Systems and equations with few monomials and their limit cycles

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Based on works done in collaboration with J.L. Bravo (UEX), M. Fernández (UEX), A. Gasull (UAB) and R. Prohens (UIB).

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Motivation

When studying polynomial differential systems one focuses on their degree.

One of the most important problems in the area is Hilbert 16th problem, that has the degree as its only data:

$$\mathcal{H}(2) = 4?, \quad \mathcal{H}(3) \geq 13, \quad \mathcal{H}(n) \geq \frac{n^2 \ln(n)}{2 \ln 2} \text{ for } n \text{ large }.$$

The higher the degree, the greater the variety and richness of behaviors.

But for some simple equations, the degree is no so decisive when studying the number of limit cycles.

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But for some simple equations, the degree is no so decisive when studying the number of limit cycles.

Main objective of the talk

Analyzing the role of monomials in the generation of limit cycles in systems of low dimension.

We will focus on equations in the complex plane of the form

$$\dot{z} = az^k\bar{z}^l + bz^m\bar{z}^n, \quad (1)$$

with $z \in \mathbb{C}$, $k, l, m, n \in \mathbb{Z}^+ \cup \{0\}$, $0 \leq k + l < m + n$, and $a, b \in \mathbb{C} \setminus \{0\}$

and on equations in the cylinder of the form

$$\dot{x} = A(t)x^m + B(t)x^n, \quad (2)$$

with $x \in \mathbb{R}$, $m, n \in \mathbb{Z}^+ \setminus \{1\}$, and $A(t), B(t)$ trigonometric functions.

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Definitions

- A simple critical point is a critical point for which the determinant of its associated Jacobian matrix is nonzero.
- A periodic orbit $\gamma(t)$ is a solution for which there exists $T \in \mathbb{R}^+$ such that $\gamma(T+t) = \gamma(t)$, for all t.
- A center is a critical point, in the planar case, or a periodic orbit, in the case of the cylinder, having a neighborhood such that all the solutions are periodic.
- A limit cycle is an isolated periodic orbit.
- In the analytic setting, limit cycles are periodic solutions that are isolated in the set of all the periodic orbits of the equation.

Contents

What is known in the complex plane

One monomial case

Consider

$$\dot{z} = az^k\bar{z}^l,$$

$z \in \mathbb{C}$, $k, l \in \mathbb{Z}^+ \cup \{0\}$ and $a \in \mathbb{C}\backslash\{0\}$.
The equation with one monomial does not have limit cycles.

Three monomial case

$$\dot{z} = az + bz^k\bar{z}^l + cz^m\bar{z}^n,$$

$z \in \mathbb{C}$, $k, l, m, n \in \mathbb{Z}^+ \cup \{0\}$, $0 \le k + l < m + n$, and $a, b, c \in \mathbb{C}\backslash\{0\}$.
For each $3 \le p \in \mathbb{N}$ there exists an equation with three monomials of the above type having at least p limit cycles, see [Gasull-Li-Torregrosa].

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The two monomial case in the complex plane

We have studied the family

$$\dot{z} = az^k\bar{z}^l + bz^m\bar{z}^n, \tag{1}$$

with $0 \leq k+l < m+n$, and $a, b \in \mathbb{C}\backslash\{0\}$.

Equation (1) written in $x = \mathrm{Re}(z), y = \mathrm{Im}(z)$ variables is not as simple.

For instance: $\dot{z} = az^3 + bz\bar{z}^3$ is

$$\begin{cases} \dot{x} = b_1x^4 - b_1y^4 + 2b_2x^3y + 2b_2xy^3 + a_1x^3 - 3a_1xy^2 - 3a_2x^2y + a_2y^3, \\ \dot{y} = -2b_1x^3y - 2b_1xy^3 + b_2x^4 - b_2y^4 + 3a_1x^2y - a_1y^3 + a_2x^3 - 3a_2xy^2. \end{cases}$$

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The two monomial case in the complex plane

Equation (1), written in (x, y) variables is the sum of two homogeneous vector fields

$$\begin{cases} \dot{x} = P_p(x, y) + P_q(x, y), \\ \dot{y} = Q_p(x, y) + Q_q(x, y), \end{cases}$$

P_i, Q_i being homogeneous polynomials of degree i.

There is no upper bound for the number of limit cycles for the previous family of equations. For instance, for $p = 1, q = n$ odd, there exist vector fields of the above form having $\frac{n+1}{2}$ limit cycles.

The two monomial case in the complex plane

Theorem (MJA - A. Gasull - R. Prohens)

The differential equation

$$\dot{z} = az^k\bar{z}^l + bz^m\bar{z}^n, \tag{1}$$

with $0 \leq k + l < m + n$, *and* $a, b \in \mathbb{C}\setminus\{0\}$, *has at most one limit cycle and it exists if and only if* $k - l = m - n = 1$, $\operatorname{Re}(a)\operatorname{Re}(b) < 0$ *and* $a/b \notin \mathbb{R}^-$. *Moreover, it is the circle* $x^2 + y^2 = (-\operatorname{Re}(a)/\operatorname{Re}(b))^{n-l}$ *and it is hyperbolic.*

Ideas of the proof of the main theorem of the planar equation: $\dot{z} = az^k\bar{z}^l + bz^m\bar{z}^n$

Characterizing the critical points

Set $q = l - k + m - n$. Then:

- The origin is a critical point if and only if $k + l > 0$. Its index is $k - l$ and it has $2(k - l) - 2$ elliptic sectors when $k - l > 1$, it is a node, focus or center when $k - l = 1$, and it has $2|k - l| + 2$ hyperbolic sectors when $k - l \leq 0$.
- If $q \neq 0$, it has $|q|$ nonzero simple critical points, and all of them are located on a circle centered at the origin. When $q > 0$ (resp. $q < 0$) all them are anti-saddles (resp. saddles).
- If $q = 0$ and $a/b \in \mathbb{R}^-$, it has a circle centered at the origin filled with critical points.
- If $q = 0$ and $a/b \notin \mathbb{R}^-$, it does not have nonzero critical points.

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The main difficulty is proving that most cases do not have limit cycles.

There can exist two kind of limit cycles:

(I) the ones surrounding a nonzero critical point of index +1.

(II) The ones surrounding the origin and, eventually, also other critical points.

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(I) Limit cycles surrounding a nonzero critical point of index +1: : $\dot{z} = az^k\bar{z}^l + bz^m\bar{z}^n$

With a result about Rotated Vector Fields, one rules out (I) possibility.

Proposition (RVF)

Let the origin be a center for a smooth differential equation $\dot{z} = iF(z,\bar{z})$ and let $\mathcal{U}$ be its period annulus.
Then for $\delta \notin \{\pi/2, -\pi/2\}$ the differential equation $\dot{z} = e^{i\delta}F(z,\bar{z})$ has not periodic orbits intersecting the set $\mathcal{U}$. Moreover, if F is analytic it does not have periodic orbits surrounding only the origin.

Equation (1) can be transformed into

$$\dot{z} = e^{\delta i}\big((z+1)^k(\bar{z}+1)^l - (z+1)^m(\bar{z}+1)^n\big).$$

This equation has a center for $\delta \in \{\pi/2, -\pi/2\} \Rightarrow$ no limit cycle exist for any δ.

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(II) Limit cycles surrounding the origin and, eventually, also other critical points: $\dot{z} = az^k\bar{z}^l + bz^m\bar{z}^n$

a) With a generalization of a result about quadratic systems, one rules out the case $k-l$ even or $m-n$ even.

Proposition

Consider the differential equation

$$\dot{z} = X_N(z,\bar{z}) + X_M(z,\bar{z}),\ 0 \le N < M,$$

where, X_j is a homogeneous vector field of degree j in the variables z and $\bar{z}$. If one of the following conditions hold:

(i) *The differential equation $\dot{z} = X_N(z,\bar{z})$ has an invariant straight line through the origin and M is even,*

(ii) *The differential equation $\dot{z} = X_M(z,\bar{z})$ has an invariant straight line through the origin and N is even,*

then it does not have periodic orbits surrounding the origin.

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b) $k - l$ and $m - n$ odd, by symmetry arguments, if a limit cycle exists, it surrounds only the origin (and hence $k - l = 1$) or it surrounds all the critical points ($m - n = 1$). If one is different from 1, then Proposition on RVF applies and no limit cycles exist.

c) $k - l = m - n = 1$ the system can have a circle of critical points ($a/b \in \mathbb{R}^-$) or not ($a/b \notin \mathbb{R}^-$). If it has it, eliminating it the system becomes homogeneous and no limit cycle can exist.
If the system does not have a circle of critical points, then it has a limit cycle if and only if $\mathrm{Re}(a)\,\mathrm{Re}(b) < 0$.

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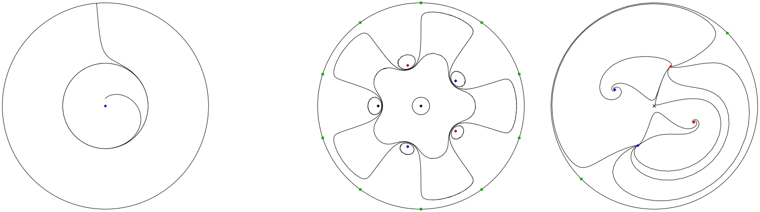
Examples

The equation $\dot{z} = (-1 + I)z^2\bar{z} - z^4\bar{z}^3$, that in x, y variables writtes as

$$\begin{cases} \dot{x} = b_1x^7 + 3b_1x^5y^2 + 3b_1x^3y^4 + b_1xy^6 - b_2x^6y - 3b_2x^4y^3 - 3b_2x^2y^5 - b_2y^7 + a_1x^3 + a_1xy^2 - a_2x^2y - a_2y^3, \\ \dot{y} = b_1x^6y + 3b_1x^4y^3 + 3b_1x^2y^5 + b_1y^7 + b_2x^7 + 3b_2x^5y^2 + 3b_2x^3y^4 + b_2xy^6 + a_1x^2y + a_1y^3 + a_2x^3 + a_2xy^2, \end{cases}$$

has a unique limit cycle surrounding the origin if $a/b \notin \mathbb{R}^-$ and $a_1/b_1 < 0$.



Byproduct: the center-focus problem

Theorem

For equation

$$\dot{z} = az^k\bar{z}^l + bz^m\bar{z}^n, \qquad (1)$$

set $q = l - k + m - n$*. Then, the following holds:*

(i) *When* $m - n = 1$ *the origin is a center if and only if* $k - l = 1$ *and* $\mathrm{Re}(a) = \mathrm{Re}(b) = 0$.

(ii) *When* $m - n \neq 1$ *the origin is a center if and only if* $k - l = 1$ *and* $\mathrm{Re}(a) = 0$.

(iii) *It has a nonzero center at* $z = Re^{i\psi}$ *if and only if the point has index* $+1$ *and the divergence vanishes at this point. More concretely, if and only if* $q > 0$ *and* $\mathrm{Re}(ae^{i(k-l-1)\psi}) = 0$*. This later condition is equivalently to* $\mathrm{Re}(be^{i(m-n-1)\psi}) = 0$.

Moreover all centers are reversible.

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Moreover all centers are reversible.

What about the degree?

$$\dot{z} = az^k\bar{z}^l + bz^m\bar{z}^n, \qquad (1)$$

Nor the lower degree, $k+l$, nor the higher degree, $m+n$, plays a role in solving the problem.

Contents

What is known in the cylinder: simplest cases

Consider $\dot{x} = f(t,x)$ being f periodic in t and polynomial in x. Let $n = deg(f)$ in x variable.

What is known

- Linear case ($n = 1$): the number of limit cycles is ≤ 1.
- Riccati case ($n = 2$): the number of limit cycles is ≤ 2.
- Abel case ($n = 3$): for each $p \in \mathbb{N}$ there are trigonometric functions $A(t), B(t)$, such that the equation

$$\dot{x} = A(t)x^3 + B(t)x^2$$

has p limit cycles.

Approaches to the $n \geq 3$ case.

Consider the generalized trigonometric Abel equation

$$\dot{x} = A(t)x^m + B(t)x^n, \quad (2)$$

$m > n \geq 2$.

Strategies to bound the number of limit cycles

- Bounding the functions $|A(t)|, |B(t)|$.
- Fixing signs of the functions $A(t), B(t)$.
- Fixing the degree of the trigonometric polynomials $A(t), B(t)$. Open problem even for degree 1.
- Studying the effects of the trigonometric monomials.

Effect of the trigonometric monomials

If $A_1, \ldots, A_l, B_1, \ldots, B_p$, are trigonometric monomials, we write eq. (2) as

$$\dot{x} \quad = \quad \left(\sum_{k=1}^{l} a_k A_k(t) \right) x^m + \left(\sum_{k=1}^{p} b_k B_k(t) \right) x^n, \qquad (2)$$

where $A(t) = \sin^{i_k} t \cos^{j_k} t, B_k = \sin^{u_k} t \cos^{v_k} t.$

Denote by $\mathcal{H}$ the Hilbert number of Eq. (2), *i.e.*, the supremum (it could be infinite) over the set of coefficients $(a_1, \ldots, a_l, b_1, \ldots, b_p) \in \mathbb{R}^{l+p}$ of its number of limit cycles.

Observe that $x = 0$ is always a periodic orbit of Equation (2).

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The context of our problem

The center problem

The case $\mathcal{H} = 0$ is the center problem (every bounded solution is periodic for all coefficients a_k, b_k).

Uniqueness of limit cycle

$\mathcal{H} = 1$ means that, for any a_k, b_k the origin is the only limit cycle, except for some of them, for which it is a center, *i.e.* if equation (2) has a limit cycle it must be the trivial one.

Non-uniqueness of limit cycles

$\mathcal{H} \geq 2$ means that there exist some values of a_k, b_k such that equation (2) has a non-trivial limit cycle.

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Notation

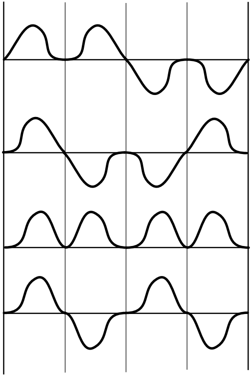
$$\dot{x} = \left(\sum_{k=1}^{l} a_k \sin^{i_k} t \cos^{j_k} t \right) x^m + \left(\sum_{k=1}^{p} b_k \sin^{u_k} t \cos^{v_k} t \right) x^n. \quad (2)$$

$$\mathcal{S} = \{\sin^i(t)\cos^j(t) : i \text{ odd}, j \text{ even}\}$$

$$\mathcal{C} = \{\sin^i(t)\cos^j(t) : i \text{ even}, j \text{ odd}\},$$

$$\mathcal{E} = \{\sin^i(t)\cos^j(t) : i, j \text{ even}\},$$

$$\mathcal{O} = \{\sin^i(t)\cos^j(t) : i, j \text{ odd}\}.$$



Few monomials result

Consider the generalized trigonometric Abel equation

$$\dot{x} = A(t)x^m + B(t)x^n = \left(\sum_{k=1}^{l} a_k A_k(t)\right) x^m + \left(\sum_{k=1}^{p} b_k B_k(t)\right) x^n, \qquad (2)$$

$m > n \geq 2$.

Lemma (MJA - J.L. Bravo - M. Fernández)

In order to characterize the center problem and the uniqueness or non-uniqueness of limit cycles of the generalized trigonometric Abel equation (2) is enough with studying the equation

$$\dot{x} = (a_1 A_1(t) + a_2 A_2(t))x^m + (b_1 B_1(t) + b_2 B_2(t))x^n,$$

with $A_k(t) = \sin^{i_k} t \cos^{j_k} t, B_k(t) = \sin^{u_k} t \cos^{v_k} t.$

Reduction of the problem

Consider

$$\dot{x} = A(t)x^m + B(t)x^n. \tag{2}$$

$$\dot{x} = (a_1A_1(t) + a_2A_2(t))x^m + b_1B_1(t)x^n, \tag{3}$$
$$\dot{x} = (a_1A_1(t) + a_2A_2(t))x^m + (b_1B_1(t) + b_2B_2(t))x^n. \tag{4}$$

To study the problems $\mathcal{H} = 0, \mathcal{H} = 1, \mathcal{H} \geq 2$, for Equation (2) it is enough to study Equations (3) and (4).

- If equation (3) has a non-trivial limit cycle then it will also exist for equation (2).
- If equation (3) does not have a non-trivial limit cycle then we have to study equation (4).

Main result in the cylinder for generalized Abel equation

$$\dot{x} = \left(\sum_{k=1}^{l} a_k A_k(t)\right) x^m + \left(\sum_{k=1}^{p} b_k B_k(t)\right) x^n. \qquad (2)$$

Theorem (MJA - J.L. Bravo - M. Fernández - R. Prohens)

Consider equation (2) not including $(SOCO)$ family.

1. $\mathcal{H} = 0$ *if and only if* $A_i, B_j \in \mathcal{S} \cup \mathcal{O}$ *(resp.* $\mathcal{C} \cup \mathcal{O}$*) for any* $1 \leq i \leq l$, $1 \leq j \leq p$.
2. $\mathcal{H} = 1$ *if and only if one of the following conditions holds:*
   - a. $A_1 \in \mathcal{E}$ *and* $A_i, B_j \in \mathcal{S} \cup \mathcal{O}$ *(resp.* $\mathcal{C} \cup \mathcal{O}$*) for any* $2 \leq i \leq l$, $1 \leq j \leq p$.
   - b. $l = p = 2$, $A_1 = B_1$, $A_2 = B_2$, $A_1 \in \mathcal{S}$ *and* $A_2 \in \mathcal{C}$.
   - c. $p = 1$, $B_1 \in \mathcal{E}$, *and* $A_i \notin \mathcal{E}$, $1 \leq i \leq l$.
   - d. $p = 1$, $B_1 \in \mathcal{C}$, $A_1 \in \mathcal{S}$, *and* $A_2, \ldots, A_l \in \mathcal{C} \cup \mathcal{O}$.
   - e. $p = 1$, $B_1 \in \mathcal{S}$, $A_1 \in \mathcal{C}$, $A_2, \ldots, A_l \in \mathcal{S} \cup \mathcal{O}$.
   - f. $p = 1$, $B_1 \in \mathcal{O}$, $A_1 \in \mathcal{S}$, $A_2 \in \mathcal{C}$, $A_3, \ldots, A_l \in \mathcal{O}$.
3. $\mathcal{H} \geq 2$ *otherwise.*

Ideas of the proof

Steps of the proof

- Determine all the sets $\{A_k, B_k\}$ for which there exist coefficients a_k, b_k such that Eq. (2) has a non-trivial limit cycle.
- Prove that for each of the remaining sets $\{A_k, B_k\}$ none of the equations inside the family (2) has a non-trivial limit cycle.

Techniques used

- Lyapunov constants.
- Hopf bifurcation.
- Moment characterization of centers: if $x = 0$ is a center then

$$\int_0^{2\pi} A(t)\,(I_B(t))^j\,dt = 0 \quad \text{for every } j \in \mathbb{Z}^+.$$

- Monotonicity of the solutions.

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- Monotonicity of the solutions.

The ruled out SOCO family

Consider the following family,

$$\dot{x} = \underset{\mathcal{S}}{(a_1 A_1(t)} + \underset{\mathcal{O}}{a_2 A_2(t))} x^m + \underset{\mathcal{C}}{(b_1 B_1(t)} + \underset{\mathcal{O}}{b_2 B_2(t))} x^n,$$

Theorem

Let us consider the SOCO equation with an additional technical hypothesis (H). There exist a_1, a_2, b_1, b_2 such that the equation has non-trivial limit cycle (an Alien one).

$$\frac{d}{dt}\left(\ln \frac{A_2(t)}{B_2(t)}\right) \in \mathbb{R} \iff \frac{A_2(t)}{B_2(t)} \in (0, \infty). \qquad \text{(H)}$$

Conclusions

$$\dot{z} = az^k\bar{z}^l + bz^m\bar{z}^n, \qquad (1)$$

has at most 1 limit cycle, independently of the degree,

$$\dot{x} = A(t)x^m + B(t)x^n, \qquad (2)$$

the existence of limit cycles can be characterized independently of the degree (of both, x and the trigonometric polynomials).

Summarizing

For some simple equations or systems the degree is not decisive.

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Summarizing

For some simple equations or systems the degree is not decisive.

Systems and equations with few monomials and their limit cycles

M.J. Álvarez (Universitat de les Illes Balears)