Distribution of limit cycles in quadratic systems

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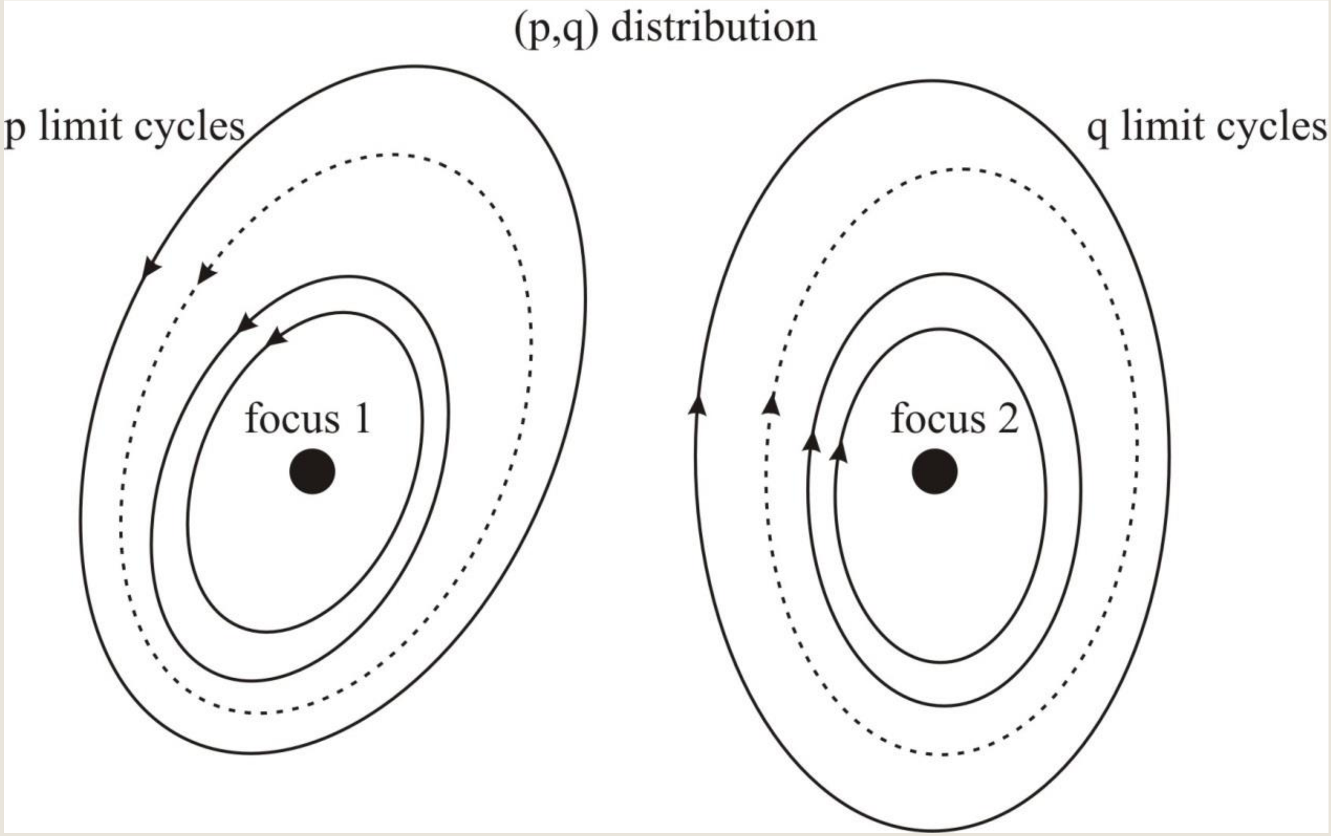
What is a quadratic system QS?

$$\frac{dx}{dt} = a_{00} + a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + a_{02}y^2$$
$$\frac{dy}{dt} = b_{00} + b_{10}x + b_{01}y + b_{20}x^2 + b_{11}xy + b_{02}y^2$$

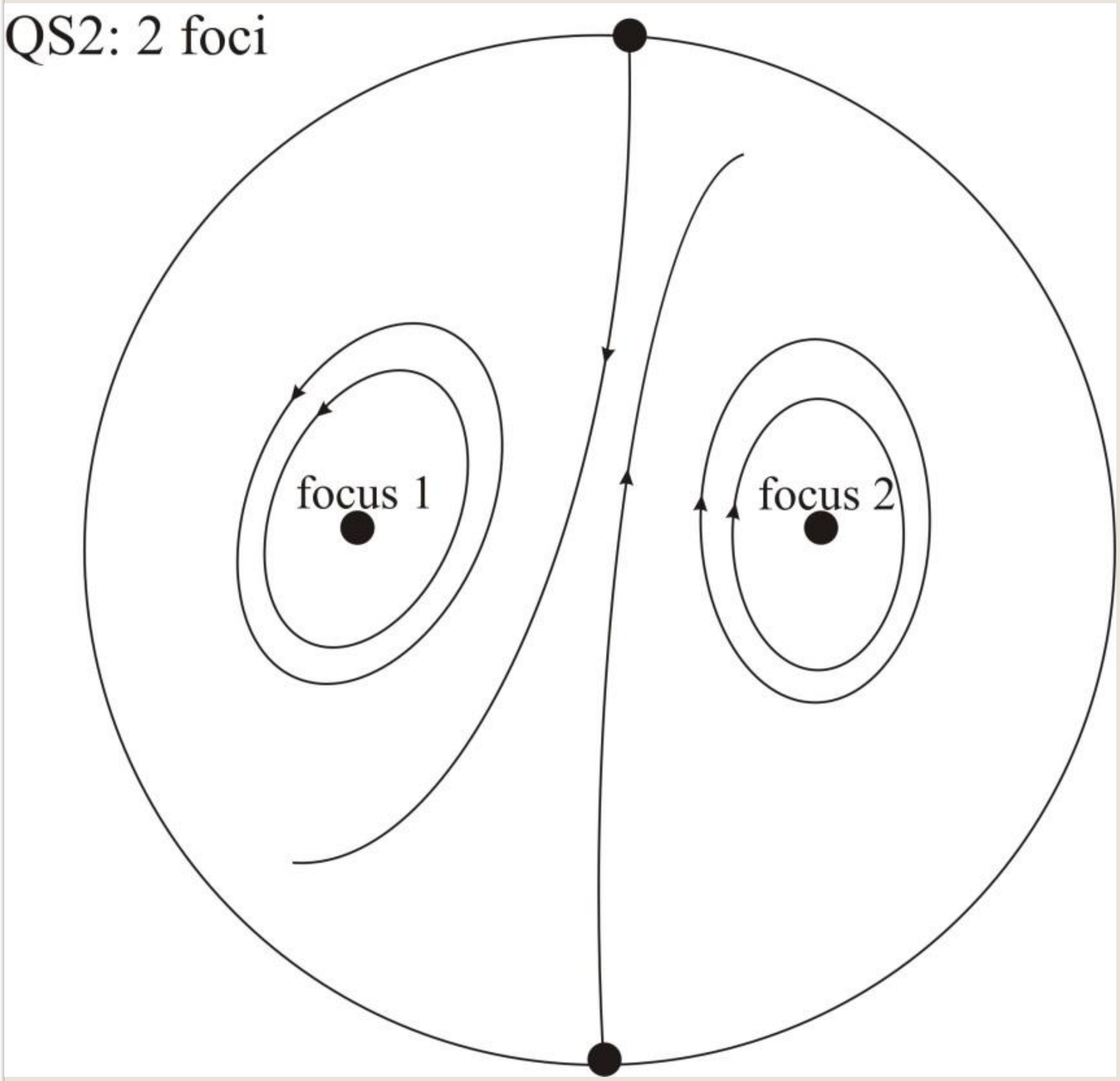
Part of Hilbert's 16^{th} problem asks for the upper bound on the number of limit cycles in a quadratic system: $H(2)$.

Currently: $H(2) \geq 4$.

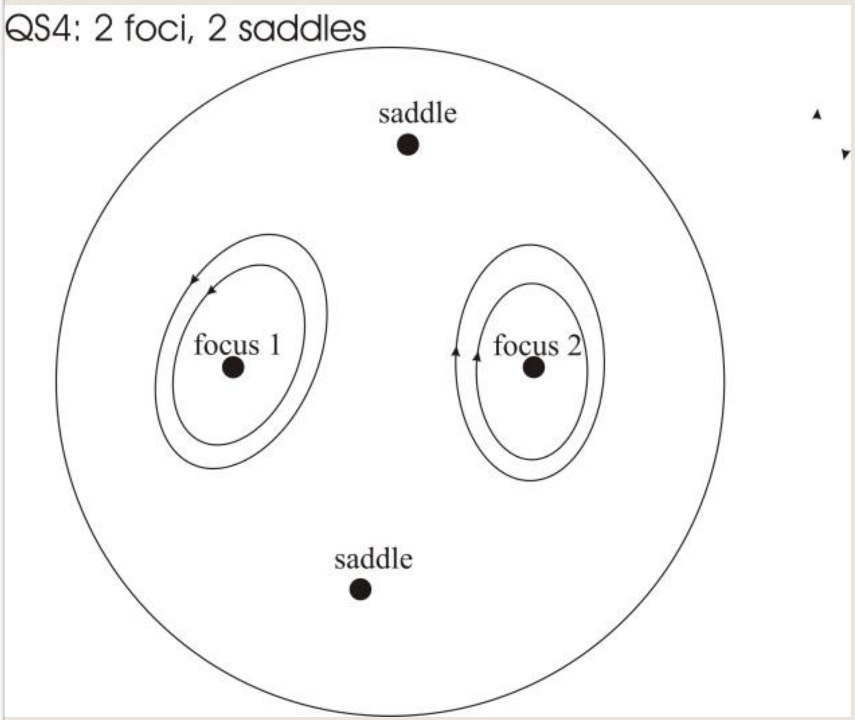
What is a distribution of limit cycles?

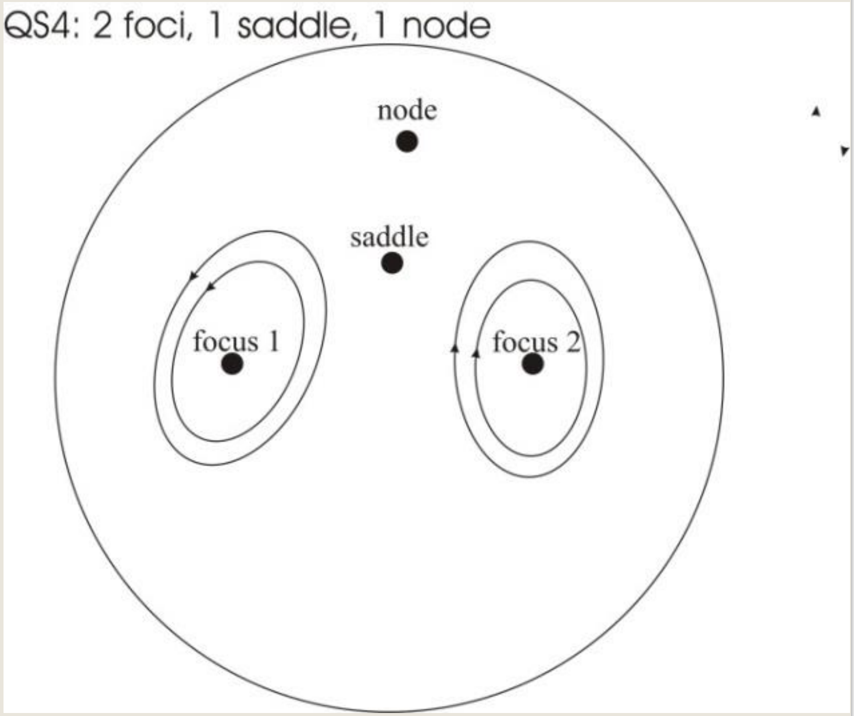


What is a distribution of limit cycles?



What is a distribution of limit cycles?





History of the problem for quadratic systems.

The first famous examples of a distribution in QS in the 70's were with $(p \geq 3, q \geq 1)$ distribution.

- **Chen Lan-sun and Wang Ming-Shu**, *Relative position and number of limit cycles of a quadratic differential system*, Acta Math. Sinica 22 (1979), 751-758.
- **Shi Song-ling**, *A concrete example of the existence of four limit cycles for plane quadratic systems*, Sci. Sinica 23 (1980), 153-158.

The result by Zhang Pingguang.

Zhang Pingguang proved in 2002 that the only possible distributions are $(p_1, 1)$ and $(p_2, 0)$, i.e. $H(2) = max\{p_1 + 1, p_2\}$.

The collection of papers was summarized in:

On the Distribution and Number of Limit Cycles for Quadratic Systems with two Foci, Zhang Pingguang, Qualitative theory of dynamical systems 3, 437–463 (2002).

The aim of this talk is to simplify and fill gaps in his proof.

The result by Zhang Pingguang.

A calculation shows that

$$\left[\int_0^x f(s)ds \cdot \frac{f(x)}{g(x)}\right]' = \frac{f^2(x)}{g(x)} + \int_0^x f(s)ds \cdot \left[\frac{f(x)}{g(x)}\right]'$$
$$= y'(x)/(-\lambda)(a\lambda^2 + b\lambda)r_1 r_3 - \tilde{F}(0)\left[\frac{f(x)}{g(x)}\right]'.$$

Thus, by (42) and $\int_0^x f(s)ds > 0$ for $x \in (x_{02}, x_{20})$ it is easy to show that

$$\left[\int_0^x f(s)ds \cdot \frac{f(x)}{g(x)}\right]' < 0,$$

for $x \in (x_{02}, x_{20})$, where x_{20} is defined as in Lemma 9. Therefore, system (13) satisfies condition (iv) of Lemma 9 (the case $x_0 \in (x_{02}, 0)$. This proves the theorem in subcase (a.22).

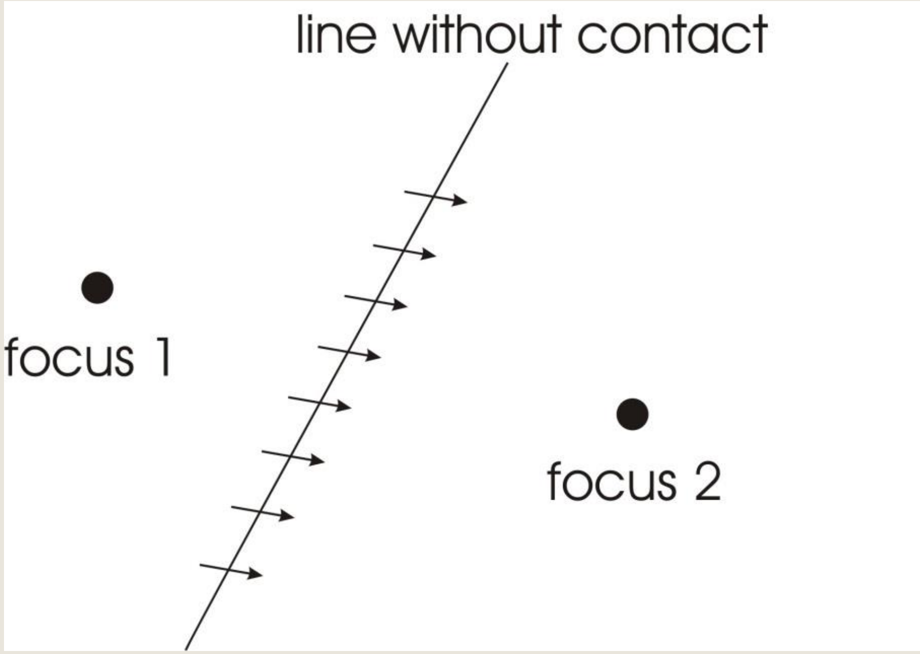
There are two questions:

1) is it easy?

2) is the inequality correct?

Structure of the proof

- Prove that in at least one of the two half planes separated by the line without contact there is *at most one limit cycle*.



Quadratic systems with 2 strong foci

First I will discuss the case of exactly two strong foci QS2.

- This is the case where (3,1) distributions have been found.

Transformation to a Liénard system:

- Zhang Pingguang used class III in the Chinese classification of quadratic systems.
- It is more natural to use the Russian classification by Cherkas et al.

Quadratic systems with 2 strong foci

The Russian classification consists of two families:

Type I:

$$\frac{dx}{dt} = y + x^2,$$

$$\frac{dy}{dt} = b_{00} + b_{10}x + b_{01}y + b_{20}x^2 + b_{11}xy + b_{02}y^2.$$

Type II:

$$\frac{dx}{dt} = 1 + xy,$$

$$\frac{dy}{dt} = b_{00} + b_{10}x + b_{01}y + b_{20}x^2 + b_{11}xy + Q_0\, y^2.$$

- QS2 cannot appear in type I.

Quadratic systems with 2 strong foci

Transformation into a Liénard system:

$$\frac{dx}{dt} = y, \qquad \frac{dy}{dt} = -g(x) - f(x)y$$

- zeros of $g(x)$ correspond to singularities.
- zeros of $f(x)$ correspond to vertical lines in the phase plane where the divergence of the vector field changes sign: limit cycles must intersect at least one vertical line where $f(x) = 0$.

Quadratic systems with 2 strong foci

Transformation to a Liénard system

$$\frac{dx}{dt} = 1 + xy, \qquad \frac{dy}{dt} = b_{00} + b_{10}x + b_{01}y + b_{20}x^2 + b_{11}xy + Q_0\, y^2.$$

Transform $u = |x|^{-Q_0-1}(1 + xy)$.
After scaling we assume $\boldsymbol{b_{20} = -1}$ and the system becomes a Liénard system:

$$\frac{dx}{dt} = u, \qquad \frac{du}{dt} = -g(x) - f(x)u$$

with

$$g(x) = \frac{|x|^{-2Q_0-2}}{x}\left(x^2(-b_{00} - b_{10}x + x^2) + x(b_{01} + b_{11}x)\ - Q_0\ \right)$$

$$f(x) = \frac{|x|^{-Q_0-1}}{x}\left(\ -x(b_{01} + b_{11}x) + 1 + 2Q_0\ \right)$$

In this system we want to prove that at most one limit cycle exists for either $x < 0$ or $x > 0$. The line $x > 0$ is a line without contact and cannot be crossed by limit cycles.

Quadratic systems with 2 strong foci

The function $f(x)$ needs to have one positive zero $\boldsymbol{x_f}^+ > \mathbf{0}$ and one negative zero $\boldsymbol{x_f}^- < \mathbf{0}$. This implies that $\boldsymbol{b_{11}} > \mathbf{0}$.

We impose that an elementary saddle exists at infinity: $\mathbf{0} < \boldsymbol{Q_0} < \mathbf{1}$.

The function $g(x)$ has two zeroes: $\boldsymbol{x_g}^+ > \mathbf{0}, \boldsymbol{x_g}^- < \mathbf{0}$ corresponding to the two foci.

We will need the following related functions:

$F_+(x) = \int_{x_g^+}^{x} f(\bar{x})d\bar{x} = c_+ + F_0(x)$ for $x > 0$,

$F_-(x) = \int_{x_g^-}^{x} f(\bar{x})d\bar{x} = c_- + F_0(x)$ for $x < 0$,

where

$$F_0(x) = \frac{|x|^{-Q_0}}{x} R(x) = \frac{|x|^{-Q_0}}{x} \left(\frac{b_{11}}{Q_0 - 1} x^2 + \frac{b_{01}}{Q_0} x - \frac{1 + 2Q_0}{1 + Q_0}\right)$$

Quadratic systems with 2 strong foci

Theorem (non-existence)

Suppose on the interval $a < x < b$

i. $(x - x_g)g(x) > 0$ for $x \neq x_g$,

ii. $(x - x_f)f(x) < 0$ for $x \neq x_f$,

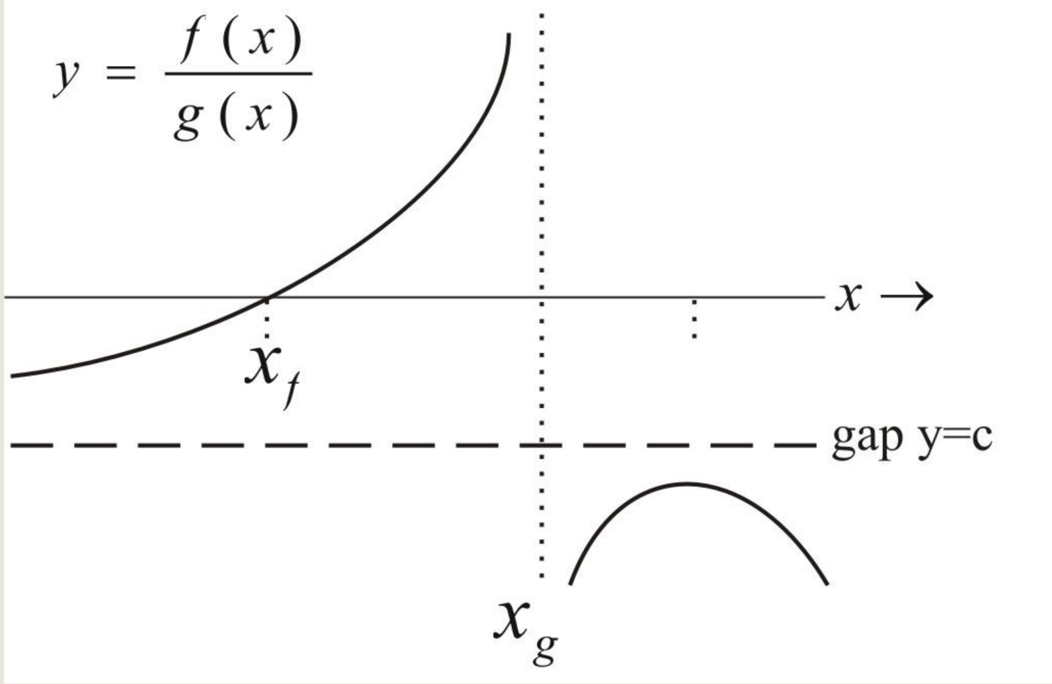
iii. If the system of equations

$$\boldsymbol{\frac{f}{g}(x_1) = \frac{f}{g}(x_2), \qquad F(x_1) = F(x_2)}$$

has **no solutions** $a < x_1 < x_g < x_2 < b$, then **no limit cycles exist** in the strip $a < x < b$.

Quadratic systems with 2 strong foci

Corollary (*"gap" theorem*) : if there is a value c such that the graph of $y = \frac{f}{g}(x)$ does not intersect $y = c$ for $a < x < b$, then no limit cycles exist in the strip $a < x < b$.



Quadratic systems with 2 strong foci

Theorem (uniqueness, *Coppel*, *Zeng*, *Zhang Zhifen*, *Zhou*)

Suppose on the interval $a < x < b$

i. $(x - x_g)g(x) > 0$ for $x \neq x_g$,

ii. $(x - x_f)f(x) < 0$ for $x \neq x_f, x_f < x_g$

iii. the system of equations

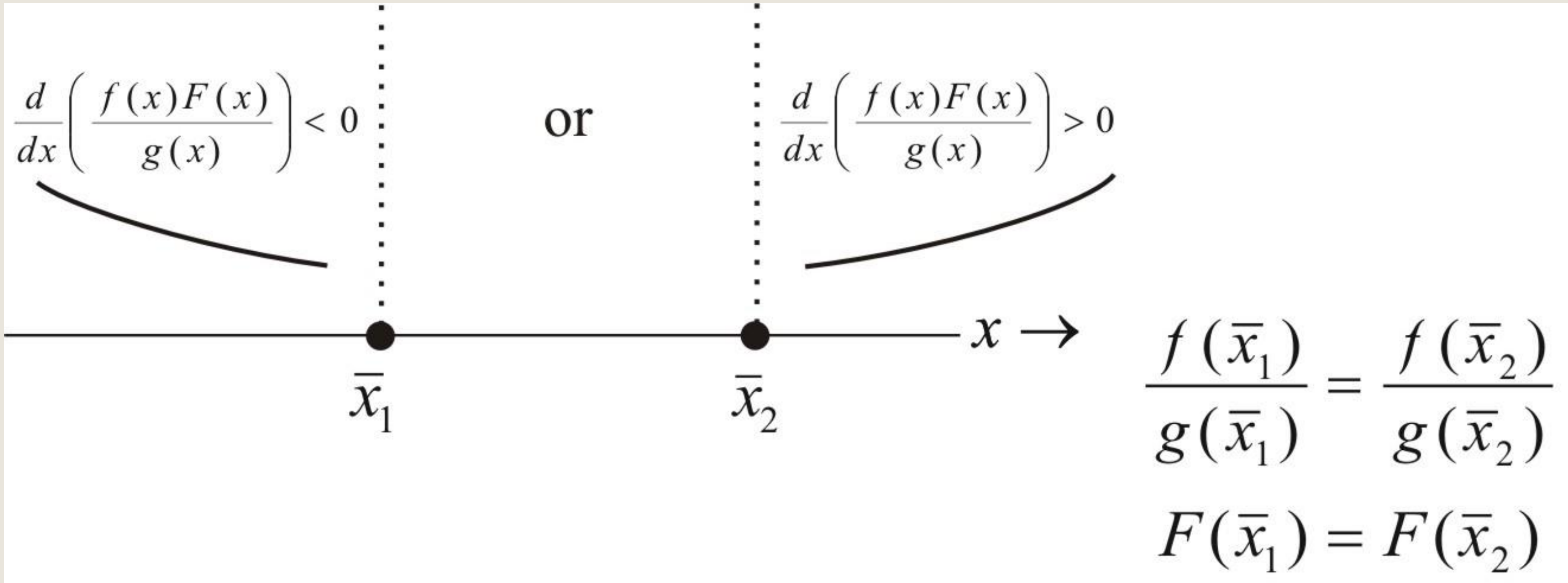
$$\boldsymbol{\frac{f}{g}(x_1) = \frac{f}{g}(x_2), \qquad F(x_1) = F(x_2)}$$

has **exactly one solution** $\bar{x}_1 < x_g < \bar{x}_2$,

iv. $\left(\frac{fF}{g}\right)'(x) < 0$ for $x < \bar{x}_1$ or $\left(\frac{fF}{g}\right)'(x) > 0$ for $x > \bar{x}_2$.

then the system has **at most one limit cycle** in the strip $a < x < b$.

Quadratic systems with 2 strong foci



Quadratic systems with 2 strong foci

The set of algebraic equations takes the form:

$$\frac{f}{g}(x_1) = \frac{f}{g}(x_2),\;\; F_0(x_1) = \;F_0(x_2)$$

$$\rightarrow$$

$$\frac{-x_1(b_{01}+b_{11}x_1)+1+2Q_0}{{x_1}^2(-b_{00}-b_{10}x_1+{x_1}^2)+x_1(b_{01}+b_{11}x_1)\;-Q_0}|x_1|^{Q_0+1} = \frac{-x_2(b_{01}+b_{11}x_2)+1+2Q_0}{{x_2}^2(-b_{00}-b_{10}x_2+x_2)+x_2(b_{01}+b_{11}x_2)\;-Q_0}|x_2|^{Q_0+1}$$

$$\left(\frac{b_{11}}{Q_0-1}{x_1}^2+\frac{b_{01}}{Q_0}x_1-\frac{1+2Q_0}{1+Q_0}\right)|x_1|^{-Q_0-1} = \left(\frac{b_{11}}{Q_0-1}{x_2}^2+\frac{b_{01}}{Q_0}x_2-\frac{1+2Q_0}{1+Q_0}\right)|x_2|^{-Q_0-1}$$

To prove: this system of equations has at most one non-trivial solution for either $x<0$ or $x>0$.

Quadratic systems with 2 strong foci

$F(x_1) = F(x_2)$ defines a curve γ_F which is a monotonically decreasing graph, because $\frac{dx_2}{dx_1} = \frac{f(x_1)}{f(x_2)} < 0$.

Define $Z(x) = \frac{fF_0}{g}(x)$ which is a rational polynomial function.

$$Z(x) = \frac{(-x(b_{01} + b_{11}x) + 1 + 2Q_0)(\frac{b_{11}}{Q_0 - 1}x^2 + \frac{b_{01}}{Q_0}x - \frac{1 + 2Q_0}{1 + Q_0})}{x^2(-b_{00} - b_{10}x + x^2) + x(b_{01} + b_{11}x) - Q_0)}$$

Then $Z(x_1) = Z(x_2)$ defines a curve $\gamma_Z$with slope $\frac{dx_2}{dx_1} = \frac{Z\prime(x_1)}{Z\prime(x_2)}$.
If $Z'(x)$ has the same sign on the relevant intervals then γ_Z is an increasing graph.
If γ_F is decreasing then the two graphs will have at most one intersection point.
Therefore **to prove: for $\boldsymbol{x < 0}$ or $\boldsymbol{x > 0}$ either $\boldsymbol{Z'(x)}$ has fixed sign or $\boldsymbol{y = Z(x)}$ has a gap.**

Quadratic systems with 2 strong foci

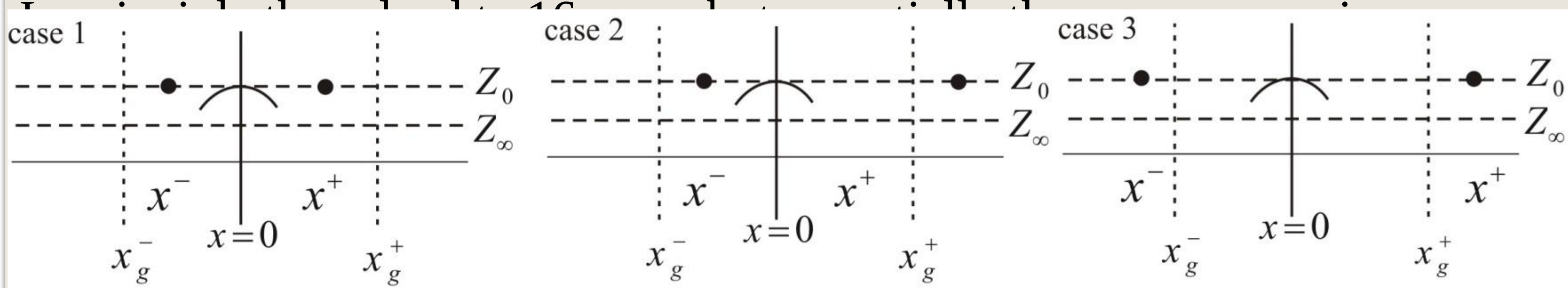
Important properties of the graph of $y = Z(x)$:

- It intersects every horizontal line $y = c$ in at most 4 points (counting multiplicity).
- It intersects $y = Z(0) = \frac{(1+2Q_0)^2}{Q_0(1+Q_0)} > 0$ for $x = 0$. It is a double contact because $Z'(0) = 0$. Then $Z(x) = Z(0)$ needs to have two solutions $x^- < 0$ and $x^+ > 0$.
- It intersects the horizontal asymptote $\lim_{x\to\pm\infty} Z(x) = Z_\infty = \frac{{b_{11}}^2}{1-Q_0} > 0$ in at most 3 points.

Quadratic systems with 2 strong foci

We distinguish the following combinations (writing $Z(0) = Z_0$):

$$Z_\infty < Z_0, Z_\infty > Z_0,$$
$$Z''(0) > 0 \,,\, Z''(0) < 0,$$
$$x^- < {x_g}^-, x^- > {x_g}^-,$$
$$x^+ < {x_g}^+, x^+ > {x_g}^+,$$

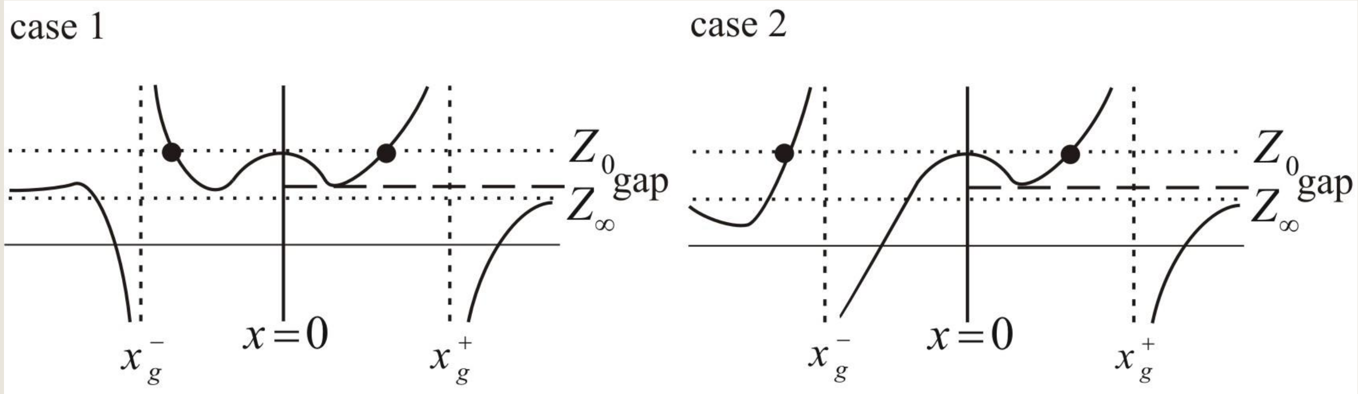


Quadratic systems with 2 strong foci

The distributions in these cases are as follows:

Case 1: $(n, 0)$
Case 2: $(n, 0)$
Case 3: $(n, 1)$

For the cases 1 and 2 we can find a "gap" to prove non-existence of limit cycles in the region $x > 0$.

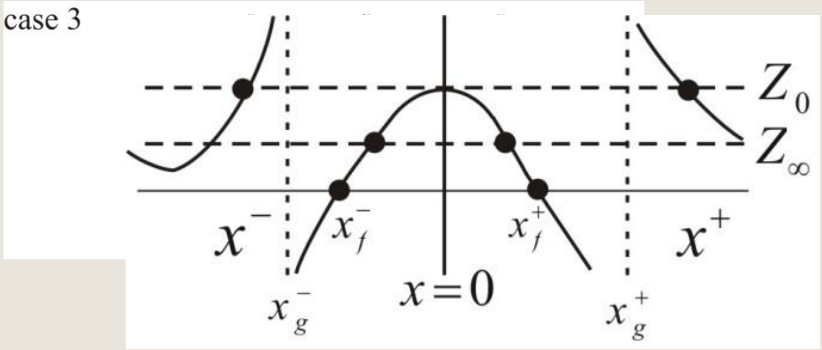


Quadratic systems with 2 strong foci

Case 3:

The function $Z(x)$ has the following restrictions:

- $x_f^- > x_g^-$, $x_f^+ < x_g^+$,
- $Z(x) > 0$ for $x < x_g^-$ and $x > x_g^+$,
- In the strip $Z_\infty < y < Z_0$ there are exactly 4 intersections of the graph of $y = Z(x)$ with any horizontal line. It follows that $Z'(x) < 0$ (< 0) in that strip.



Quadratic systems with 2 strong foci

The conclusion is: the curve γ_Z defined by the equation $Z(x_1) = Z(x_2)$ is either monotonically increasing for $x < 0$ or $x > 0$.

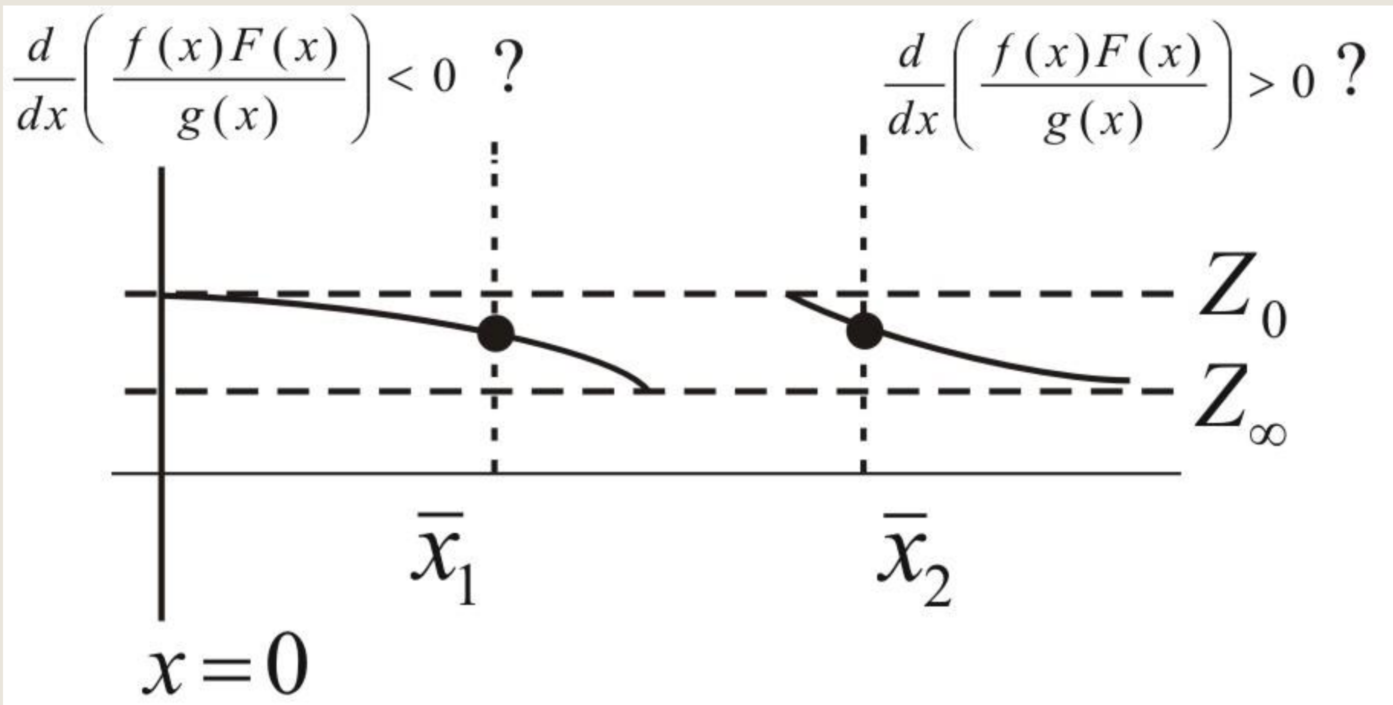
It follows that

$$\frac{f}{g}(x_1) = \frac{f}{g}(x_2), \qquad F(x_1) = F(x_2)$$

has at most one non-trivial solution for either $x < 0$ or $x > 0$.

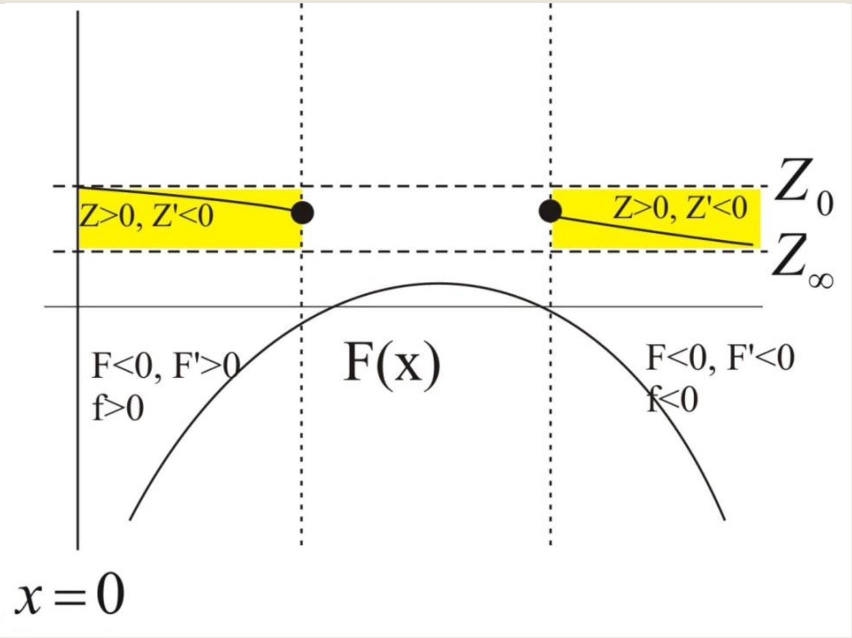
Quadratic systems with 2 strong foci

It remains to prove that $\left(\frac{fF}{g}\right)'(x)$ has the correct sign in the region where the system of equations has a unique solution.



Quadratic systems with 2 strong foci

$$\left(\frac{fF}{g}\right)'(x) = \left(\frac{ZF}{F_0}\right)'(x) = \frac{fZ(F_0 - F) + Z'FF_0}{{F_0}^2} < 0$$



This proves that at most one limit cycle exists for $x > 0$.
In a similar way the case for $x < 0$ can be done.

This completes the proof for the quadratic system with 2 real singularities QS2.

Quadratic systems with 4 real singularities

To prove the distribution property for QS4 we start by placing the other two real singularities on the y-axis: at the origin $(x=0, y=0)$, and at $(x=0, y=1)$.

$$\frac{dx}{dt} = a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy - a_{01}y^2$$

$$\frac{dy}{dt} = b_{10}x + b_{01}y + b_{20}x^2 + b_{11}xy - b_{01}y^2$$

Transform to a Liénard system according to $y = u\,x$, followed by $x = w(u)e^z$

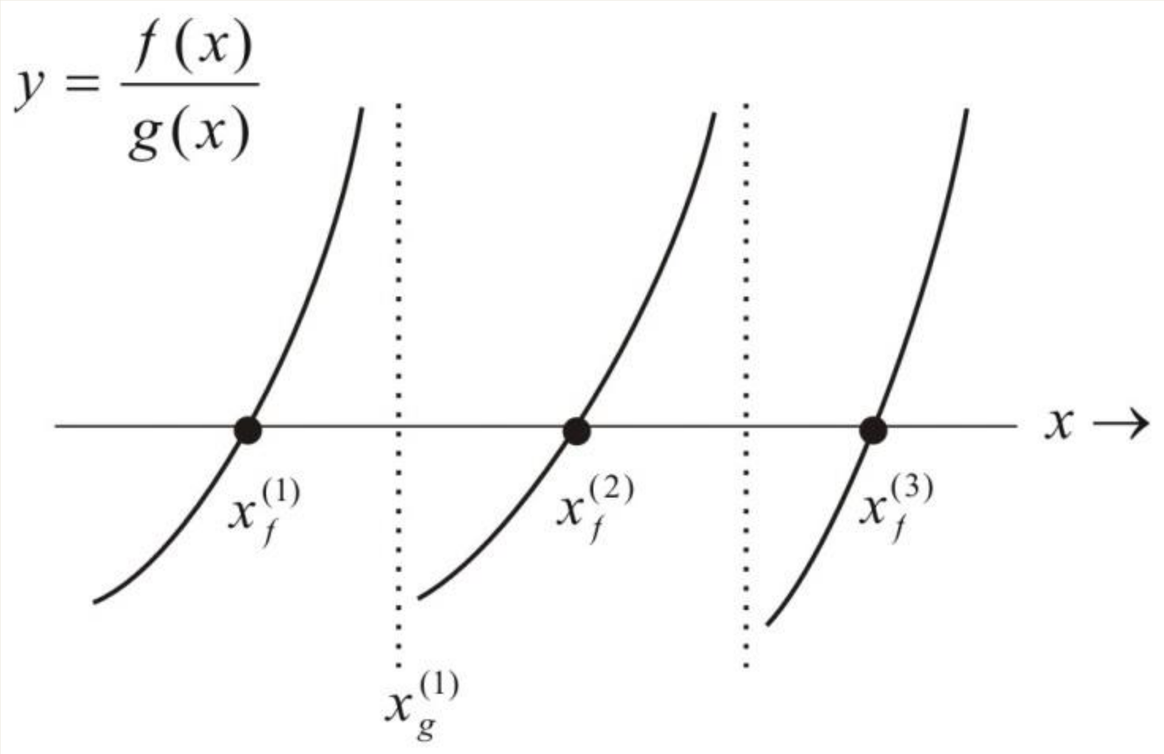
$$\frac{f}{g}(u) = \frac{c_0 + c_1u + c_2u^2 + c_3u^3}{d_0 + d_1u + d_2u^2}$$

Quadratic systems with 4 real singularities

Here f/g is cubic/quadratic.

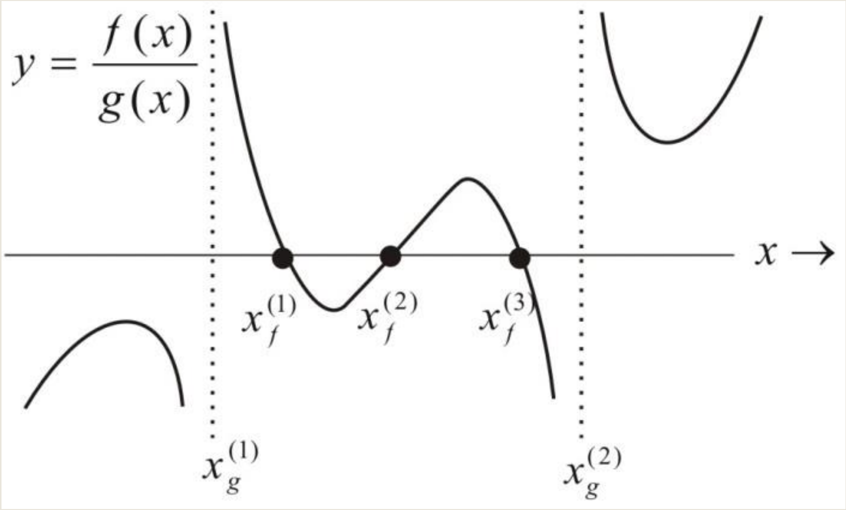
Crucial observation: there are at most 3 intersections counting multiplicity of $y = f/g$ with horizontal lines $y = c$.

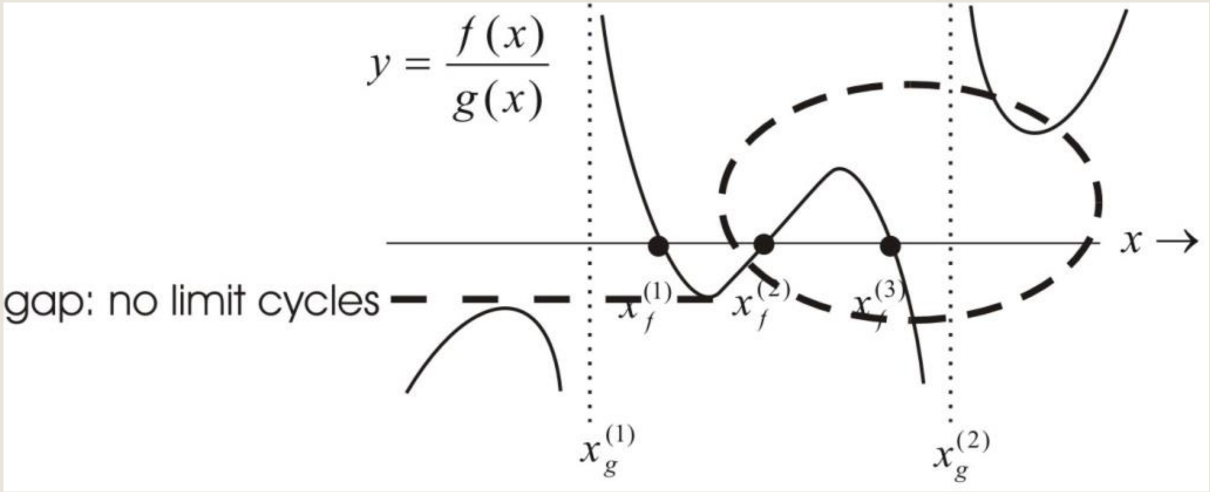
Example 1:



Quadratic systems with 4 real singularities

Example 2:





Current and future work

Is the following true?

Theorem (uniqueness)

Suppose on the interval $a < x < b$

i. $(x - x_g)g(x) > 0$ for $x \neq x_g$,

ii. $(x - x_f)f(x) < 0$ for $x \neq x_f, x_f < x_g$

iii. the system of equations

$$\frac{f}{g}(x_1) = \frac{f}{g}(x_2), \qquad F(x_1) = F(x_2)$$

has exactly one solution $\bar{x}_1 < x_g < \bar{x}_2$,

then the system has at most one limit cycle in the strip $a < x < b$.

$$\frac{f}{g}(x_1) = \frac{f}{g}(x_2), \qquad F(x_1) = F(x_2)$$

~~*iv.* $\left(\frac{fF}{g}\right)'(x) < 0$ for $x < \bar{x}_1$ or $\left(\frac{fF}{g}\right)'(x) > 0$ $x > \bar{x}_2$.~~

Current and future work

Non-existence and uniqueness of limit cycles in quadratic systems

- *QS with invariant line*:

method 1: find a Cherkas-Dulac function

method 2: transformation to Liénard system.

- *QS with a weak focus of order three*:

method: find a Dulac-function.

- *QS with a weak focus of order two*:

method 1: simplify the proof using a Liénard system.

method 2: find a Cherkas-Dulac function.

- *QS with a weak singularity*: there is at most one limit cycle surrounding other singularities.

method: simplify the proof by Zhang Pingguang.

Thank you!