Stability of singular limit cycles for Abel equations revisited

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Abel equation

Consider

$$\frac{dx}{dt} = x' = A(t)x^3 + B(t)x^2, \quad t \in [0, T].$$

Denote $u(t,x)$ to the solution determined by $u(0,x) = x$, and

$$d(x) = u(T,x) - x.$$

A solution $u(t,x)$ is said to be

- *periodic or closed* if $u(T,x) = x$ or equivalently $d(x) = 0$.
- *periodic and singular* if $d(x) = 0$, $d'(x) = 0$.
- *periodic and double* if $d(x) = d'(x) = 0$, $d''(x) \neq 0$.

Isolated periodic solutions are also called *limit cycles*.

A motivation problem

Consider the Abel equation with linear coefficients

$$x' = (a_0 + a_1 \sin\ t + a_2 \cos\ t)\, x^3 + (b_0 + b_1 \sin\ t + b_2 \cos\ t)\, x^2.$$

What is the maximum number of limit cycles for this equation?

A motivation problem

Consider the Abel equation with linear coefficients

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What is the maximum number of limit cycles for this equation?

What is the maximum number of positive limit cycles?

Some known results

In the following cases, the equation has at most one positive limit cycle:

- $A(t) \geq 0$ (Pliss 1966).
- $B(t) \geq 0$ (Gasull-Llibre 1990).
- $\alpha A(t) + \beta B(t) \geq 0$ for some $\alpha, \beta \in \mathbb{R}$ (Álvarez-Gasull-Giacomini 2007).
- $a_0, b_0 \leq 0$ (Bravo-Torregrosa 2008).
- $a_0 b_0 = 0$ (Bravo-Fernández-Gasull 2009).

The first three also holds changing ≥ 0 for ≤ 0.

Some known results

Assume we are in the region where none of the known results hold. Then

1. A has two zeros in $[0, 2\pi)$, $t_{A_1} < t_{A_2}$.
2. B has two zeroes in $[0, 2\pi)$, $t_{B_1} < t_{B_2}$.
3. The zeroes are interleaved $t_{B_1} < t_{A_1} < t_{B_2} < t_{A_2}$ or $t_{A_1} < t_{B_1} < t_{A_2} < t_{B_2}$.

Semi-stability

Our aim is two control the stability of the singular limit cycles, and then use it to bound the number of limit cycles.

Semi-stability

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For any $\alpha, \beta \in \mathbb{R}$, if $u(t, \tilde{x})$ is a singular limit cycle

$$\operatorname{sgn}(d_{xx}(\tilde{x})) = \operatorname{sgn}\left(\int_0^T F(t,\alpha)G(t,\beta)\,dt\right),$$

where

$$F(t,\alpha) := (2-\alpha)B(t) + 2(3-\alpha)A(t)\tilde{u}(t),$$
$$G(t,\beta) := u_x(t,\tilde{x}) - \beta\tilde{u}(t).$$

Lower bound

By changes of variables,

$$x' = (a_0(1 - \cos t) - \sin t)\, x^3 + (b_0 + b_1 \sin t + b_2 \cos t)\, x^2,$$

with $b_0 + b_2 > 0$. That is, $A(0) = 0$, $A'(0) = -1$, $B(0) > 0$.
Developing in series

$$\begin{aligned} d(x) =& 2b_0\pi x^2 + (2a_0\pi + 4b_0^2\pi^2)x^3 \\ & + \pi(3a_0 b_1 - b_2 + 8b_0^3\pi^2 + 2b_0(1 + 5a_0\pi))x^4 + \mathcal{O}(x^5). \end{aligned}$$

For $a_0 = b_0 = 0$,

$$d(x) = -\pi b_2 x^4 + \mathcal{O}(x^5).$$

Therefore two positive limit cycles bifurcate form $u(t, x) = 0$ in a neighbourhood of $a_0 = b_0 = 0$, with $b_0 < 0 < a_0$.

Stability of double closed solutions

Consider

$$x' = A(t)x^3 + B(t)x^2, \quad t \in [0, T]$$

Denote

$$P(t) = 4(B(t)A'(t) - B'(t)A(t)) - B^3(t)$$

and

$$v(t, x) = B(t)(2A(t)x + B(t))^2 + P(t).$$

Stability of double closed solutions

Theorem

If

(C_1) $A(0) = 0$, $A(t)$ *has a simple zero* $t_A \in (0, T)$ *and* $B(t)$ *has two simple zeroes* $t_{B_1}, t_{B_2} \in [0, T]$ *with* $0 < t_{B_1} < t_A < t_{B_2} \leq T$;

and, for any positive singular closed solution $\tilde{u}(t) := u(t, \tilde{x})$ *of the Abel equation,*

(C_2) *the function* $2A(t)\tilde{u}(t) + B(t)$ *has at most a simple zero in each of the intervals* $[0, t_A]$ *and* $[t_A, T]$,

(C_3) $v(t, \tilde{u}(t)) < 0$ *for every* $t \in [0, T]$,

then

$$d''(\tilde{x}) < 0.$$

Bounding the number of positive limit cycles

Assume the Abel differential equation deppends on a parameter,

$$x' = A(t,\lambda)x^3 + B(t,\lambda)x^2,$$

and denote

$$F(t,x,\lambda) = A(t,\lambda)x^3 + B(t,\lambda)x^2.$$

Denote

- $u(t,x,\lambda)$ the solution determined by $u(0,x,\lambda) = x$,
- $d(x,\lambda) = u(T,x,\lambda) - x$,
- $\mathcal{H}(\lambda)$ the number of positive isolated closed solutions (limit cycles) for that value of the parameter.

Bounding the number of positive limit cycles

Theorem

Assume that

1. $F_\lambda(t,x,\lambda) > 0$ *for every* $\lambda \in (\lambda_1,\lambda_2)$, $t \in (0,T)$ *and* $x > 0$,
2. $d_{xx}(\tilde{x},\lambda) < 0$, *for every positive singular closed solution* $u(t,\tilde{x},\lambda)$ *with* $\lambda \in [\lambda_1,\lambda_2]$.

Then

$$\mathcal{H}(\lambda) \leq \mathcal{H}(\lambda_2) + 2, \quad \text{for every } \lambda \in (\lambda_1,\lambda_2).$$

Moreover, the two possible aditional closed solutions correspond to a Hopf bifurcation of the origin or a Hopf bifurcation of infinity.

Stability of double closed solutions

Assume $u(t, \tilde{x})$ is a double closed solution.

Proposition (Bravo-Fernández-Gasull 2015)

For any $\alpha, \beta \in \mathbb{R}$,

$$\operatorname{sgn}\left(d_{xx}(\tilde{x})\right) = \operatorname{sgn}\left(\int_0^T F(t,\alpha)G(t,\beta)\,dt\right),$$

where

$$F(t,\alpha) := (2-\alpha)B(t) + 2(3-\alpha)A(t)\tilde{u}(t),$$
$$G(t,\beta) := u_x(t,\tilde{x}) - \beta\tilde{u}(t).$$

Stability of double closed solutions

By (C_2) and (C_3), for every $\alpha \in \mathbb{R}$, $F(t,\alpha)$ has at most two changes of sign in $(0, T)$. Moreover, $F(t,\alpha) = 0$ is the graph of a smooth function $\alpha(t)$,

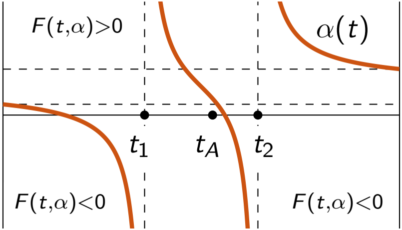

Figure: Sketch of $\alpha(t)$.

Stability of double closed solutions

By (C_2), there exist $\beta_0, \beta_1, \beta_2$, such that $G(t,\beta)$ has two changes of sign in $(0, T)$ for every $\beta \in (\beta_1, \beta_2)$, $\beta \neq \beta_0$, and no zeroes for $\beta \notin [\beta_1, \beta_2]$. More precisely, $G(t,\beta) = 0$ is the graph of a positive closed smooth function $\beta(t)$,

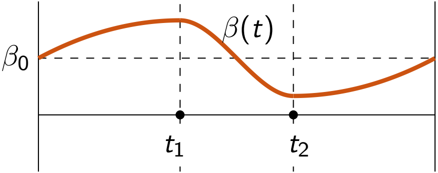

Figure: Sketch of $\beta(t)$.

Some sufficient conditions

Define

$$\phi(t) = -B(t)/(2A(t)).$$

By condition (C_1), $\phi(t) \geq 0$ if and only if $t \in [0, t_{B_1}] \cup [t_A, t_{B_2}]$.

Proposition

Let $u(t)$ be a positive singular closed solution and suppose that (C_1) holds. Let $J_1 = (0, t_{B_1})$, $J_2 = (t_A, t_{B_2})$. If the function P, has at most one zero in each J_i, $i = 1, 2$, then $u(t) - \phi(t)$ has a unique simple zero in each $J_i, i = 1, 2$. That is, condition (C_2) holds.

Some sufficient conditions

Denote

$$\dot{v}(t,x) = v_t(t,x) + v_x(t,x)(A(t)x^3 + B(t)x^2).$$

Proposition

If $v(t,0) = A'(t)B(t) - A(t)B'(t) < 0$ *for every* $t \in [0,T]$, $v(0,x) < 0$ *and* $v(T,x) < 0$, *for every* $x \geq 0$, *and* $v^{-1}(0) \cap \dot{v}^{-1}(0) = \emptyset$, *then condition* (C_3) *holds.*

Some sufficient conditions

Let

$$Q(t) = B(t)(A(t)B''(t) - B(t)A''(t)) + 3B'(t)(B(t)A'(t) - A(t)B'(t))$$

Corollary

If $Q(t)$ has no zeroes in $(0, T)$ or $v(\bar{t}, x) = 0$ has not positive solutions for each zero $\bar{t}$ of $Q(t)$ in $(0, T)$, then $v^{-1}(0) \cap \dot{v}^{-1}(0) = \emptyset$.

A simple family

Consider $t_A, t_B \in \mathbb{R}$, and

$$x' = t(t - t_A)x^3 + (t - t_B)(t - 1)x^2, \quad t \in [0, 1].$$

It has at most one positive isolated closed solution if

1. $t_A \notin (0, 1)$ or $t_B \notin (0, 1)$.
2. $t_A \in (0, 1), t_B \in (0, 1)$, and $t_A \in (0, t_B)$.

Theorem

Abel equation has at most two positive closed solutions, taking into account their multiplicities, and this upper bound is sharp.

Thank you!

Moltes gràcies!