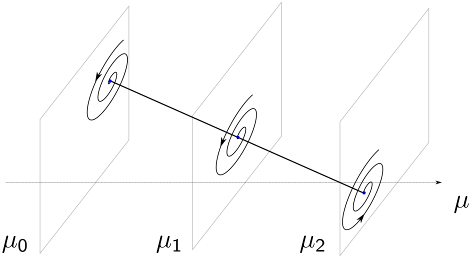

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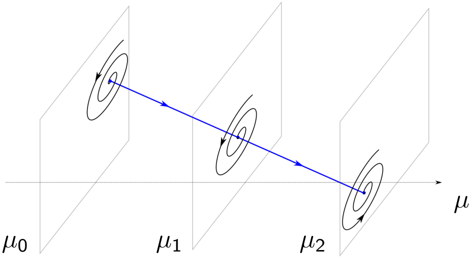

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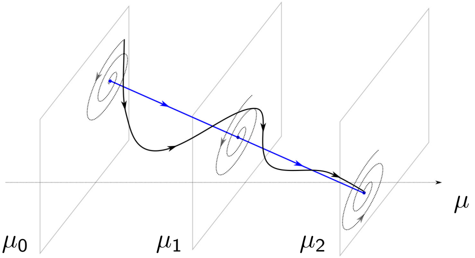
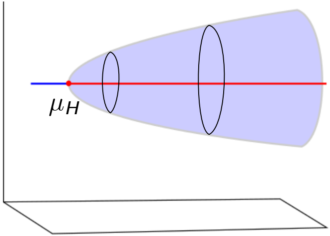


Slow passage phenomenon

- Nevertheless, some unpredictable behaviours can appear when the fast subsystem undergoes a bifurcation.
- Slow passage through a supercritical Hopf bifurcation.



- Maximal delay $z_d(\varepsilon)$: depends on regularity of **F**

$$\lim_{\varepsilon \searrow 0} z_d(\varepsilon) = 0; \quad z_d(\varepsilon) \text{ unbounded}, \forall \varepsilon; \quad \lim_{\varepsilon \searrow 0} z_d(\varepsilon) = C.$$

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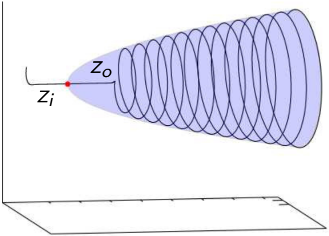

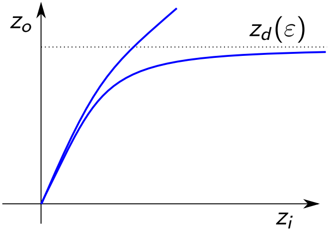

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Slow passage through a transcritical bifurcation

- In A. Pérez,A.E.T: "*Slow passage through a transcritical bifurcation by PWL systems*" work in progress.
- We revisit the transcritical scenario, where y is the bif. parameter.

$$\dot{x} = x^2 - y^2 + \varepsilon\lambda,$$
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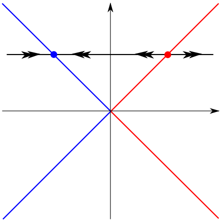

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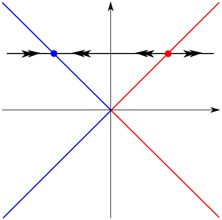

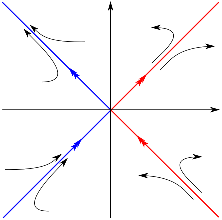

Slow passage through a transcritical bifurcation

- Slow passage through a PWL transcritical bifurcation

$$\dot{x} = |x| - |y| + \varepsilon\lambda,$$
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  Lipschitz vector field consisting of 4 four linear systems each defined in a quadrant.

- The first coordinate of the flow $\varphi(t;(x_0,y_0),\lambda,\varepsilon)$ is locally given by:

$$\varphi_1(t;(x_0,y_0),\lambda,\varepsilon) =$$
$$\begin{cases} (x_0 - y_0 + (\lambda-1)\varepsilon)e^{t} + \varepsilon t + y_0 - (\lambda-1)\varepsilon & (x_0,y_0)\in Q_1, \\ (x_0 + y_0 + (1+\lambda)\varepsilon)e^{t} - \varepsilon t - y_0 - (\lambda+1)\varepsilon & (x_0,y_0)\in Q_2, \\ (x_0 - y_0 + (1-\lambda)\varepsilon)e^{-t} + \varepsilon t + y_0 + (\lambda-1)\varepsilon & (x_0,y_0)\in Q_3, \\ (x_0 + y_0 - (1+\lambda)\varepsilon)e^{-t} - \varepsilon t - y_0 + (\lambda+1)\varepsilon & (x_0,y_0)\in Q_4, \end{cases}$$

- whereas the second one is

$$\varphi_2(t;(x_0,y_0),\lambda,\varepsilon) = y_0 + \varepsilon t.$$

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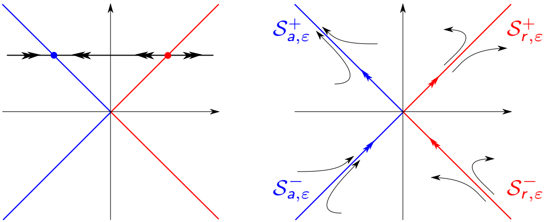
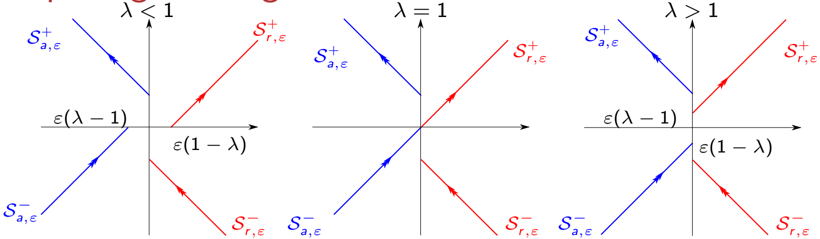


- $\mathcal{S}^-_{a,\varepsilon} = \{(x,y) : y = x + \varepsilon(1-\lambda), \quad x < \min\{0, \varepsilon(\lambda - 1)\}\}$
- $\mathcal{S}^+_{a,\varepsilon} = \{(x,y) : y = -x + \varepsilon(1+\lambda), \; x < 0\}$
- $\mathcal{S}^-_{r,\varepsilon} = \{(x,y) : y = -x - \varepsilon(1+\lambda), \; x < 0\}$
- $\mathcal{S}^+_{r,\varepsilon} = \{(x,y) : y = x - \varepsilon(1-\lambda), \quad x > \max\{0, \varepsilon(1 - \lambda)\}\}$

Slow passage through a transcritical bifurcation



$$\Delta^{in} = \{(-\rho, -\rho + \varepsilon(1-\lambda) + \delta)\} \quad \begin{array}{l} \xrightarrow{\Pi_a} \quad \Delta^{out}_a = \{(-\rho, \rho + \varepsilon(1+\lambda) + \delta)\} \\ \xrightarrow{\Pi_e} \quad \Delta^{out}_e = \{(\rho, \delta)\} \end{array}$$

Theorem: *Fixed $\lambda > 0$, exists $\varepsilon_0(\lambda) > 0$ such that for $\varepsilon \in (0, \varepsilon_0(\lambda)]$.*

a) *If $\lambda > 1$: $\mathcal{S}^-_{a,\varepsilon}$ passes through Δ^{out}_e at $(\rho, h(\varepsilon))$ where $h(\varepsilon) = O(\varepsilon|\ln(\varepsilon)|)$. Π_a maps Δ^{in} to an interval containing $\mathcal{S}^-_{a,\varepsilon} \cap \Delta^{out}_e$ of size $O(e^{-c/\varepsilon})$.*

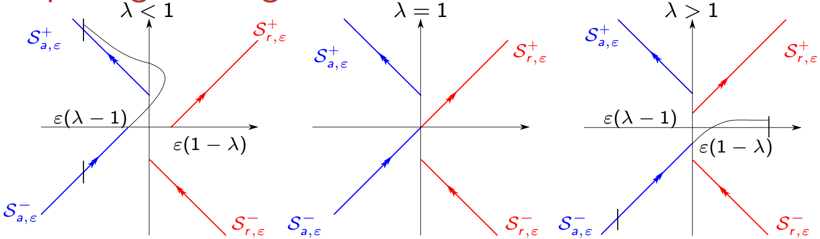
b) *If $\lambda < 1$: Π_e maps Δ^{in} to an interval cont. $\mathcal{S}^+_{a,\varepsilon} \cap \Delta^{out}_a$ of size $O(e^{-c/\varepsilon})$.*

$$\varepsilon_0(\lambda) = \begin{cases} \frac{e^{1-C(\lambda)}}{C(\lambda)-1} & \lambda < 1, \\ \left(\frac{2(e^{\lambda-1}-1)}{\rho+\lambda-1}\right)^{\frac{1}{2}} & \lambda > 1, \end{cases}$$

where $\lim_{\lambda\to 1} C(\lambda) = +\infty$

and $\lim_{\lambda\to 1} \varepsilon_0(\lambda) = 0$.

Slow passage through a transcritical bifurcation



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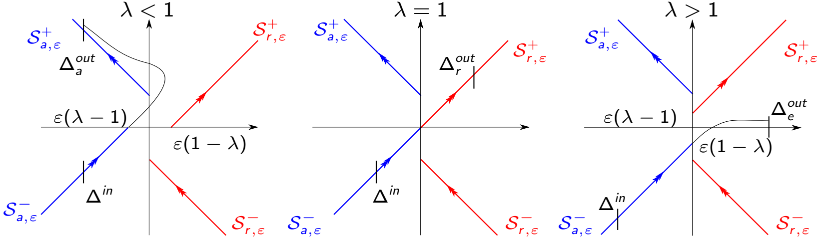
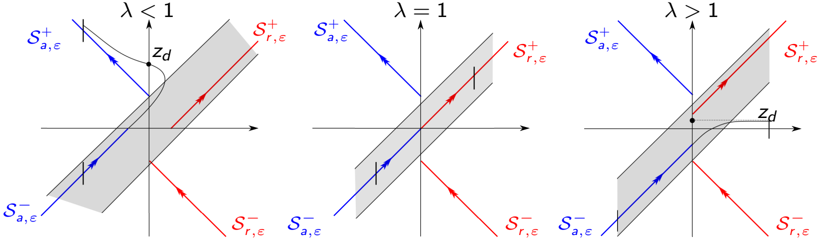


$$\Delta^{in} = \{(-\rho, -\rho + \varepsilon(1-\lambda) + \delta)\} \quad \xrightarrow{\Pi_r} \quad \Delta^{out}_r = \{(\rho - 2|r|, \rho - 2|r| + \delta)\}$$

Theorem: *In the degenerate case* $\lambda = 1$ *transition map* Π_r *maps* Δ^{in} *to* Δ^{out}_r.

Way-in/way-out function and maximal delay



Theorem: *Fixed $\lambda > 0$.*

a) *If $\lambda > 1$, then maximal delay is $z_d(\varepsilon) = O(\varepsilon \ln(|\varepsilon|))$.*

b) *If $\lambda < 1$, then maximal delay is $z_d(\varepsilon) = \varepsilon(\lambda + c)$ with $c \in (1, C(\lambda))$ where $C(\lambda)$ is a differentiable function satisfying:*

$$C(\lambda) > 1, \quad C'(\lambda) > 0, \quad \lim_{\lambda \nearrow 1} C(\lambda) = +\infty, \quad \lim_{\lambda \nearrow 1} C'(\lambda) = +\infty.$$

c) *If $\lambda = 1$, then maximal delay is unbounded.*

Maximal delay: Trivial behaviour.

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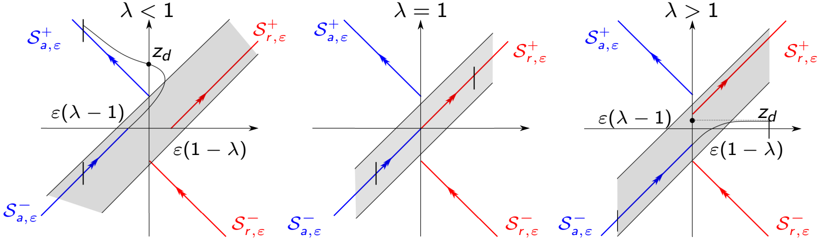

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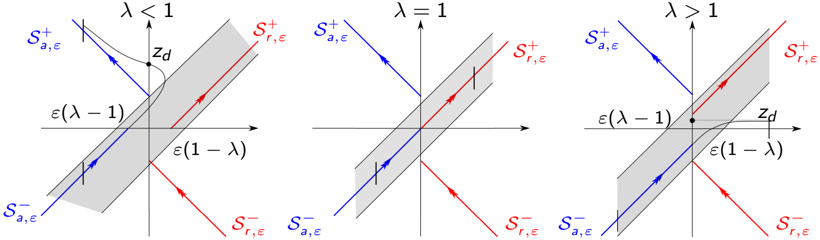

Assuming $\lambda = 1 \pm e^{-\frac{c}{\varepsilon}}$ slow manifolds $\mathcal{S}^-_{a,\varepsilon}$, $\mathcal{S}^+_{r,\varepsilon}$ are exponentially close.

Theorem:

a) *If $\lambda = 1 - e^{-\frac{c}{\varepsilon}}$, then the maximal delay $z_d(\varepsilon) = c + \varepsilon h(\varepsilon)$ with $\lim_{\varepsilon \searrow 0} \varepsilon h(\varepsilon) = 0$.*

b) *If $\lambda = 1 + e^{-\frac{c}{\varepsilon}}$, then the maximal delay $z_d(\varepsilon) = c + \varepsilon g(\varepsilon)$ with $\lim_{\varepsilon \searrow 0} \varepsilon g(\varepsilon) = 0$.*

Maximal delay: Non trivial behaviour.

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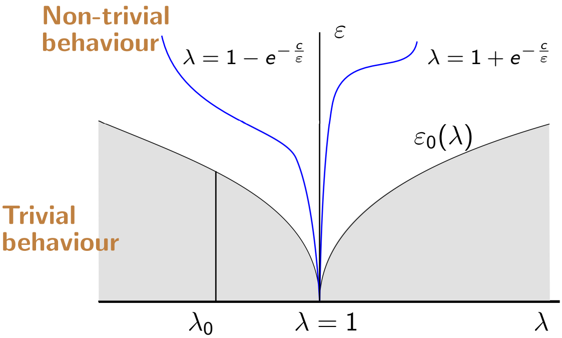

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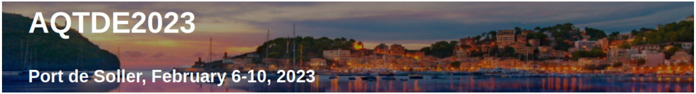

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