DARBOUX RELATIVE EXACTNESS AND PSEUDO-ABELIAN INTEGRALS

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AQTDE,
Port de Soller, 7/02/2023.

Center problem (generic case)

$$\omega = P(x,y)dx + Q(x,y)dy, \quad P, Q \in \mathbb{R}[x,y],$$

$$\omega = xdx + ydy + \cdots .$$

Foliation given by

$$\omega = 0$$

Center or focus?

Displacement function:

$$\Delta(t) = \sum a_i(P,Q)t^i.$$

Center: $\Delta \equiv 0$, i.e. $a_i \equiv 0$, for all i. It is an *algebraic set.*
Focus: $\Delta \not\equiv 0$.
Problem: Determine *irreducible components* corresponding to center in parameter space.

Quadratic case

The center focus problem is solved for quadratic vector fields (Dulac). There are four irreducible components.

Infinitesimal center problem

Let $\omega = 0$ have a center at the origin, F first integral near the center surrounded by a family of closed trajectories $\gamma(t)$, $\gamma(t) \subset F^{-1}(t)$
Conisider deformations:

$$\omega + \epsilon\eta = 0$$

Infinitesimal center problem:
Determine deformations η such that the center is preserved (i.e. $M_i \equiv 0$, for every i). M_i *Melnikov functions*
Displacement function

$$\Delta_\epsilon(t) = \sum_{i=1}^{\infty} M_i(t)\epsilon^i.$$

Tangential center problem: Conditions for $M_1 \equiv 0$.

Solution of tangential center problem in generic Hamiltonian case: Ilyashenko's results

Hamiltonian case $\omega = dH$, $H \in \mathbb{R}[x, y]$, η polynomial form.
Then $M_1(t) = -\int_{\gamma(t)} \eta$ is an *abelian integral*.
η is *relatively exact* if $\eta = PdF + dR$, $P, R \in \mathbb{R}[x, y]$.

Theorem (Ilyashenko)

Under generic conditions on F, the first Melnikov function M_1 vanishes identically if and only if η is relatively exact.

Idea of proof: $(\Leftarrow)$ obvious.
$(\Rightarrow)$: Complexify! Show that by monodromy the cycle γ generates all the cycles of the complex fiber $F^{-1}(t)$. Then obtain P by integration of $\frac{d\eta}{dF}$ using the vanishing of its integral on all the cycles in the fibers $F^{-1}(t)$.

Hamiltonian case: Corollaries

Corollary 1

Hamiltonians form an irreducible component of the space of centers.

Corollary 2

Inferior bound for the number of limit cycles bifurcating from a center. At least $\frac{1}{2}(n^2 - n - 2)$ *limit cycles can bifurcate from Hamiltonian centers in degree* n *deformations.*

Corollary 3

Françoise algorithm for calculating the first nonzero Melnikov function.

Darboux integrable system:

Let $F = \prod_{i=0}^{\ell} f_i^{\lambda_i}$, $M = \prod_{i=0}^{\ell} f_i$, $f_i \in \mathbb{R}[x, y]$, $\lambda_i \in \mathbb{R}$.

Let $\omega = M\frac{dF}{F}$. It is a polynomial form, with first integral F and integrating factor $\frac{1}{M}$.
Assume $\omega = 0$ has a center at the origin surrounded by closed cycles $\gamma(t) \subset F^{-1}(t)$.

We study *deformations*

$$\omega + \epsilon\eta = 0.$$

Note that now

$$M_1(t) = \int_{\gamma(t)} \frac{\eta}{M},$$

where $\gamma(t) \subset F^{-1}(t)$. It is a *pseudo-abelian integral*

Darboux relatively exact forms

With Colin Christopher, we impose generic conditions on F, prove a theorem generalizing Ilyashenko's theorem for deformation of Hamiltonian centers, as well as the three corollaries.

Definition

A form $\frac{\eta}{M}$ is *Darboux relatively exact* if

$$\frac{\eta}{M} = \frac{P}{M}\frac{dF}{F} + d\left(\frac{R}{M}\right) + \sum_{i=0}^{\ell} a_i \frac{df_i}{f_i}.$$

for some polynomials P and R and coefficients a_i.

Note first that if $\frac{\eta}{M}$ is Darboux relatively exact and γ does not wind around $f_i = 0$ for any i, then

$$\int_{\gamma(t)} \frac{\eta}{M} \equiv 0.$$

The converse is our main theorem generalizing Ilyashenko's theorem to deformations of Darboux centers under some genericity conditions.
We complexify F.

Let L_i be the separatrices given by $f_i = 0$ in $\mathbb{CP}^2$ and L_∞ be the line at infinity.

Conditions:

(G1) L_i are all smooth and together with L_∞ intersect two by two transversally (normal crossing) and no three in the same point.

(G2) All quotients of exponents $\frac{\lambda_i}{\lambda_j}$ are irrational (including the exponents at points at infinity).

(G3) All critical points are of Morse type and all critical values of F outside $F = 0$ are different.

Main Theorem

Theorem (C. Christopher, P. Mardešić)

Assume $F = \prod_{i=0}^{\ell} f_i^{\lambda_i}$ *verifies (G1),(G2) and (G3). Let* $M = \prod_{i=O}^{\ell} f_i$, $\omega = M\frac{dF}{F}$, $\gamma(t) \subset F^{-1}(t)$ *a family of vanishing cycles at a center p, (with* $F(p) \neq 0$*), of the foliation given by* $\omega = 0$ *.*
Then

$$\int_{\gamma(t)} \frac{\eta}{M^k} = 0$$

if and only if

$$\frac{\eta}{M^k} = \frac{P}{M^{k+1}}\omega + d\left(\frac{R}{M^k}\right) + \sum_{i=0}^{\ell} a_i \frac{df_i}{f_i}.$$

For $k = 1$:
$M_1 \equiv 0$ if and only if the form $\frac{\eta}{M}$ is Darboux relatively exact.

Corollaries

Corollary 1

Tangential Darboux centers form an irreducible component.

Corollary 2

At least $n^2 - 2$ limit cycles can be created by deformations of Darboux centers in the space of vector fields of degree n.

Corollary 3

Darboux-Françoise algorithm for calculating the first non-zero Melnikov function.

Idea of the proof of the Main Theorem (line case) I

We assume $\int_{\gamma(t)} \frac{\eta}{M} \equiv 0$. We search for $a_i \in \mathbb{R}$, R and P in $\mathbb{R}[x, y]$.

- Complexify
- Solve locally
- Extend

There exists at least one node p_0 at the line at infinity. Let L_0 be the corresponding separatrix given by $f_0 = 0$ and $\{p_0\} = L_0 \cap L_\infty$.
We put $a_0 = 0$, $a_i = \text{Res}\left(\frac{\eta}{M^k}, p_i\right)$, $\{p_i\} = L_i \cap L_0$.
Put

$$\tilde{\eta} = \frac{\eta}{M^k} - \sum_{i=0}^{\ell} a_i \frac{df_i}{f_i}.$$

We want first to construct the function $G = \frac{R}{M^k}$.

Idea of the proof of the Main Theorem (line case) II

G is first constructed in a neighborhood of p_0 by integration term by term of $\tilde{\eta}$. There are no convergence problems due to the choice of a node (no small divisors).
Take a small transversal Σ to L_0 in a neighborhood of the node p_0, where the function G is already defined. Let $\{p_\Sigma\} = L_0 \cap \Sigma$. The function G will be given by integrating $\tilde{\eta}$ along lifts of paths $\sigma \in \pi_1(L_0 \setminus \cup_{i=1}^{\ell,\infty}, p_\Sigma)$.
In order to get G univalued, it must verify the homological equation:

$$G(\sigma_q(t)) - G(q) = \int_{\sigma_q} \tilde{\eta},$$

where σ_q is a lift of σ from q and $G(\sigma_q)$ is analytic extension of G along σ_q.

Monodromy and weighted variation

In the classical Hamiltonian case, Ilyashenko obtains an analogous condition by showing that by monodromy one generates all the cycles. More precisely, he uses the variation

$$\mathrm{Var}_\Gamma(\gamma(t_*)) = \mathrm{Mon}_\Gamma(\gamma)(t_*) - \gamma(t_*) = \gamma(t_* e^{2\pi i}) - \gamma(t_*),$$

where t_* is a generic value.
Here we use *weighted variation* Var_λ:

$$\mathrm{Var}_\lambda(\gamma(t_*)) = \gamma(t_* e^{2\pi i\lambda}) - \gamma(t_*).$$

for convenient λ.

Pochhammer cycles are lifts of commutator cycles above $L_0 \setminus \cup_{i=1}^{\ell,\infty} L_i$ to the leaves of the foliation $\omega = 0$.

Proposition (key Proposition)

All Pochhammer cycles above $L_0 \setminus \cup_{i=1}^{\ell,\infty} L_i$ are in the orbit by weighted variation of γ. Hence, by analytic continuation the integral of $\tilde{\eta}$ along all Pochhamer cycles vanishes.

End of the proof of the Main Theorem (in the line case)

The vanishing of the integral of $\tilde{\eta}$ along all Pochhammer cycles above $L_0 \setminus \cup_{i=0}^{\ell,\infty} L_i$ give a *univalued meromorphic function* G defined in a neighborhood of L_0.

Next one uses a Stein extension theorem,which shows that this function extends meromorphically to the whole $\mathbb{CP}^2$. One verifies that the poles are of order at most k at the separatrices L_i.

By construction, the integral of the form $\tilde{\eta} - dG$ along any path in $\omega = 0$ vanishes. It is hence proportional to ω. One obtains the proportionality factor $\frac{P}{M^{k+1}}$.

Concluding remarks

- We think that our theorem will be very important to study *bifurcations starting from Darboux integrable systems*.
- Many results are obtained for the number of zeros of abelian integrals, but very few for *number of zeros of pseudo abelian integrals*.
- The Françoise-Darboux algorith will give *iterated pseudo abelian integrals.* What can be said about their length, number of zeros?
- Can one obtain some kind of *Picard-Fuchs equations* which would help studying zeros of pseudo abelian integrals?