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Complex dimensions of orbits of dynamical systems

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- The *standard Hurwitz (Riemann) zeta function*

$$\zeta_a(s) := \sum_{j=0}^{\infty} \frac{1}{(j+a)^s},\ a > 0,\ \mathsf{Re}(s) > 1$$

  - converges absolutely for $\mathsf{Re}(s) > 1$
  - meromorphically extendable to $\mathbb{C} \setminus \{1\}$
  - single pole at 1 with residue $\mathrm{Res}(\zeta_a(s), s = 1) = 1$
  - for $a = 1$: the Riemann zeta function

'Geometric generalizations' - **fractal zeta functions** in the sense of *Lapidus*

- $\mathcal{L} := \{\ell_j : j \in \mathbb{N}\}$
  a disjoint union of intervals on the real line with lengths ℓ_j

(1) The *geometric zeta function* of a *fractal string* (Lapidus, Frankenhuijsen, 2000)

$$\zeta_{\mathcal{L}}(s) := \sum_{j=1}^{\infty} \ell_j^s, \ \ s \in \mathbb{C}, \text{ s.t. the sum converges absolutely}$$

$\star$ $\ell_j := \frac{1}{j}$ standard zeta function

(2) The *distance zeta function* of a bounded set $A \subseteq \mathbb{R}^N$

$$\zeta_A(s) := \int_{A_\delta} d(x,A)^{s-N}\,dx$$

- $\delta > 0$ inessential (up to a holomorphic function)

(3) The *tube zeta function* of a bounded set $A \subseteq \mathbb{R}^N$:

- the tube function of A:

$$\varepsilon \mapsto V_A(\varepsilon) := |A_\varepsilon| \text{ (the Lebesgue measure)}$$

- $V_A(\varepsilon) \sim M\varepsilon^{N-s},\ \varepsilon \to 0 \Rightarrow \dim_B(A) = s,\ \mathcal{M}^s(A) = M$.

$$\tilde{\zeta}_A(s) := \int_0^\delta t^{s-N-1} V_A(t)\,dt,$$

$$\mathrm{Re}(s) > \dim_B(A),\ \delta > 0 \text{ inessential}$$

(Lapidus, Frankenhuijsen 2000, 2006; Lapidus, Radunović, Žubrinić, 2017)

For fractal strings, all three equal up to a holomorphic function

$$\mathcal{L} \Rightarrow \; A := \{a_j : \; j \in \mathbb{N}_0\}, \; \ell_j := a_{j-1} - a_j$$

The functional equations on domains of definition (up to holomorphic functions):

- $\zeta_A(s) = \frac{2^{N-s}}{s}\zeta_{\mathcal{L}}(s)$,
- $\tilde{\zeta}_A(s) = \frac{2}{s} + \zeta_A(s)$, $\mathrm{Re}(s) > \dim_B(A)$.

Definition

Let

- $A \subseteq \mathbb{R}^N$ *bounded,*
- $\zeta_A(s)$ *admits the meromorphic extension to whole* $\mathbb{C}$.

The set of all poles is called the set of complex dimensions of A, $\Omega(A)$.

- $\zeta_A(s)$ holomorphic for $\mathrm{Re}(s) > \dim_B(A)$,
- *simple pole at* $s = \dim_B(A)$.

Complex dimensions (and their residues i.e. principal parts) *talk* about the geometry of the set! Similarly as the tube function!

$\star$ The box dimension of the set is the *first complex dimension*, with Minkowski content directly related to its residue!

One example of a self-similar set: the ternary Cantor set

Example 1 (The complex dimensions of the ternary Cantor set, LRŽ 2017)

⋆ viewed as a fractal string, the order of intervals not important

$\zeta_{\mathcal{L}_\mathcal{C}}(s) = \sum_{j=1}^{\infty} \ell_j^s = \sum_{k=0}^{\infty} 2^k \left(\frac{1}{3^{k+1}}\right)^s = \frac{1}{3^s-2}, \ \left|\frac{2}{3^s}\right| < 1$

- holomorphic for $\mathrm{Re}(s) > \log_2 3 = \dim_B \mathcal{C}$
- unique meromorphic extension to $\mathbb{C}$ by the above formula with poles:

$$\Omega(\mathcal{C}) = \{\omega_k := \log_3 2 + i\frac{2k\pi}{\log 3}, \ k \in \mathbb{Z}\}.$$

Example 2 (The tube function of the Cantor set (LRŽ 2017))

$$V_\mathcal{C}(\varepsilon) = \varepsilon^{1-\log_3 2}\big(G(-\log\varepsilon) + o(1)\big), \ \varepsilon \to 0,$$

G a nonconstant periodic function.

A conjecture (LRŽ):
Strong oscillations in the first term indication of self-similarity; non-real complex dimensions;
possible definition of *fractality* of a set as possessing non-real complex dimensions?

Complex dimensions vs. asymptotics of the tube function

(formally proven in LRŽ, 2017)

$\star$ $\tilde{\zeta}_A$ the tube zeta function of set $A \subseteq \mathbb{R}^N$, meromorphically extendable to $\mathbb{C}$.

$\star$ $t \mapsto V_A(t) = |A_t|$, $t \in (0, \delta)$, the tube function of A

- $\tilde{\zeta}_A(s) = \mathcal{M}(\chi_{(0,\delta)} V_A/\mathrm{id}^N)(s) = \int_0^\delta V_A(t) t^{s-1-N}\, dt$
- Conversely,

$$V_A(t) = \frac{t^N}{2\pi i} \mathcal{M}^{-1}(\tilde{\zeta}_A)(t) = \frac{1}{2\pi i} \int_{\Gamma_c} \zeta_A(s) t^{N-s}\, ds,\ \ t \in (0, d).$$

$\Gamma\ldots$ a vertical line at around $s = c$, $\ c > \dim_B A$

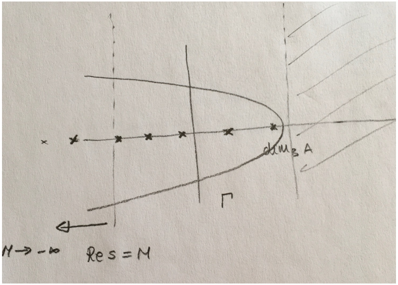
$\star$ the basis is the residue theorem: **the complex dimensions and their residues correspond to asymptotic terms and their coefficients in an asymptotic expansion of the tube zeta function of the set**

Important ingredient for relating

The *k-th primitive tube function* $V_A^{[k]}$, $k \in \mathbb{N}$:
$V_A^{[1]}(t) := \int_0^t V_A(s)\,ds,\ t \in (0,\delta)\ldots$

Changing the order of integration, $N > \mathrm{Re}(s) > \dim_B A$:

$$V_A^{[k]}(t) = \frac{1}{2\pi i}\int_{\Gamma_c} \frac{t^{N-s+k}}{(N-s+1)_k}\tilde{\zeta}_A(s)\,ds,\ k \in \mathbb{N}_0.$$



$*$ Heuristically, the residue theorem 'gives' expansions of $t \mapsto V_A(t)$ or $t \mapsto V_A^{[k]}(t)$, $k \in \mathbb{N}$, from poles and residues of ζ_A:

$\star$ e.g. $\Omega_A = \{\omega_n,\ n \in \mathbb{N}\}$ only first-order poles

$$(**)\ \ V_A(t) = \frac{1}{2\pi i}\int_{\Gamma_c} t^{N-s}\tilde{\zeta}_A(s)\,ds =$$

$$= \sum_{\omega\in\Omega_A,\ \mathrm{Re}(\omega)>-M} t^{N-\omega}\mathrm{Res}(\tilde{\zeta}_A,\omega) + O(t^{N+M}),\ t \to 0,\ M \in \mathbb{N}.$$

(in case of higher-order poles logarithmic terms in the expansion)

Idea of proof of (**) (LRŽ)

- to get asymptotic remainder $O(t^{N+M})$, $M \in \mathbb{N}$, bounds needed on zeta function along vertical lines $\mathrm{Re}(s) = -M$, $M \to \infty$
- so-called *languidity bounds* of $\tilde{\zeta}_A(s)$ along vertical lines $s = \sigma + i\tau$, as $\tau \to \pm\infty$
- **pointwise asymptotics** as long as bounds *rational*

$$|\tilde{\zeta}_A(\sigma + i\tau)| \sim \tau^{-\gamma},\ \ \gamma > 0, \tau \to \pm\infty$$

- *polynomial bounds* $(\gamma < 0) \Rightarrow$ only **distributional asymptotics** (there exists some primitive of tube function $t \mapsto V_A^{[k]}$) that expands pointwise up to this term, but differentiation of asymptotic expansions can be done just distributionally!
- $\frac{t^{N-s+k}}{(N-s+1)_k}\tilde{\zeta}_A(\sigma + i\tau)$, as $\tau \to \pm\infty$, becomes rational for k sufficiently big!

Relation to dynamical systems

Orbits of local diffeomorphisms ($\equiv$ germs) on the real line $\mathbb{R}_+$

- (attracting) *parabolic* germ
  $f(z) = z - ax^{k+1} + \ldots \in \mathrm{Diff}(\mathbb{R}_+, 0)$, $a > 0$, $k \in \mathbb{N}$

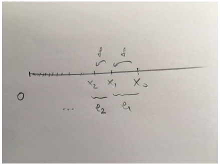
$$a_j \sim j^{-1/k},\ \ell_j \sim j^{-\frac{k+1}{k}},\ j \to \infty,$$

- (attracting) *hyperbolic* germ $f(x) = \lambda x + \ldots$, $0 < \lambda < 1$

$$a_j \sim \lambda^j,\ \ell_j \sim \lambda^j,\ j \to \infty.$$

Orbit of f with initial point $x_0 \in (\mathbb{R}_+, 0)$:

$$\mathcal{O}_f(x_0) := \{x_n := f^{\circ n}(x_0) :\ n \in \mathbb{N}_0\},\ g := \mathrm{id} - f$$



Box dimension and Minkowski content of orbits

Žubrinić, Županović 2005, MRŽ 2012

- a parabolic orbit of *multiplicity* k

$$V_{\mathcal{O}^f(x_0)}(\varepsilon) \sim (2/a)^{\frac{1}{k+1}} \frac{k+1}{k} \varepsilon^{\frac{1}{k+1}} + \ldots + c(\rho, a)\varepsilon(-\log\varepsilon) + \\ + o\left(\varepsilon(-\log\varepsilon)\right), \ \varepsilon \to 0,$$

$$\dim_B(\mathcal{O}^f(x_0)) = 1 - \frac{1}{k+1}, \ \mathcal{M}(\mathcal{O}^f(x_0)) = (2/a)^{\frac{1}{k+1}} \frac{k+1}{k},$$

- a hyperbolic orbit

$$V_{\mathcal{O}^f(x_0)}(\varepsilon) \sim a(\lambda) \cdot \varepsilon(-\log\varepsilon) + o(\varepsilon(-\log\varepsilon)), \ \varepsilon \to 0,$$

$$\dim_B(\mathcal{O}^f(x_0)) = 1 - 1 = 0, \ \mathcal{M}(\mathcal{O}^f(x_0)) = +\infty,$$

Later: R [2013]

- formal class of f using asymptotic expansion of function $\varepsilon \mapsto V_{\mathcal{O}^f(x_0)}(\varepsilon)$, as $\varepsilon \to 0$
- further (finitely many!) complex dimensions needed

Orbits as fractal strings and complex dimensions: the parabolic case

$$\zeta_{\mathcal{L}_f}(s)\text{"} \sim \text{"} \sum_{j\in\mathbb{N}} j^{-s\frac{k+1}{k}}$$

* holomorphic for $\mathrm{Re}(s) > \frac{k}{k+1} = \dim_B \mathcal{O}^f(x_0)$
* however, too coarse approximations for meromorphic extensions - info on poles and residues lost
* notation: $\zeta_{\mathcal{L}_f}$, ζ_f, $\tilde{\zeta}_f$

Precise computations tedious even in the simplest model case of germs, $k=1$, $\rho=0$ (MRR 2020)

$*$ *Model cases* with residual invariant $\rho=0$ and multiplicity $k\in\mathbb{N}$
$*$ time-one maps of simple vector fields $x'=-x^{k+1}$:

$$f_k(x):=\mathrm{Exp}(x^{k+1}\frac{d}{dx})=\frac{x}{(1+kx^k)^{1/k}}=x-x^{k+1}+o(x^{k+1}),\ k\in\mathbb{N}.$$

Proposition (The complex dimensions of orbits, MRR 2020)

$\zeta_{f_k}(s)$, $\mathrm{Re}(s)>\frac{k}{k+1}$, *the distance zeta function of an orbit* $\mathcal{O}_{f_k}(x_0)$ *of a model parabolic germ.*

1. $\zeta_{f_k}(s)$ *can be meromorphically extended to* $\mathbb{C}$,
2. *the poles of* $\zeta_{f_k}(s)$ *located at* $\frac{k}{k+1}$ *and at (a subset of) the set of points* $\frac{-mk}{k+1}$, $m\in\mathbb{N}_0$, *all simple*
3. *the Minkowski (box) dimension of* $\mathcal{O}_{f_k}(x_0)$ *is* $D=\frac{k}{k+1}$, *the only pole of* $\zeta_{f_k}(s)$ *with a positive real part*

Heuristical proof in the simplest model case $k = 1,\ \rho = 0$

Putting $X := x_0^{-1}$,

$$\ell_j = \frac{1}{(j+X)(j+1+X)} = \frac{1}{(j+X)^2} \cdot \left(1 + \frac{1}{j+X}\right)^{-1},$$

$$\ell_j^s = \frac{1}{(j+X)^{2s}} \cdot \left(1 + \frac{1}{j+X}\right)^{-s} =$$

$$= \sum_{m=0} \binom{-s}{m} \frac{1}{(j+X)^{2s+m}}.$$

Heuristically (formal change of order of summation),

$$\zeta_{\mathcal{L}_{f_1}}(s) = \sum_{j=0} \ell_j^s \text{ `` } \sim \text{ '' } \sum_{m=0} \binom{-s}{m} \zeta_X(2s+m). \tag{1}$$

Complex dimensions: $\omega_n := \frac{1-n}{2},\ n \in \mathbb{N}_0$, with residues: $\mathrm{Res}(\zeta_{\mathcal{L}_{f_1}}, \omega_n) = \binom{\frac{n-1}{2}}{n}$. Zero residue for n odd.

What to do in the case $\rho \neq 0$ or even non-model case?

Arbitrary **parabolic** germ

$$f(x) = x - ax^{k+1} + o(x^{k+1}) \in \mathrm{Diff}(\mathbb{R}_+, 0)$$

Theorem B (MRR 2020, Complex dimensions for arbitrary parabolic orbits)

$f \in \mathrm{Diff}(R_+, 0)$, *of formal class* (k, ρ), $k \in \mathbb{N}$, $\rho \in \mathbb{R}$.

1. *The distance zeta function* $\zeta_f(s)$ *can be meromorphically extended to* $\mathbb{C}$.
2. *In any open right half-plane* $W_M := \{s > 1 - \frac{M}{k+1}\}$, *where* $M \in \mathbb{N}$, $M > k + 2$, *given as:*

Theorem B

For $s \in W_M := \{s > 1 - \frac{M}{k+1}\}$*:*

$$\zeta_f(s) = (1-s)\sum_{m=1}^{k} \frac{a_m}{s - \left(1 - \frac{m}{k+1}\right)} + (1-s)\left(\frac{b_{k+1}(x_0)}{s} + \frac{a_{k+1}}{s^2}\right) +$$

$$+ (1-s)\sum_{m=k+2}^{M-1} \sum_{p=0}^{\lfloor \frac{m}{k} \rfloor + 1} \frac{(-1)^p p! \cdot c_{m,p}(x_0)}{\left(s - \left(1 - \frac{m}{k+1}\right)\right)^{p+1}} + g(s),$$

$g(s)$ *holomorphic in* W_M*.*

$*$ the coefficients in principal parts of poles real, with dependence on x_0, as noted!

$*$ related to the coefficients of the asymptotic expansion of the tube function of the orbit!

$*$ **new** wrt model: **higher-order poles** correspond to *logarithmic terms* in the asymptotic expansion of the tube function due to $\rho \neq 0$

generalized asymptotic expansion of tube function-coefficients oscillatory functions

Proposition (MRR 2020)

A **generalized asymptotic expansion** *of the tube function with full description of* **oscillatory** *coefficients:*

$$V_f(\varepsilon) \sim 2^{\frac{1}{k+1}} a^{-\frac{1}{k+1}} \frac{k+1}{k} \cdot \varepsilon^{\frac{1}{k+1}} + \sum_{m=2}^{k} a_m \cdot \varepsilon^{\frac{m}{k+1}} + 2\rho \frac{k-1}{k} \cdot \varepsilon \log \varepsilon + b_{k+1}(x_0)\varepsilon +$$

$$+ \sum_{m=k+2}^{2k} \sum_{p=0}^{\lfloor \frac{m}{k} \rfloor + 1} c_{m,p} \varepsilon^{\frac{m}{k+1}} \log^p \varepsilon + \sum_{p=1}^{\lfloor \frac{2k+1}{k} \rfloor + 1} c_{2k+1,p} \varepsilon^{\frac{2k+1}{k+1}} \log^p \varepsilon +$$

$$+ \tilde{P}_{2k+1}(G(\tau_\varepsilon)) \cdot \varepsilon^{\frac{2k+1}{k+1}} + \sum_{m=2k+2}^{\infty} \sum_{p=0}^{\lfloor \frac{m}{k} \rfloor + 1} \tilde{Q}_{m,p}(G(\tau_\varepsilon)) \cdot \varepsilon^{\frac{m}{k+1}} \log^p \varepsilon, \ \varepsilon \to 0^+.$$

$(*)$ $\varepsilon \mapsto \tau_\varepsilon$ *the so-called* **continuous critical time** *(MRRZ 2019)*, $\tau_\varepsilon \sim \varepsilon^{-\frac{k}{k+1}}$
$(*)$ $G : [0, +\infty) \to \mathbb{R}$ 1-**periodic**, $G(s) = 1 - s, \ s \in (0, 1), \ G(0) = 0$

$(*)$ $\tilde{P}_{2k+1}$ *resp.* $\tilde{Q}_{m,p}$, **polynomials** *whose coefficients in general depend on coefficients of* f *and initial condition* x_0.

The model hyperbolic case

- ⋆ $\mathcal{O}_f(x_0)=\{x_0\lambda^n:\ n\in\mathbb{N}_0\}$,
- ⋆ $\mathcal{L}_f:=\{\ell_j:=f^{\circ j}(x_0)-f^{\circ(j+1)}(x_0)=x_0(1-\lambda)\lambda^j:\ j\in\mathbb{N}_0\}$,
- ⋆

$$\zeta_f(s):=\frac{2^{1-s}}{s}\sum_{j=0}^{\infty}\ell_j^s=\frac{2^{1-s}x_0^s\cdot(1-\lambda)^s}{s}\frac{1}{1-\lambda^s},$$

- ⋆ extends meromorphically from $\{s\in\mathbb{C}:\ \Re(s)>0\}$ to $\mathbb{C}$: double pole $s_0=0$ and the simple poles:

$$s_k:=\frac{2k\pi}{\log\lambda}i,\ \ k\in\mathbb{Z}.$$

- ⋆

$$V_f(\varepsilon)=-\frac{2}{\log\lambda}\varepsilon(-\log\varepsilon)+H(\log_\lambda\frac{2\varepsilon}{x_0(1-\lambda)})\cdot\varepsilon,$$

$H:\ [0,+\infty)\to\mathbb{R}$ a 1-periodic bounded function.

Parabolic orbits vs. hyperbolic orbits and fractality

- the hyperbolic case: poles of zeta function as **non-real complex dimensions**, similarly as for Cantor sets (LF 2013, LRZ 2017), but in further terms
- the parabolic case: no non-real complex dimensions
- indication of self-similarity of hyperbolic orbits?

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