Effective applications of Lyapunov and Bendixson-Dulac approaches

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Advances in Qualitative Theory of Differential Equations, 4th Ed.

Port de Sóller, Mallorca,
February 9th 2023

Talk based on the papers:

A. G., H. Giacomini. *Effectiveness of the Bendixson-Dulac theorem.* J. Differential Equations, **305**, 347-367. 2021.

A. G., H. Giacomini. *Number of limit cycles for planar systems with invariant algebraic curves*. To appear in Qual. Theory Dyn. Syst.

This work is supported by Spanish State Research Agency, through the projects PID2019-104658GB-I00 grant and the Severo Ochoa and María de Maeztu Program for Centers and Units of Excellence in R&D (CEX2020-001084-M) and by the grant 2017-SGR-1617 from AGAUR, Generalitat de Catalunya.

Outline of the talk

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Methods

We will present some results about either non-existence or maximum number of limit cycles for some families of planar vector field, reducing the problem to the control of the sign of some function.

More concretely:

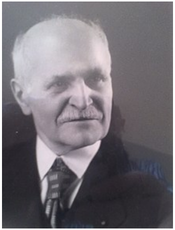
- Some new applications of the "extended" Lyapunov approach for proving non-existence of limit cycles.
- Several applications of the Bendixson–Dulac method to give non-existence and upper bounds of the number of limit cycles.

MAIN GOAL

SIGN OF A FUNCTION $\Longrightarrow$ NUM. OF PERIODIC ORBITS

Aleksandr M. Lyapunov (1857-1919)

Ivar Bendixson (1861-1935)



Henri Dulac (1870-1955)

Results

- We will study the non-existence of limit cycles for polynomial systems having some invariant algebraic curve.
- We will provide a simple 1-parametric differential system of “degree” $m+3$ for which we prove that it has limit cycles only for the values of the parameter that are in a non empty subset of an interval which length decreases exponentially when m grows.
- We will study several Liénard systems.
- We will obtain some results about number of limit cycles by using a Dulac function related with the curvature of the orbits of the vector field.

Outline of the talk

Lyapunov approach

Consider a polynomial differential system

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y), \tag{1}$$

and denote by $X = (P, Q)$ its associated vector field.
Recall that system (1) has an invariant algebraic curve $f(x, y) = 0$ if f is irreducible and it holds that

$$\dot{f}(x, y) = \frac{\partial f(x, y)}{\partial x} P(x, y) + \frac{\partial f(x, y)}{\partial y} Q(x, y) = k(x, y) f(x, y), \tag{2}$$

for some polynomial $k(x, y)$, called the cofactor of f.

Lyapunov approach

Theorem

Consider the polynomial differential system

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y),$$

and assume that it has an invariant algebraic curve $f(x, y) = 0$ *with cofactor* $k(x, y)$. *For each* $\alpha \in \mathbb{R}$ *and each polynomial* $g(x, y)$ *define the new polynomial*

$$N_{\alpha,g}(x, y) = \alpha k(x, y) g(x, y) + \frac{\partial g(x, y)}{\partial x} P(x, y) + \frac{\partial g(x, y)}{\partial y} Q(x, y).$$

Then, if for some α *and* g, $N_{\alpha,g}$ *does not change sign and vanishes only on some algebraic curve that it is not invariant by the flow, then the only limit cycles of the system (if any) are included in the invariant algebraic curve* $f(x, y) = 0$.

Lyapunov approach. Proof:

We will use the following well know fact: If for some open set $\mathcal{U} \subset \mathbb{R}^2$, there exists a class $\mathcal{C}^1$ function such that $v : \mathcal{U} \to \mathbb{R}$, and

$$\dot{v}(x, y) = \frac{\partial v(x, y)}{\partial x} P(x, y) + \frac{\partial v(x, y)}{\partial y} Q(x, y) \tag{3}$$

does not vanish then the system (1) does not have periodic orbits totally contained in $\mathcal{U}$.

This is so because while the solution is in $\mathcal{U}$ the function $t \to v(x(t), y(t))$, where $(x(t), y(t))$ is any solution of the differential equation, is monotonous.

This fact prevents the existence of periodic orbits. If the right hand side of (3) vanishes on some curve, but does not change it sign, then the same holds, unless this curve is invariant by the flow.

Lyapunov approach. Proof:

Since $f(x,y) = 0$ is an invariant algebraic curve, each of the connected components of $\mathcal{U} \setminus \{f(x,y) = 0\}$ is invariant.
For proving the theorem we apply the above result by taking

$$v(x,y) = g(x,y)|f(x,y)|^{\alpha}$$

and $\mathcal{U}$ any of these components.
Some computations give that

$$\dot{v}(x,y) = |f(x,y)|^{\alpha} N_{\alpha,g}(x,y)$$

and hence the result follows under the hypotheses on $N_{\alpha,g}$.

Clearly when $N_{\alpha,g}(x,y) \equiv 0$ then $\dot{v}(x,y) \equiv 0$ and v is a first integral of the differential system (1).

□

Outline of the talk

Lyapunov approach. QS with an algebraic limit cycle

Theorem

The eight families of quadratic systems detailed in the proof, and that nowadays include all the known cases of quadratic systems with an algebraic limit cycle, have at most one limit cycle and when it exists it is this algebraic limit cycle.

The known families are: Qin; Yablonskii; Filiptsov; Chavarriga; Chavarriga, Llibre and Sorolla; Christopher, Llibre and Świrszcz (two cases); and Alberich-Carramiñana, Ferragut and Llibre.

We do not study here the hyperbolicity of the limit cycles considered in the above theorem. This question is studied by Giacomini and Grau.

Lyapunov approach. QS with an algebraic limit cycle

Case 1: Qin system with algebraic limit cycle of degree 2:

$$P = -y(ax + by + c) - (x^2 + y^2 - 1), \quad Q = x(ax + by + c),$$
$$f = x^2 + y^2 - 1, \quad k = -2x, \quad \alpha = b/2, \quad g = c + by, \quad N = abx^2.$$

We remark that the corresponding QS has only $x^2 + y^2 - 1 = 0$ as a limit cycle when $a \neq 0$, $c^2 + 4(b+1) > 0$ and $a^2 + b^2 < c^2$ but the proof that there are no other limit cycles works for all values of the parameters. A similar fact also holds for all the other cases.

Y.-X. Qin, *On the algebraic limit cycles of second degree of the differential equation* $dy/dx = \sum_{0\leq i+j\leq 2} a_{ij}x_iy_j / \sum_{0\leq i+j\leq 2} b_{ij}x_iy_j$, Acta Math. Sinica **8** (1958), 23–35.

Lyapunov approach. QS with an algebraic limit cycle

Case 2: Yablonskii system with algebraic limit cycle of degree 4:

$$
\begin{aligned}
P &= -4abcx - (a+b)y + 3(a+b)cx^2 + 4xy, \\
Q &= (a+b)abx - 4abcy + (4abc^2 - 3(a+b)^2/2 + 4ab)x^2 \\
&\quad + 8(a+b)cxy + 8y^2, \\
f &= (y+cx^2)^2 + x^2(x-a)(x-b), \quad k = 4(-2abc + 3c(a+b)x + 4y), \\
g &= 2c(a-3b)(3a-b)(y+cx^2) - ab(a+b-4x)^2, \\
\alpha &= -1/2, \quad N = -c\big(2ab(a+b) + (2ab - 3a^2 - 3b^2)x\big)^2.
\end{aligned}
$$

A. I. Yablonskii, *On the limit cycles of a certain differential equation (in Russian),* Diff. Uravneniya **2** (1966), 335–344, translated in Differential Equations **2** (1966), 164–168.

Lyapunov approach. QS with an algebraic limit cycle

Case 3: Filipstov system with algebraic limit cycle of degree 4:

$$
\begin{aligned}
P &= 6(1+a)x + 2y - 6(2+a)x^2 + 12xy, \\
Q &= 15(1+a)y + 3a(1+a)x^2 - 2(9+5a)xy + 16y^2, \\
f &= 3(1+a)(ax^2+y)^2 + 2y^2(2y - 3(1+a)x), \\
k &= 6(5(1+a) - (8+4a)x + 8y), \\
\alpha &= -1/2, \quad g = 2y + (3+5a)x^2, \quad N = -\big(3(1+a)x - 4y\big)^2.
\end{aligned}
$$

V. F. Filipstov, *Algebraic limit cycles (in Rusian)*, Diff. Uravneniya **9** (1973), 1281–1288, translated in Differential Equations **9** (1973), 983–988.

Lyapunov approach. QS with an algebraic limit cycle

Case 4: Chavarriga system with algebraic limit cycle of degree 4:

$$P = 5x + 6x^2 + 4(1+a)xy + ay^2, \quad Q = x + 2y + 4xy + (2+3a)y^2,$$
$$f = x^2 + x^3 + x^2y + 2axy^2 + 2axy^3 + a^2y^4, \quad a \neq -10/3,$$
$$k = 2\big(5 + 9x + (5+6a)y\big), \quad \alpha = -\frac{7+3a}{3(10+3a)},$$
$$g = -5 + (21+9a)x - (35+15a)y,$$
$$N = \frac{2(7+3a)}{3(10+3a)}\big(5 + 9x + (5+6a)y\big)^2.$$

J. Chavarriga, J. Llibre, J. Sorolla, *Algebraic limit cycles of degree 4 for quadratic systems*, J. Differential Equations **200** (2004), 206–244.

Lyapunov approach. QS with an algebraic limit cycle

Case 5: Chavarriga, Llibre and Sorolla system with algebraic limit cycle of degree 4:

$$
\begin{aligned}
&P = 2(1 + 2x - 2ax^2 + 6xy), \quad Q = 8 - 3a - 14ax - 2axy - 8y^2, \\
&f = 1/4 + x - x^2 + ax^3 + xy + x^2y^2, \quad k = 4(2 - 3ax + 2y), \\
&\alpha = 1/3, \quad g = -5 + 3ax/2 + y, \quad N = -4(2 - 3ax + 2y)^2/3.
\end{aligned}
$$

J. Chavarriga, J. Llibre, J. Sorolla, *Algebraic limit cycles of degree 4 for quadratic systems*, J. Differential Equations **200** (2004), 206–244.

Lyapunov approach. QS with an algebraic limit cycle

Case 6: Christopher, Llibre and Świrszcz system with algebraic limit cycle of degree 5:

$$P = 28x + 2(16 - a^2)(a+12)x^2 + 6(3a-4)xy - \frac{12}{a+4}y^2, \quad a \neq -4,$$

$$Q = 2(16 - a^2)x + 8y + (16 - a^2)(a+12)xy + 2(5a-12)y^2,$$

$$f = x^2 + (16 - a^2)x^3 + (a-2)x^2y - \frac{2}{a+4}xy^2 - \frac{1}{4}(4-a)(a+12)x^2y^2 + \frac{8-a}{a+4}xy^3 + \frac{1}{(a+4)^2}y^4 + \frac{a+12}{a+4}xy^4 - \frac{6}{(a+4)^2}y^5,$$

$$k = 56 + 6(16 - a^2)(a+12)x + 4(13a-24)y, \; \alpha = -\frac{3+4a}{15(3+a)}, a \neq -3,$$

$$g = 28 + (-144 - 192a + 9a^2 + 12a^3)x + (42 + 56a)y,$$

$$N = -\frac{2(3+4a)}{15(3+a)}\left(28 + (576 + 48a - 36a^2 - 3a^3)x + (-48 + 26a)y\right)^2.$$

Lyapunov approach. QS with an algebraic limit cycle

Case 7: Christopher, Llibre and Świrszcz system with algebraic limit cycle of degree 6:

$$
\begin{aligned}
P &= 28a(a-30)x + y + 168a^2x^2 + 3xy,\\
Q &= 16a(a-30)\big(14a(a-30)x + 5y + 84a^2x^2\big) + 24a(17a-6)xy + 6y^2,\\
f &= 48a^3(a-30)^4x^2 + 24a^2(a-30)^3xy + 3a(a-30)^2y^2\\
&\quad + 64a^3(a-30)^3(9a-4)x^3 + 24a^2(a-30)^2(9a-4)x^2y\\
&\quad + 18a(a-30)(a-2)xy^2 - 7y^3 + 576a^3(a-30)^2(a-2)^2x^4\\
&\quad + 144a^2(a-30)(a-2)^2x^3y + 27a(a-2)^2x^2y^2\\
&\quad - 3456a^3(a-30)(a-2)^2(2a+3)x^5 - 432a^2(a-2)^2(2a+3)x^4y\\
&\quad + 3456a^3(a-2)^2(a+12)(2a+3)x^6,\\
k &= 168a(a-30) + 1008a^2x + 18y, \quad \alpha = -1/3,\\
g &= -16a(a-30)^2 - 24a(a-30)(7a-30)x\\
&\quad - 72a(360 - 78a + 5a^2)x^2 + 3(30+a)y,\\
N &= 896a^2(a-30)(a-30+6ax)^2.
\end{aligned}
$$

Lyapunov approach. QS with an algebraic limit cycle

Case 8: Alberich-Carramiñana, Ferragut and Llibre system with algebraic limit cycle of degree 5:

$$P = -8x + \frac{a}{2}(a-16)y - (5a-64)x^2 + \frac{a}{8}(a^2-256)xy,$$

$$Q = -28y + \frac{24}{a}x^2 - 3(3a-32)xy + \frac{a}{4}(a^2-256)y^2,$$

$$f = ay^2 - 4x^2y + \frac{a}{2}(a-12)xy^2 - \frac{a^2}{4}(a-16)y^3 + \frac{4}{a}x^4 + (24-a)x^3y + \frac{a}{16}(a^2-256)x^2y^2 - \frac{24}{a}x^5 + (a+16)x^4y,$$

$$k = -56 - 2(13a-152)x + \frac{3a}{4}(a^2-256)y, \quad \alpha = \frac{26-4a}{15(a-2)}, \; a \neq 2,$$

$$g = 112 + 56(2a-13)x + 3a(2a-13)(a-16)y,$$

$$N = \frac{2a-13}{60(a-2)}\big(-224 + (1216-104a)x + a(3a^2-768)y\big)^2.$$

Lyapunov approach. QS with an algebraic limit cycle

Cases 6 and 7:

C. J. Christopher, J. Llibre, G. Świrszcz, *Invariant algebraic curves of large degree for quadratic systems*, J. Math. Anal. Appl. **303(2)** (2005) 450–461.

Case 8:

M. Alberich-Carramiñana, A. Ferragut, J. Llibre, *Quadratic planar differential systems with algebraic limit cycles via quadratic plane Cremona maps*, Adv. Math. **389** (2021), 107924:1–38.

In fact, for completeness, our proof studies all the eight cases, although from the results of this last paper it can be seen that these eight cases can be reduced to four of them because the other ones can be obtained from these four via suitable Cremona transformations (birational automorphisms) and changes of time.

Outline of the talk

QS with an invariant parabola

It is known that QS with an invariant parabola can have limit cycles. Moreover when they have a limit cycle, Kooij and Zegeling proved in 1994 that it is unique and hyperbolic.

By using our approach, for some cases of the general family of QS with an invariant parabola ($y - x^2 = 0$),

$$\begin{cases} \dot{x} = a + bx + hy + c(y - x^2) + exy, \\ \dot{y} = 2x(a + bx + hy) + d(y - x^2) + 2ey^2, \end{cases}$$

we will prove a non-existence result.

QS with an invariant parabola. Sketch of an alternative proof

Theorem

A quadratic system with an invariant parabola and a limit cycle can be transformed into another quadratic system with an invariant straight line. As a consequence, quadratic systems with an invariant parabola have at most one limit cycle and when it exists it is hyperbolic.

This proof is based of the use of suitable birrational transformations, also called Cremona maps and gives an alternatice proof to that of Kooij and Zegeling.

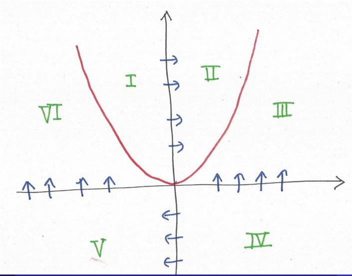
Lyapunov approach. QS with an invariant parabola.

IDEA OF THE PROOF: Consider system

$$\begin{cases} \dot{x} = bx + hy + c(y - x^2) + exy, \\ \dot{y} = 2x(bx + hy) + d(y - x^2) + 2ey^2, \end{cases}$$

and let γ be one of its limit cycles. It is totally contained in one of the six connected components of $\mathbb{R}^2 \setminus (\mathcal{P} \cup \{x = 0\} \cup \{y = 0\})$. This is so because $\mathcal{P}$ is invariant and moreover

$$\dot{x}|_{x=0} = (c + h)y, \quad \text{and} \quad \dot{y}|_{y=0} = (2b - d)x^2.$$



QS with an invariant parabola

Call $\mathcal{V}$ this connected component which, of course, must contain a critical point of index $+1$. Hence in particular the new birrational change of variables

$$X = \frac{x}{y - x^2}, \quad Y = \frac{y}{y - x^2}$$

is well defined on the region $\mathcal{V}$ where γ lies, and its inverse is

$$x = \frac{Y-1}{X}, \quad y = \frac{Y(Y-1)}{X^2}.$$

It transforms the parabola $y - x^2 = 0$ into the straight line $Y - 1 = 0$. By introducing a new time s such that $ds/dt = X$, and writing $Z' = dZ/ds = XdZ/dt = X\dot{Z}$, after some computations we get that

$$\begin{cases} X' = X\dot{X} = -cX + eY + (b-d)X^2 + (2c+h)XY - eY^2, \\ Y' = X\dot{Y} = (Y-1)\big((2b-d)X + 2(c+h)Y\big), \end{cases}$$

which is a quadratic system with the invariant straight line $Y - 1 = 0$, as we wanted to prove. □

Lyapunov approach. QS with an invariant parabola

Proposition

Consider the QS with the invariant parabola $y - x^2 = 0$,

$$\begin{cases} \dot{x} = a + bx + hy + c(y - x^2) + exy, \\ \dot{y} = 2x(a + bx + hy) + d(y - x^2) + 2ey^2. \end{cases}$$

If

$$\begin{aligned} \Delta = \Big(&16a^2e^3 + ((-8bc - 16hb - 12cd - 8dh)a + (2b - d)^3\big)e^2 \\ &- 2(c + h)\big(4c(c + 2h)a - 8b^2c + 6bcd - 4bdh - 3cd^2\big)e \\ &+ 8c(c + h)^2(bc + dh)\Big)\Big((2b - d)e + 2c^2 + 2hc\Big)e \geq 0. \end{aligned}$$

the system does not have periodic orbits.

Lyapunov approach. QS with an invariant parabola

Proof.

The invariant parabola is $f(x,y) = y - x^2 = 0$, and its cofactor is $k(x,y) = -2cx + 2ey + d$. When $\Delta \geq 0$, it can be seen that there exist suitable α and $g(x,y) = g_0 + g_1x + g_2y$ such that in our theorem we get that

$$N_{\alpha,g} = \sum_{0\leq i+j\leq 2} n_{i,j}x^iy^j = r\left(\sum_{0\leq i+j\leq 1} w_{i,j}x^iy^j\right)^2.$$

□

Outline of the talk

Lyapunov approach. A Liénard system

In this example, instead of using invariant algebraic curves we will use exponential factors, that are a particular case of generalized invariant curves. An exponential factor is a function $f(x, y) = \exp(h(x, y))$, with h a polynomial and such that it holds that

$$\dot{f}(x, y) = k(x, y) f(x, y),$$

as in the condition of invariant algebraic curve, with k being also a polynomial, also called its cofactor.

Our results also apply if the f of its statement is an exponential factor, instead of an invariant algebraic curve. A simple example of exponential factor for the Liénard system

$$\begin{cases} \dot{x} = y - xH(x), \\ \dot{y} = -x \end{cases}$$

is $f(x, y) = \exp(y)$, because $\dot{f}(x, y) = -xf(x, y)$ and $k(x, y) = -x$.

Lyapunov approach. A Liénard system

Proposition

If there exists $\alpha \in \mathbb{R}$ such that $\alpha x + 2H(x) \not\equiv 0$ does not change sign then the Liénard system

$$\begin{cases} \dot{x} = y - xH(x), \\ \dot{y} = -x \end{cases}$$

does not have periodic orbits.

Proof.

We can apply our approach by taking f as the exponential factor $f(x,y) = \exp(y)$. We also take $g(x,y) = 2 - 2\alpha y - \alpha^2 x^2$. Then some computations give that

$$N_{\alpha,g}(x,y) = \alpha^2 x^2 \big(\alpha x + 2H(x)\big),$$

and the result follows. □

Lyapunov approach. A Liénard system

Corollary

Liénard system

$$\begin{cases} \dot{x} = y - a_1 x - a_2 x^2 - x^3, \\ \dot{y} = -x \end{cases}$$

does not have periodic solutions when $a_1 \geq 0$.

Proof.

By using the above Proposition with $\alpha = 4\sqrt{a_1} - 2a_2$ and $H(x) = a_1 + a_2 x + x^2$ we get that

$$\alpha x + 2H(x) = 2a_1 + (\alpha + 2a_2)x + 2x^2 =$$
$$= 2a_1 + 4\sqrt{a_1}x + 2x^2 = 2\left(\sqrt{a_1} + x\right)^2 \geq 0,$$

and the result follows. □

Outline of the talk

Bendixson-Dulac theorem

Recall the version of the Bendixson–Dulac theorem for multiply connected regions:

Theorem

Consider a $\mathcal{C}^1$ planar differential system

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y),$$

defined on $\mathcal{U} \subset \mathbb{R}^2$, an open connected subset such that $\mathbb{R}^2 \setminus \mathcal{U}$ has $\ell = \ell(\mathcal{U})$ bounded components, and denote by $X = (P, Q)$ its associated vector field. Let $B : \mathcal{U} \to \mathbb{R}$ be a $\mathcal{C}^1$ function such that

$$\operatorname{div}(BX) = (BP)_x + (BQ)_y$$

does not change sign and vanishes only on a null measure set which is not invariant by the flow. Then the system has at most ℓ limit cycles in $\mathcal{U}$.

Bendixson-Dulac theorem. Definition of $L(V)$

Given an open connected subset $\mathcal{U} \subset \mathbb{R}^2$, with finitely many holes, we have denoted by $\ell = \ell(\mathcal{U})$ this number of holes, that is, the number of bounded components of $\mathbb{R}^2 \setminus \mathcal{U}$. Notice that if $\mathcal{U}$ is simply connected then $\ell(\mathcal{U}) = 0$.

Definition

Given a function $V : \mathbb{R}^2 \to \mathbb{R}$ of class $\mathcal{C}^1$ we will say that it is admissible if:

(i) The vector ∇V vanishes on $\{V(x, y) = 0\}$ at finitely many points.

(ii) The set $\{V(x, y) = 0\}$ has finitely many connected components.

(iii) The set $\mathbb{R}^2 \setminus \{V(x, y) = 0\}$ has j connected components, $\mathcal{U}_i, i = 1, 2, \ldots j$, and for all of them $\ell(\mathcal{U}_i) < \infty$.

Associated to V, we define the non negative integer number

$$L(V) := \sum_{i=1}^{j} \ell(\mathcal{U}_i).$$

Bendixson-Dulac theorem

Theorem (A version of Bendixson–Dulac theorem)

Consider a $\mathcal{C}^1$ planar differential system

$$\dot{x} = P(x,y), \quad \dot{y} = Q(x,y),$$

and denote by $X = (P,Q)$ its associated vector field. Let $V : \mathbb{R}^2 \to \mathbb{R}$ be an admissible function such that there exists $s \in \mathbb{R}^+$ for which the function

$$M_s := M_{s,V} = \frac{\partial V}{\partial x} P + \frac{\partial V}{\partial y} Q - s \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) V$$

does not change sign and vanishes only on a null measure set, not invariant by the flow of X. Then the system has at most $L_X(V) := N + L(V)$ limit cycles, where N is the number of periodic orbits of X contained in the set $\mathcal{V} = \{V(x,y) = 0\}$ and $L(V)$ the introduced computable number that depends on the shape of the set $\{V(x,y) = 0\}$.

Bendixson-Dulac theorem, 2nd version. Idea of the proof

In a few words we apply the classical Bendison-Dulac theorem with $B = |V|^{-1/s}$ to each of the connected components of $\mathbb{R}^2 \setminus \{V(x,y)=0\}$. The key points are:

- The formula:

$$\operatorname{div}\left(|V|^{-1/s}X\right) = -\frac{1}{s}\operatorname{sign}(V)|V|^{-1/s-1}M_s$$

  where

$$M_s = M_{s,V} = \frac{\partial V}{\partial x}P + \frac{\partial V}{\partial y}Q - s\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right)V.$$

- The fact that

$$M_s\Big|_{V=0} = \frac{\partial V}{\partial x}P + \frac{\partial V}{\partial y}Q$$

  does not change sign implies that the periodic orbits of the system not contained in $\{V(x,y)=0\}$ can not cut this set.

Outline of the talk

Break: a comparison Lyapunov/B-D

In both cases we need that a given function does not change sign. More concretely:

- Lyapunov (for systems with an invariant algebraic curve $f = 0$ with cofactor k):

$$N_{\alpha,g} = \alpha k + \frac{\partial g}{\partial x} P + \frac{\partial g}{\partial y} Q.$$

- Bendixson-Dulac:

$$M_{s,V} = \frac{\partial V}{\partial x} P + \frac{\partial V}{\partial y} Q - s\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) V.$$

Outline of the talk

Bendixson-Dulac theorem. First example of application

Consider the family of differential systems introduced by Villari and Zanolin in 2020,

$$\begin{cases} \dot{x} = y - \lambda |y|^m (x^3 - x). \\ \dot{y} = -x, \end{cases} \tag{4}$$

where $1 < m \in \mathbb{N}$ and $\lambda \in \mathbb{R}$.

Theorem

For the differential system (4) the following holds:

(i) *For $|\lambda| \neq 0$ small enough it has at least one limit cycle.*

(ii) *For $|\lambda| \geq \dfrac{3}{\sqrt{2}} \left(\dfrac{3}{m} \right)^{m/2}$ it has no limit cycle.*

Notice that the limit cycles exist only for some values of λ contained in the interval of length $3\sqrt{2}(3/m)^{m/2}$, centered at the origin. Notice that for m big it is extremely thin. This interval decreases exponentially with m.

Bendixson-Dulac theorem. First example of application

(i) Recall that given any perturbed Hamiltonian systems,

$$\begin{cases} \dot{x} = \quad \dfrac{\partial H(x,y)}{\partial y} + \varepsilon R(x,y), \\ \dot{y} = -\dfrac{\partial H(x,y)}{\partial x} + \varepsilon S(x,y), \end{cases} \tag{5}$$

where ε is a small parameter, each simple zeroe of its associated Melnikov–Poincaré–Pontryagin function

$$M(h) = \int_{\gamma(h)} S(x,y)\,\mathrm{d}x - R(x,y)\,\mathrm{d}y,$$

where the curves $\gamma(h)$ are a continuum of ovals contained in $\{H(x,y) = h\}$, gives rise to hyperbolic limit cycle of (5) that tends, when $\epsilon \to 0$, to some of these ovals.

Bendixson-Dulac theorem. First example of application

Consider the differential system

$$\begin{cases} \dot{x} = y - \lambda|y|^m(x^3 - x). \\ \dot{y} = -x, \end{cases}$$

with $\lambda = \varepsilon$. By applying the above result with $H(x, y) = x^2 + y^2 = h = r^2$, with $r \in (0, \infty)$, and taking the parameterization of the level sets as $x = r\cos\theta$, $y = r\sin\theta$, we get that

$$M(r^2) = \int_{x^2+y^2=r^2} |y|^m(x^3 - x)\,\mathrm{d}y = \int_0^{2\pi} r^m|\sin\theta|^m(r^4\cos^4\theta - r^2\cos^2\theta)\,\mathrm{d}\theta$$

$$= \frac{\sqrt{\pi}}{2}\frac{\Gamma\big((m+1)/2\big)}{\Gamma\big((m+6)/2\big)} r^{m+2}\big(3r^2 - (m+4)\big),$$

where Γ is the Euler Gamma function. Hence, for each m, this function has a simple positive zero $r = \sqrt{(m+4)/3}$, that gives rise to the desired limit cycle.

Bendixson-Dulac theorem. First example of application

(ii) Take $s = 1/3$ and $V(x,y) = \exp\left(\frac{\lambda^2 y^{2m}}{9m}\right)\left(3 + \lambda xy|y|^{m-2}\right)$ in Bendixson-Dulac theorem. Some calculations give that

$$M_{1/3} = -\frac{1}{9}\exp\left(\frac{\lambda^2 y^{2m}}{9m}\right)\lambda x^2 |y|^{m-2}\left(2\lambda^2 y^{2m} - 27y^2 + 9(m-1)\right).$$

We need that $M_{1/3}$ does not change sign. Writing $y^2 = z$ we want that

$$z^m - \frac{27}{2\lambda^2} z + \frac{9(m-1)}{2\lambda^2} \geq 0 \quad \text{for} \quad z \geq 0.$$

It can be proved that for $P(z) = z^m + bz + c$, with $m \geq 2$, it holds that $P(z) \geq 0$ for all $z \geq 0$ if and only if $b \geq -m\big(c/(m-1)\big)^{(m-1)/m}$. This inequality gives the condition of the statement.

Since $\{V(x,y) = 0\}$ does not contain ovals and all the connected components of $\mathbb{R}^2 \setminus \{V(x,y) = 0\}$ are simply connected, we have that $L_X(V) = 0$ and the system does not have limit cycles, as we wanted to prove. □

Outline of the talk

Bendixson-Dulac theorem. Second example of application

Proposition

Consider the $\mathcal{C}^1$ *differential system*

$$\begin{cases} \dot{x} = y - C(x)B(x), \\ \dot{y} = -C(x)C'(x), \end{cases}$$

with $C(0) = 0$ *and* $C'(x) \neq 0$ *for* $x \neq 0$. *If* $C(x)B'(x)$ *does not change sign and vanishes at isolated points, then this system has at most one limit cycle and when it exists it is hyperbolic.*

Notice that the van der Pol equation corresponds to $C(x) = x$ and $B(x) = \lambda(x^2/3 - 1)$. Then $C(x)B'(x) = 2\lambda x^2/3$, which does not change sign.

Bendixson-Dulac theorem. Second example of application

$$\begin{cases} \dot{x} = y - C(x)B(x), \\ \dot{y} = -C(x)C'(x), \end{cases}$$

Proof.

We take

$$V(x,y) = y^2 - C(x)B(x)y + C^2(x), \quad \text{and} \quad s = 1$$

in our version of Bendixson-Dulac theorem. Then some computations give that

$$M_1(x) = C^3(x)B'(x).$$

Hence, it does not change sign and vanishes at isolated points. Moreover it can be seen that the set $\{V(x,y) = 0\}$ has only one bounded connected component and then $L_X(V) = 1$ as we wanted to prove □

Bendixson-Dulac theorem. Second example of application

With similar ideas it can be proved:

Theorem

Consider planar differential equations of the form

$$\begin{cases} \dot{x} = y - |y|^m F(x), \\ \dot{y} = -G'(x)/2, \end{cases}$$

where F *and* G' *are* $\mathcal{C}^1$ *functions satisfying* $F(0) = 0$ *and* $G(x) = x^{2k} + o(x^{2k})$, $m \in \mathbb{N} \cup \{0\}$ *and* $k \in \mathbb{N}$.
If the function

$$(m-1)F(x)G'(x) + 2F'(x)G(x)$$

does not change sign and vanishes at isolated points, then the system has at most J *limit cycles, all of them hyperbolic, where* J *is the number of zeroes of* G'.

Outline of the talk

Bendixson-Dulac theorem. Third example of application

We study a family of Liénard type equations introduced recently that includes the classical Wilson family of Liénard equatins, which gave the first example of such equations having an algebraic limit cycle. More concretely, we consider systems

$$\begin{cases} \dot{x} = y - (x^2 - 1)B(x), \\ \dot{y} = -x(1 + yB(x)), \end{cases}$$

where B is a $\mathcal{C}^1$ function. They have the invariant algebraic curve $f(x, y) = x^2 + y^2 - 1 = 0$, because

$$f_x(x, y)P(x, y) + f_y(x, y)Q(x, y) = -2xB(x)f(x, y).$$

The example studied in previous works corresponds to $B(x) = x^3 - bx$.

Bendixson-Dulac theorem. Third example of application

Theorem

Consider the system

$$\begin{cases} \dot{x} = y - (x^2 - 1)B(x), \\ \dot{y} = -x(1 + yB(x)), \end{cases}$$

with

$$B(x) = x \int_0^x W(t)/t \, \mathrm{d}t - bx,$$

where W is any function that does not change sign, vanishes at isolated points, and such that $B \in \mathcal{C}^1$. Then it has at most $L + 2$ limit cycles, where L is the number of bounded connected components of the set $\mathcal{B} = \{x \in \mathbb{R} : (B(x) + 2x)(B(x) - 2x) \geq 0\}$.

When $B(x) = x^3 - bx$ we have that $W(t) = 2t^2 \geq 0$.

Bendixson-Dulac theorem. Third example of application

Proof.

We apply our version of Bendixson-Dulac theorem with

$$V = (1 - x^2 - y^2)\left(x^2 + y^2 + B(x)y\right) \quad \text{and} \quad s = 1.$$

Then,

$$M_1(x, y) = x(x^2 + y^2 - 1)^2\left(B(x) - xB'(x)\right) = -x^2(x^2 + y^2 - 1)^2 W(x).$$

Hence, we can apply Bendixson-Dulac theorem. To get $L(V)$ we must study the bounded connected components of $\mathcal{V}$. Notice that these components are formed by the oval $x^2 + y^2 - 1 = 0$ together with the components of $x^2 + y^2 + B(x)y = 0$. This curve also writes as

$$y = \frac{-B(x) \pm \sqrt{(B(x) + 2x)(B(x) - 2x)}}{2}.$$

Outline of the talk

Bendixson-Dulac theorem. Fourth example of application

Consider the Liénard system

$$\begin{cases} \dot{x} = y - bx + x^3 + \frac{4b}{3}x^3 - \frac{6}{5}x^5, \\ \dot{y} = -x + b^2x^3 - b(2+b)x^5 + (1+2b)x^7 - x^9, \end{cases} \tag{6}$$

By taking $s = 1$ and $V(x,y) = A(x,y)B(x,y)$, where

$$\begin{aligned} A(x,y) = & -225 + 225x^2 + 25b^2x^6 - 30bx^8 + 9x^{10} \\ & + (150bx^3 - 90x^5)y + 225y^2, \\ B(x,y) = & 225x^2 - 75b^2x^4 + 5b(24+5b)x^6 - 15(3+2b)x^8 + 9x^{10} \\ & + (-225bx + 25(9+6b)x^3 - 90x^5)y + 225y^2, \end{aligned}$$

we get that $M_1 = 2x^4A^2(x,y) \geq 0$ and BD theorem can be applied.

Outline of the talk

Bendixson-Dulac theorem. Fifth example of application

It is known that the function

$$K^{\perp} := Q^2 P_x + P^2 Q_y - PQ(P_y + Q_x),$$

that is the numerator of the curvature of the orbits of the vector field $X^{\perp} = (-Q, P)$, orthogonal to the vector field $X = (P, Q)$, can be used to know the stability of the periodic orbits of X. For instance, Diliberto in 1950 proved that a limit cycle is hyperbolic and stable (resp. unstable) if and only if

$$\int_0^l K^{\perp}(\gamma(s))\,\mathrm{d}s < 0 \quad (\text{resp. } > 0),$$

where $\gamma(s)$ is its parameterization by the arc length and l is its length. We will show that the function

$$K := Q^2 P_y - P^2 Q_x + PQ(P_x - Q_y),$$

proportional to the numerator of the curvature of the orbits of X is, together with $s = 1$, a good candidate for a suitable V in Bendixson-Dulac theorem.

Bendixson-Dulac theorem. Fifth example of application

By taking $V = K$ and $s = 1$ in Bendixson-Dulac theorem we prove:

Theorem

Consider a class $\mathcal{C}^2$ planar system

$$\dot{x} = P(x,y), \quad \dot{y} = Q(x,y),$$

and assume that the function

$$D := P^2 Q\big(P_{xx} - 2Q_{xy}\big) + PQ^2\big(2P_{xy} - Q_{yy}\big) + Q^3 P_{yy} - P^3 Q_{xx}$$

does not change sign and vanishes on a null measure set not invariant by the flow of X. Then it has at most $L_X(V)$ limit cycles, where

$$V = K = Q^2 P_y - P^2 Q_x + PQ(P_x - Q_y)$$

and $L_X(V)$ is defined in Bendixson-Dulac Theorem.

Bendixson-Dulac theorem. Fifth example of application

Consider the rigid systems

$$\begin{cases} \dot{x} = -y + xF(x,y), \\ \dot{y} = \quad x + yF(x,y), \end{cases} \tag{7}$$

introduced by Conti. Here F is an arbitrary smooth function. Notice that in polar coordinates $\dot{\theta} = 1$. A consequence of the previous result is:

Theorem

Let X be the vector field associated to (7). If $F \in \mathcal{C}^2$,

$$H := F_{xx}F_{yy} - F_{xy}^2 \geq 0,$$

and H vanishes on a null measure set, not invariant by the flow of X, then (7) has at most $L_X(V)$ limit cycles, where

$$V = (x^2 + y^2)\left(xFF_x + yFF_y + xF_y - yF_x - 1 - F^2\right)$$

and $L_X(V)$ is defined in Bendixson-Dulac Theorem.

Bendixson-Dulac theorem. Fifth example of application

We apply previous theorem (BD with $V = K$ and $s = 1$) when $P = -y + xF$ and $Q = x + yF$. We get that V is as in the statement and

$$
\begin{aligned}
M_1 = D = (x^2+y^2)\Big(& \big(x^2 F_{xx} + 2xyF_{xy} + y^2 F_{yy}\big) F^2 \\
& + 2\big((x^2-y^2)F_{xy} + xy(F_{yy} - F_{xx})\big) F \\
& + \big(x^2 F_{yy} - 2xyF_{xy} + y^2 F_{xx}\big)\Big).
\end{aligned}
$$

To control the sign of M_1 we first remove the factor $x^2 + y^2$. Notice that the discriminant of the remaining part, thinking it as a second degree polynomial in F, $AF^2 + BF + C$, is $B^2 - 4AC = -4(x^2+y^2)^2 H \leq 0$. Moreover, looking to A and B as quadratic homogenous polynomials of the form $ax^2 + bxy + cy^2$, we get that their corresponding discriminants coincide and are given by $b^2 - 4ac = -4H \leq 0$. Therefore, our condition on H implies that M_1 does not change sign and vanishes only on a null measure set and hence our result follows.

Bendixson-Dulac theorem. Fifth example of application

Notice that the upper bound for the number of limit cycles given in the above theorem essentially depends on the shape of the set $\{V(x,y)=0\}$. To get the actual value of $L_X(V)$ for each case this set must be carefully studied. We present now a concrete application.

Corollary

Consider the rigid cubic systems, that have

$$F = a + bx + cy + dx^2 + exy + hy^2.$$

If $4dh - e^2 > 0$ *they have at most one (hyperbolic) limit cycle.*

This result is not new. It was proved by G., Prohens and Torregrosa by transforming the system into an Abel differential equation and then by applying know results about these equations. In that work it was also proved that when $4dh - e^2 < 0$ there are systems with at least two limit cycles.

Do you want to know an actual application of knowing the number of limit cycles?

Thank you very much for your attention