About Dulac's problem on non-accumulation of limit cycles in dimension three

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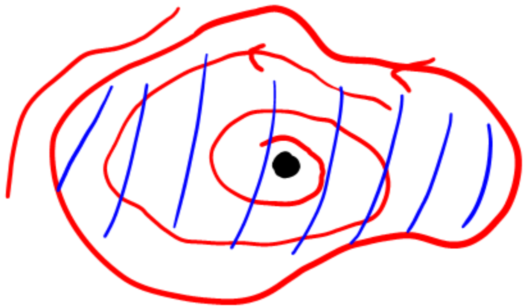
Motivation: Dulac's problem

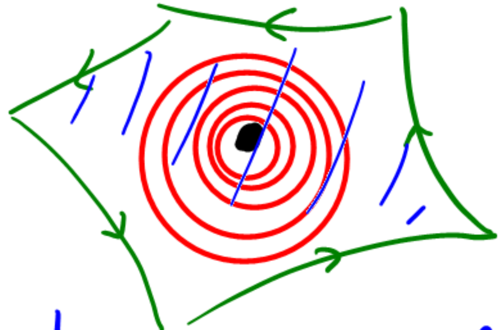
ξ = analytic v.f. at $0 \in \mathbb{R}^2$

<u>**Theorem.**</u>- There are no infinitely many limit cycles of ξ accumulating to 0.



no local l.c.

or



continuum of l.c.
(center configuration)

<u>History</u>: Dulac's proof (1923) $\rightsquigarrow$ Discover of a gap by Ilyashenko (1981)
$\rightsquigarrow$ Two independent <u>proofs?</u> by Ilyashenko and Écalle (1991-92)

Dulac's problem: the easiest case

ξ is a non-degenerated center-focus (or Hopf singularity) if $\operatorname{Spec} D\xi(0)=\{\pm bi\}$, $b\neq 0$

Poincaré first return map $\quad P:\Sigma' \longrightarrow \Sigma$, $\Sigma' \subset \Sigma$ $\quad$ analytic in this case

local limit cycle $\Longleftrightarrow$ isolated fixed point of P

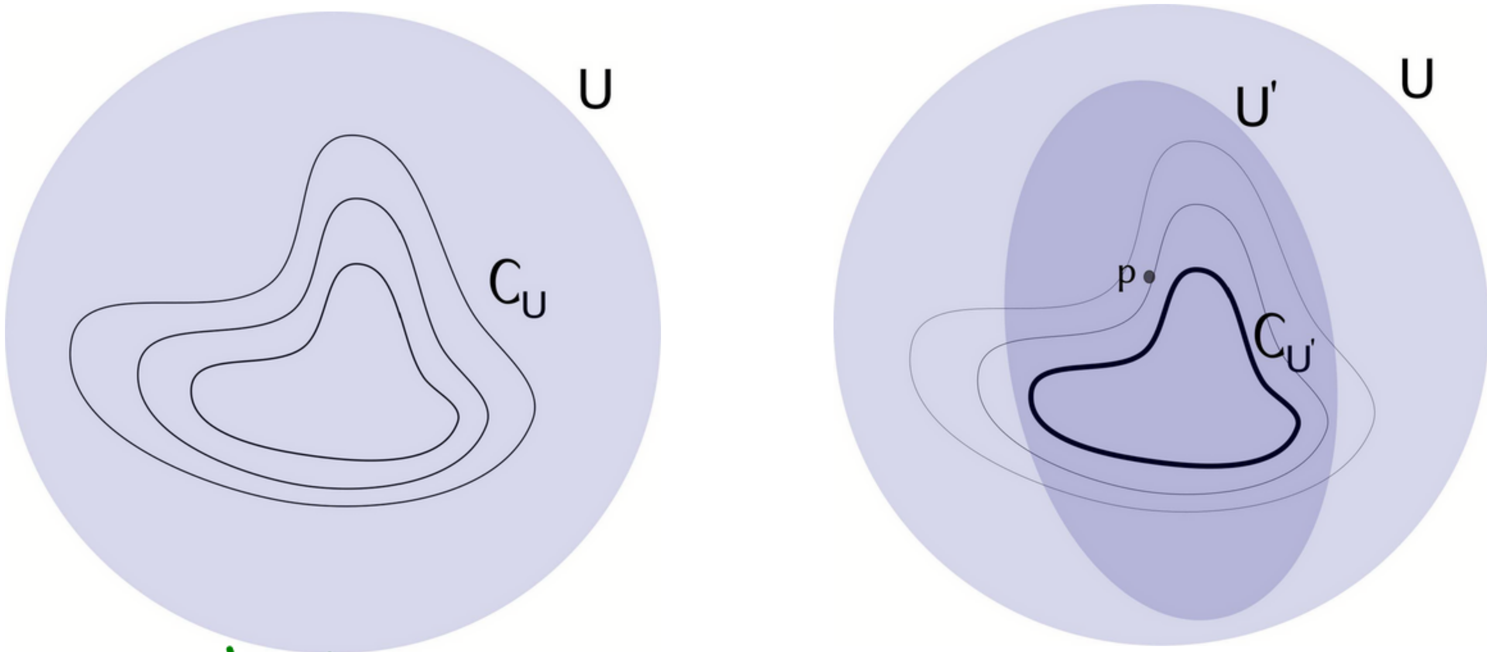
finitely many unless $P \equiv id$

Local set of periodic orbits

In general, if ξ is an analytic v.f. at $0 \in \mathbb{R}^n$ and $\mathcal{U} \ni 0$ is a nbhd,

$$\mathcal{C}_{\mathcal{U}} := \bigcup_{\substack{\gamma \text{ p.o. of } \xi, \\ \gamma \subset \mathcal{U}}} \gamma$$

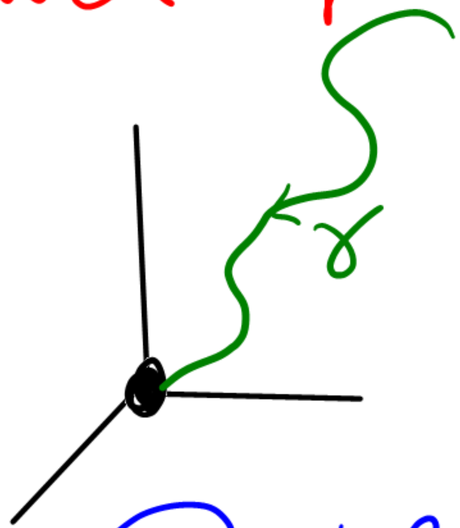
$$\boxed{\mathcal{C}_{\mathcal{U}'} \neq \mathcal{C}_{\mathcal{U}} \cap \mathcal{U}'}$$



It is not "germifiable" !!

No "centers" in dimension 3

Brunella's Theorem.- If $0 \in \mathbb{R}^3$ is an isolated singularity then there exists a trajectory of ξ accumulating to 0 with a well defined tangent (in fact with I.T.)



$\Longrightarrow$ $G_U \neq U \setminus \{0\}$ for any nbhd U.

Dulac type vector fields

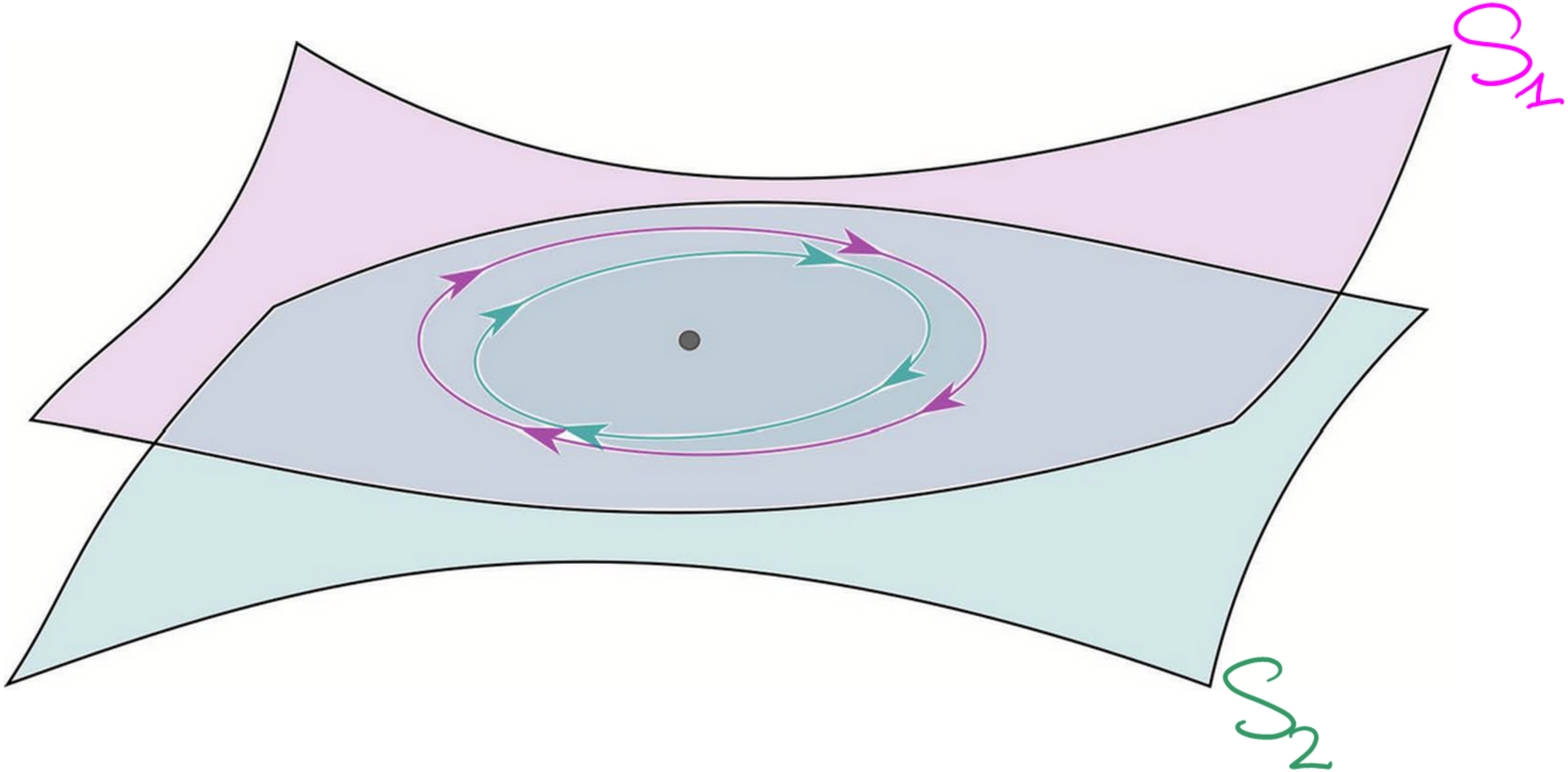
Definition.- ξ analytic v.f. at $0 \in \mathbb{R}^3$. We say that ξ is of **Dulac type** if

(1) Either there exists $\mathcal{U} \ni 0$ s.t. $\boxed{\mathcal{C}_{\mathcal{U}} = \emptyset}$

(2) Or there are finitely many topological surfaces $S_1, S_2, \dots, S_r$ with $S_i \cap S_j = \{0\}$ if $i \neq j$ and there exists $\mathcal{U} \ni 0$ s.t.

$$\boxed{\mathcal{C}_{\mathcal{U}} = (S_1 \cup S_2 \cup \dots \cup S_r) \cap (\mathcal{U} \setminus \{0\})}$$

In case (2), $S_1, \dots, S_r$ are called **central-center surfaces**



Remark.- If ξ is of Dulac type, there are NO local limit cycles accumulating to $0\in\mathbb{R}^3$.

The result

Let $\mathcal{H}_3 := \{\xi$ an. v.f. at $0 \in \mathbb{R}^3 \,/\, \mathrm{Spec}(D\xi(0)) \supset \{\pm bi\}$ for some $b \neq 0\}$

("3D-Hopf singularities")

In other words: $\xi \in \mathcal{H}_3$ if it can be written as

$$\xi = (-by + A)\frac{\partial}{\partial x} + (bx + B)\frac{\partial}{\partial y} + (cz + C)\frac{\partial}{\partial z}$$

with $c \in \mathbb{R}$, $A, B, C = O(\|x,y,z\|^2)$

- Semi-hyperbolic ($c \neq 0$)
- Completely non-hyperbolic ($c = 0$)

Theorem 1 (Corral-Martín-S.) If $\xi \in \mathcal{H}_3$ and 0 is an isolated singularity $\Rightarrow$ ξ is of Dulac type.

(Moreover, the central-center surfaces are subanalytic)

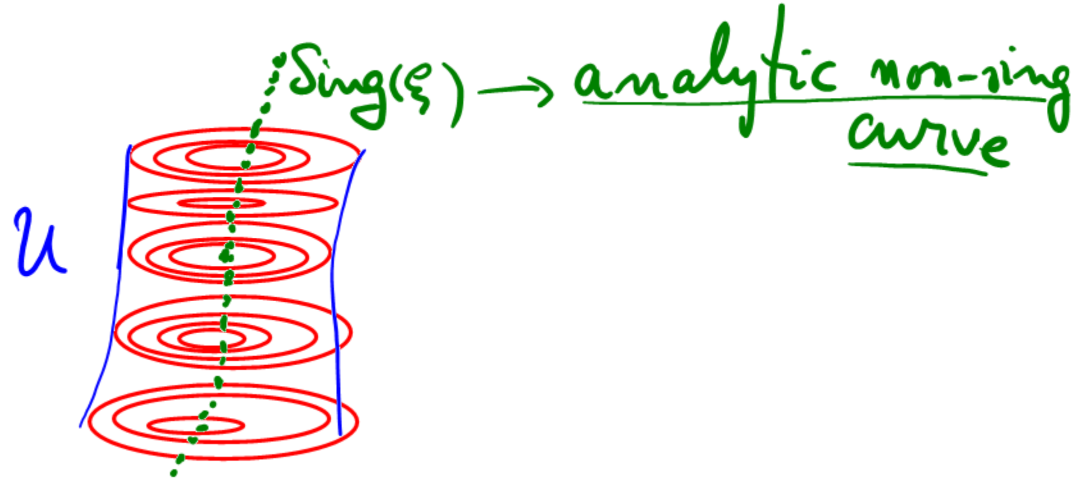
in case (2)

The result

Theorem 2 If $\xi \in \mathcal{H}_3$ and $\mathrm{Sing}(\xi)$ is a curve, then

- either ξ is of Dulac type.
- or there exists a nbhd $\mathcal{U} \ni 0$ such that

$$\boxed{\mathcal{C}_{\mathcal{U}} = \mathcal{U} \setminus \mathrm{Sing}(\xi)}$$



Background: Dumortier's work 1985

$\xi \in \mathcal{H}_3^\infty$ (class C^∞) with Łojasiewicz inequalities

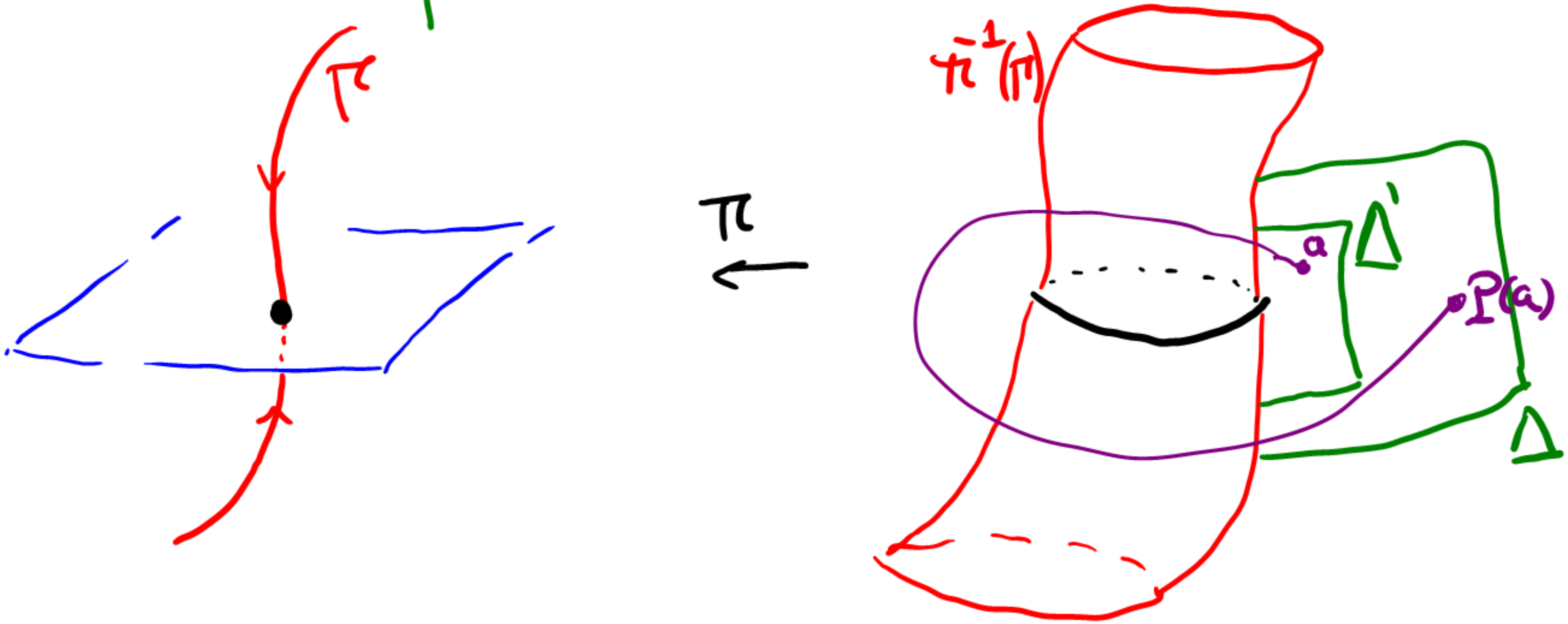
(1) $\|\xi(\underline{x})\| \geq \|\underline{x}\|^k$ for some $k \Rightarrow$ isolated sing.

(2) After blowing-up the invariant curve Γ the infinitesimal generator $\hat{\eta}$ of the Poincaré map $P : \Delta' \longrightarrow \Delta$ satisfies also a Łojasiewicz ineq.

$\Downarrow$

no local p.o.

$\boxed{\mathcal{C}_u = \emptyset}$



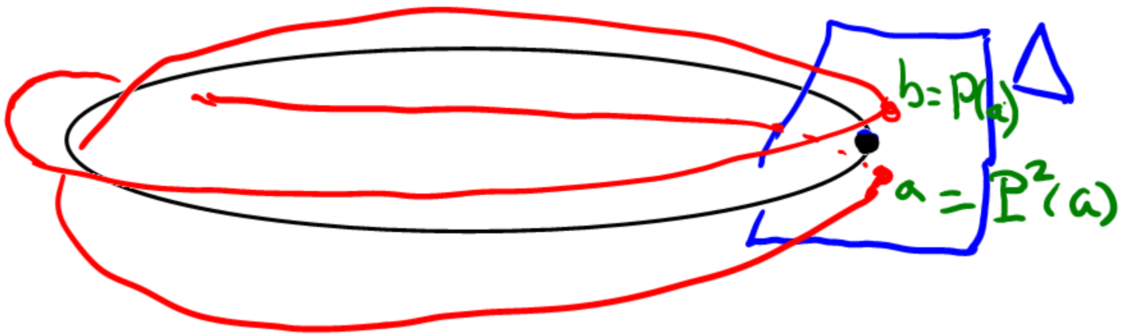
Warning

For a Poincaré first return map P around a periodic orbit γ in dimension $\geqslant 3$

Fixed points of P $\Rightarrow$ periodic orbits

BUT

Fixed points of some iterate P^m $\Leftarrow$ periodic orbits



Background: I. García's work 2021

Family $\{\xi_\lambda\}_{\lambda\in\Omega} \subset \mathcal{H}_3$ such that

- ξ_λ polynomial of some degree $\leq d$
- $D\xi_\lambda(0) = -y\partial_x + x\partial_y + 0\cdot z\partial_z$ (completely non-hyperbolic)
- ξ_λ has an analytic invariant curve Γ (tangent to $c=0$)
- Ω is some open semi-algebraic set in the space of coefficients of polynomial vector fields.

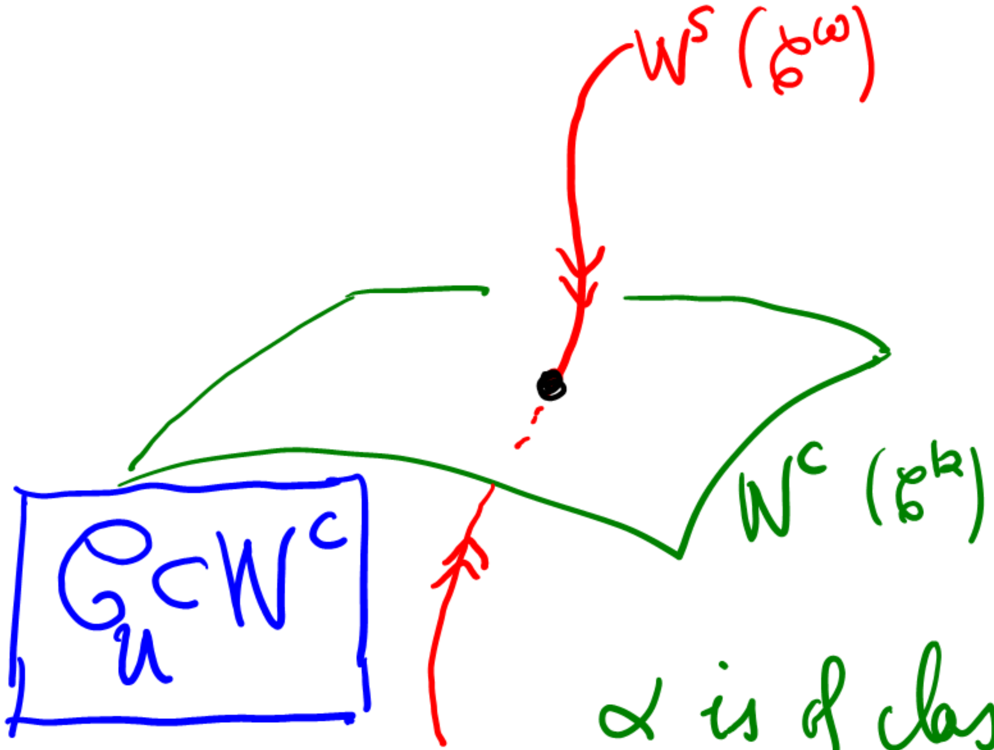
<u>Result</u>.- (Finite m-cyclicity). Given $\lambda_0\in\Omega$ and $m\in\mathbb{N}_{\geq 1}$ <u>fixed</u>, there exists $N=N(\lambda_0,m)$ and nbhds $0\in\mathcal{U}\subset\mathbb{R}^3$ and $\lambda_0\in V_0\subset\Omega$ such that $\forall\lambda\in V_0$, ξ_λ has at most N limit cycles in $\mathcal{U}$ that make <u>m turns</u> around Γ ($\Longleftrightarrow$ <u>isolated zeros</u> of $P^m:\Delta^m\to\Delta$)

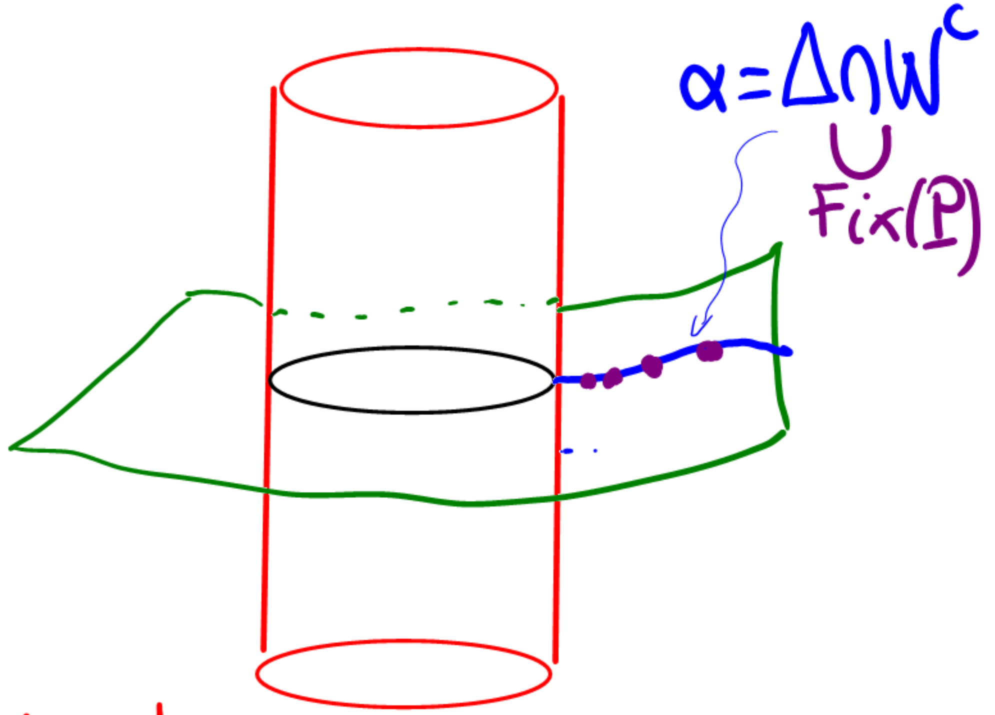
Background: the semi-hyperbolic case

$(c \neq 0)$

This case is already solved (Smidcht '78, Kirchgraber '80, Aulbach '85, ...)

<u>A pictorial proof:</u>



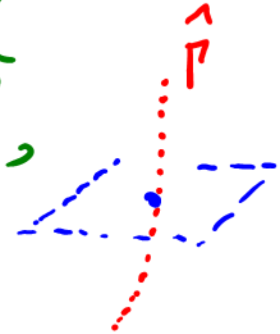


α is of class $\mathcal{C}^k$

but $\mathrm{Fix}(P)$ is analytic!

General case: Ideas of the proof. FNF

Assume that $D\xi(0) = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$. Use:



- There is a unique formal invariant curve $\hat{\Gamma}$, non-singular and tangent to the z-axis

- Formal normal form: There are formal coordinates $(\hat{x},\hat{y},\hat{z})$ in which ξ can be written as the formal v.f.

$$\hat{\xi} := \left(-\hat{y} + f(\hat{x}^2+\hat{y}^2,\hat{z})\right)\frac{\partial}{\partial \hat{x}} + \left(\hat{x} + g(\hat{x}^2+\hat{y}^2,\hat{z})\right)\frac{\partial}{\partial \hat{y}} + h(\hat{x}^2+\hat{y}^2,\hat{z})\frac{\partial}{\partial \hat{z}}$$

where $f,g,h \in \mathbb{R}[[u,v]]$ have order ≥ 2 and $h(0,\hat{z}) \not\equiv 0$

- Truncated normal form: For any $\ell \geq 2$, there are analytic coordinates $(x,y,z) = (x_\ell, y_\ell, z_\ell)$ in which

$$j_\ell(\xi) = j_\ell(\hat{\xi})$$

Ideas of the proof. First blowing-up

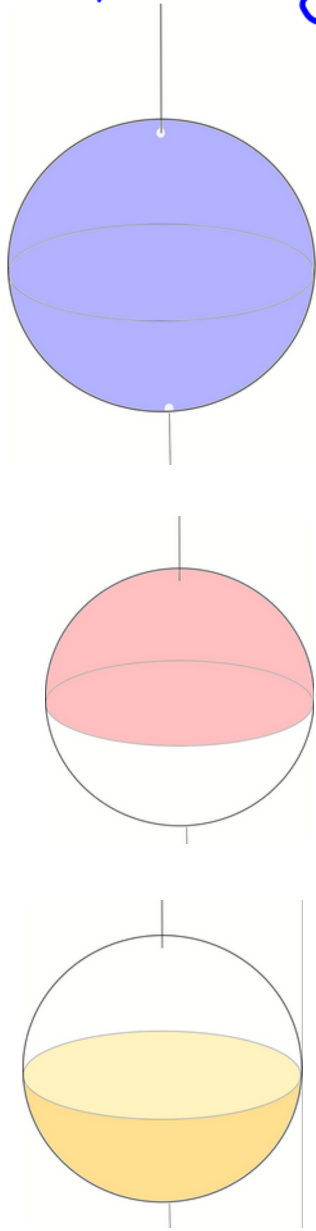
We blow-up the origin $0 \in \mathbb{R}^3$, using three charts

$(\pi_1 : M_1 \to \mathbb{R}^3, 0)$

$$C_0 : \begin{cases} x = \rho \cos\theta \\ y = \rho \operatorname{sen}\theta \\ z = z \end{cases}$$

$$C_N : \begin{cases} x = x' z' \\ y = y' z' \\ z = z' \end{cases}$$

$$C_S : \begin{cases} x = x'' z'' \\ y = y'' z'' \\ z = -z'' \end{cases}$$

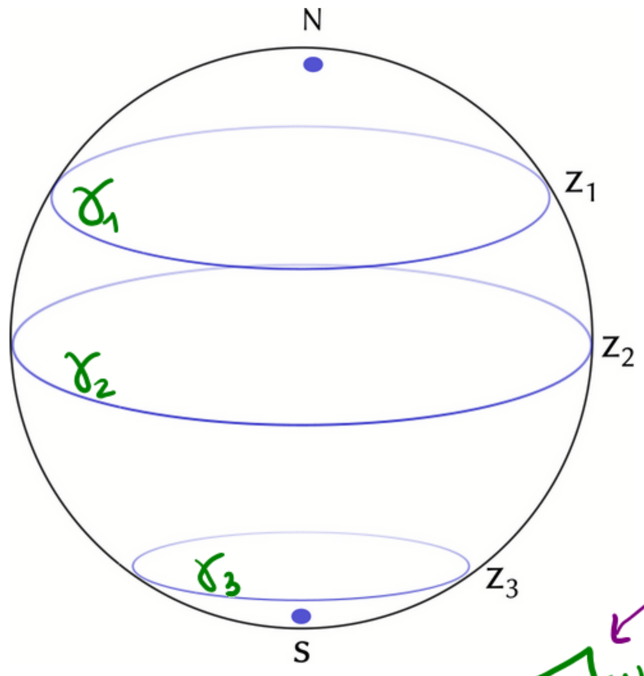


First exceptional divisor

$\pi_1^{-1}(0) = E_1 \simeq S^2$

Ideas of the proof. Characteristic cycles

In the central chart we consider the **Characteristic cycles**



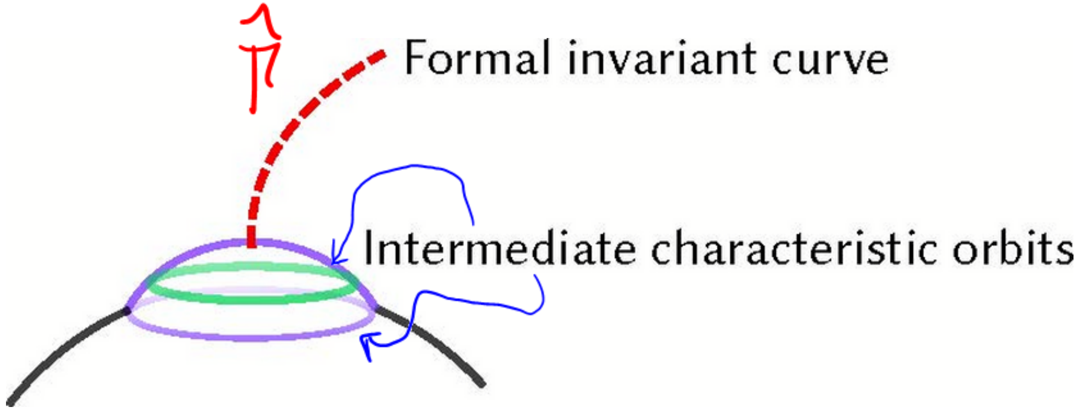
strict transforms by π_1

$\{\gamma_j\}$: periodic orbits of $\widehat{\xi}^{(1)}\big|_{E_1}$ or $\xi_\ell^{(1)}\big|_{E_1}$

(Finitely many!!) if $\ell \gg 0$

Ideas of the proof: No cycles "near" $\hat{\Gamma}$

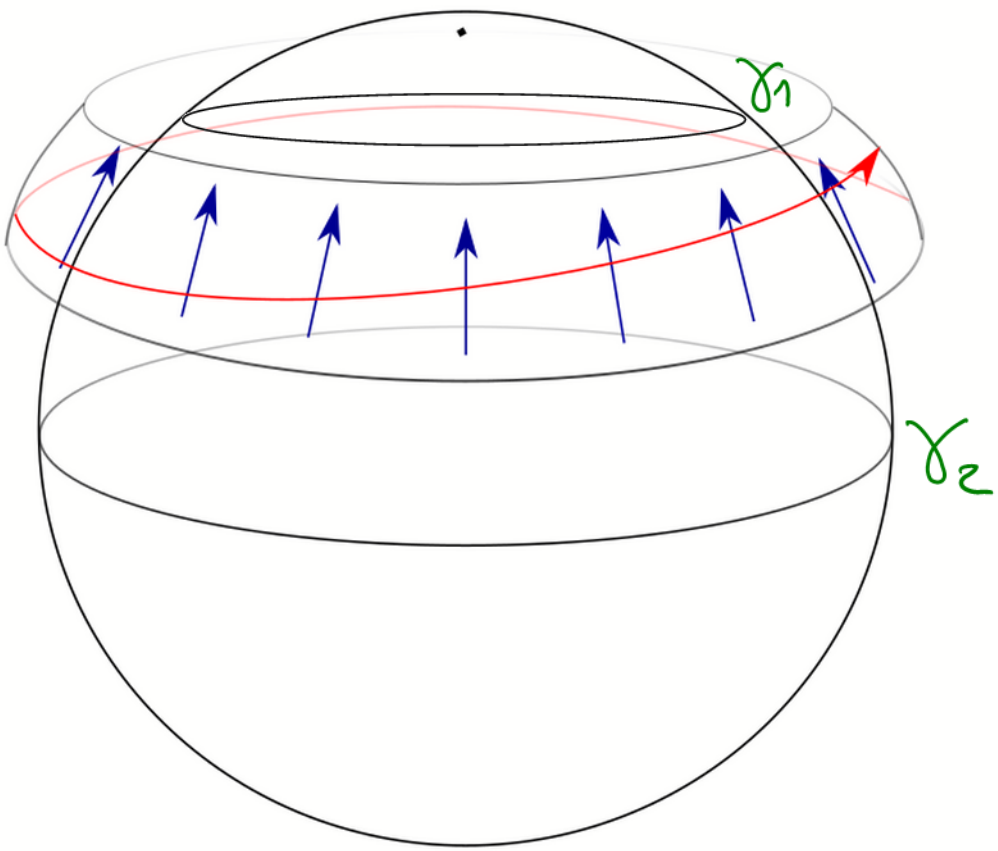
At the north pole N and at the south pole S we continue making punctual blowing-ups along $\hat{\Gamma}$



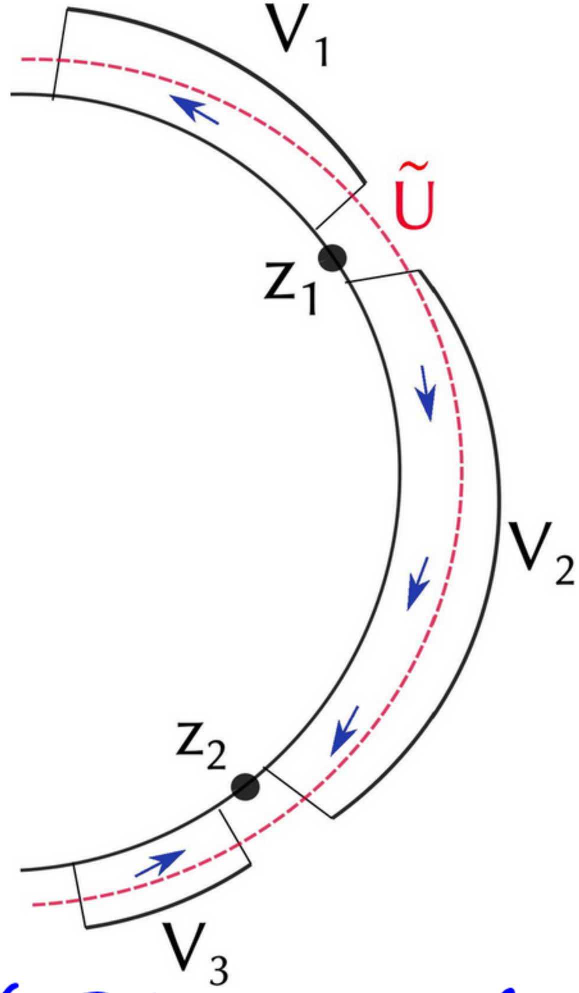
<u>Fact</u>: By the hypothesis $h(0,\hat{z})\neq 0$, after several of these blowing-ups, the strict transform of ξ_ℓ satisfies

$$\dot{z} = z^k\Big(\underset{\neq 0}{\alpha} + o(1)\Big) \Rightarrow \begin{cases} \cdot\ z \text{ is monotonous along solutions} \\ \cdot\ \text{No cycles in} \end{cases}$$

Ideas of the proof. No cycles between char. cycles



Flow boxes between characteristic cycles



$\pi^{-1}(\mathcal{C}_y)$ can only accumulate along the char. cycles

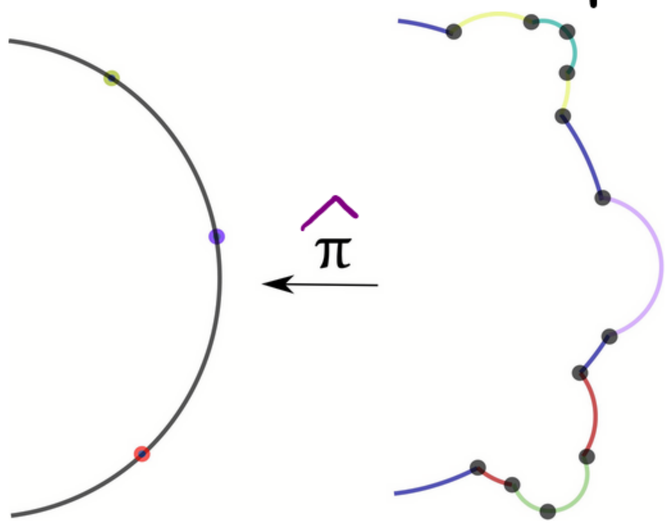
Ideas of the proof. Desingularize the formal v.f. $\hat{\xi}$

Near each characteristic cycle, $\hat{\xi}$ can be written as:

$$\hat{\xi}: \begin{cases} \dot{\rho} = \hat{A}(\rho, z) \\ \dot{z} = \hat{B}(\rho, z) \\ \dot{\theta} = 1 + \hat{C}(\rho, z) \end{cases}$$

$\rightsquigarrow$ two-dimensional formal v.f. $\hat{\eta} := \left(\frac{\hat{A}}{1+\hat{C}}\right)\frac{\partial}{\partial \rho} + \left(\frac{\hat{B}}{1+\hat{C}}\right)\frac{\partial}{\partial z}$

Reduction of singularities of $\hat{\eta}$



$\hat{\pi}$ = sequence of punctual two-dimensional blowing-ups

$\updownarrow$

π = sequence of three-dim blowing-ups along char. cycles

Ideas of the proof. Final singularities

Corner (no cycles)

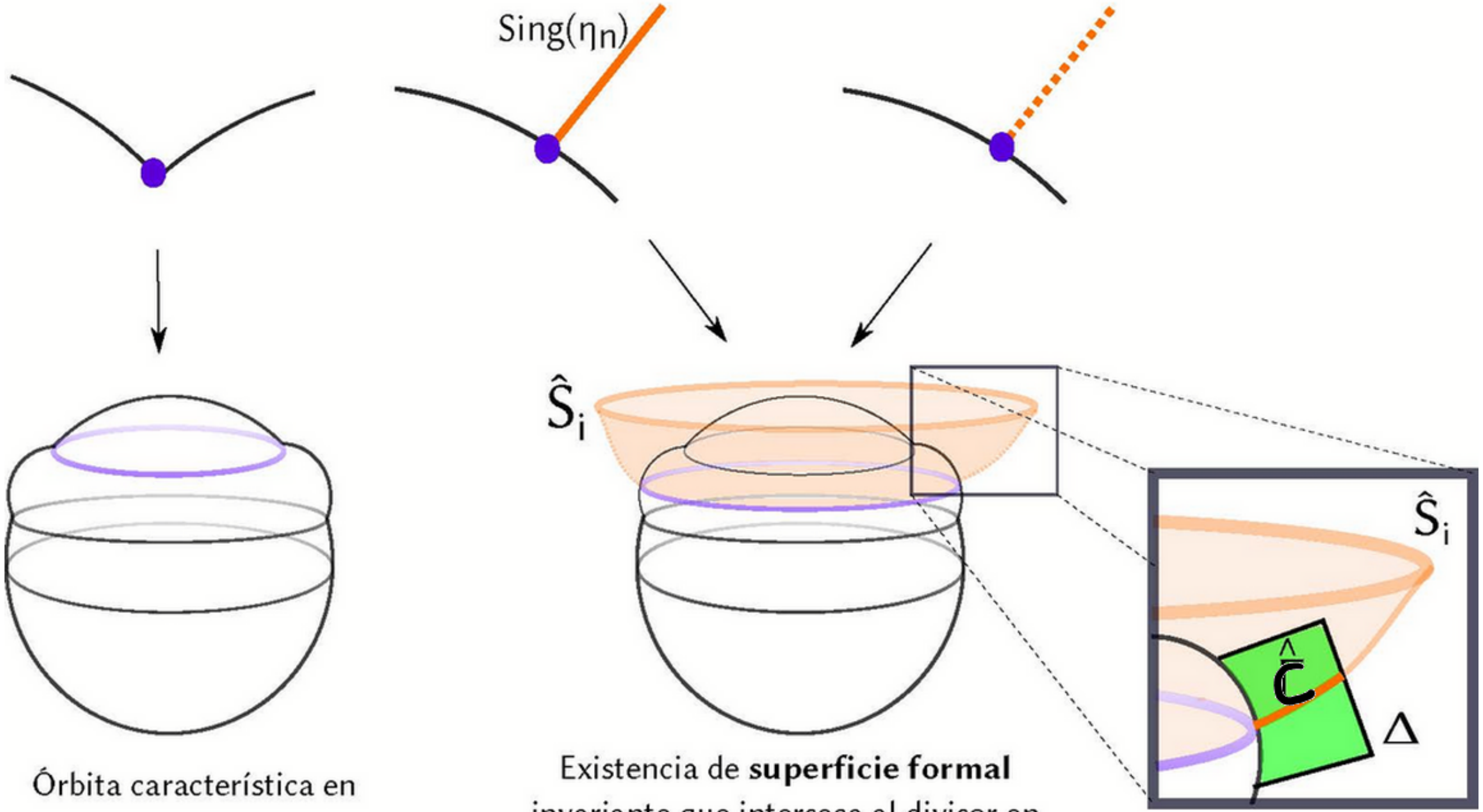
Singularidad tipo esquina

Analytic curve of singularities

Singularidad en curva de singularidades

Formal separatrix $\notin \mathrm{Sing}(\hat{\eta})$

Singularidad con separatriz formal transversal



Órbita característica en esquina

Existencia de **superficie formal** invariante que interseca al divisor en la órbita característica correspondiente

Definición de una aplicación de Poincaré P: Δ→Δ

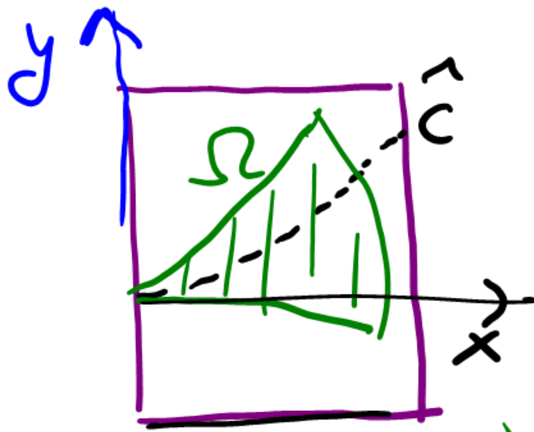
$\hat{C} = \hat{S}_i \cap \Delta$ is a formal curve, invariant for $\hat{P}$

Formal invariant surface $\hat{S}_i$ along a char. cycle

Ideas of the proof. Periodic orbits near a separatrix

The final situation is that we have

- $P : \Delta' \longrightarrow \Delta$ analytic diffeo.
- $\hat{C}$ = formal invariant curve of P (non-sing)

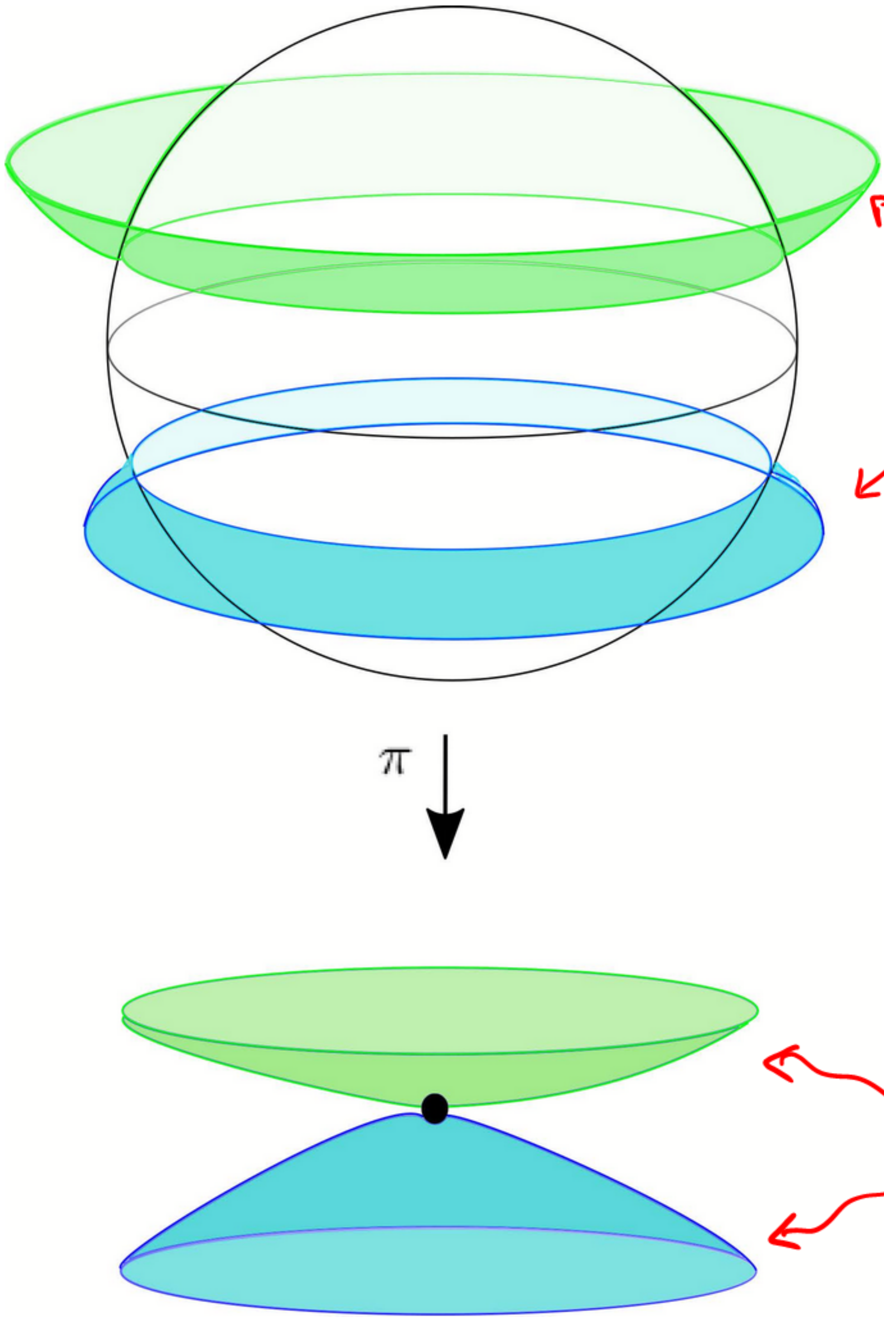


Two cases:

(1) $\hat{C} \not\subset \mathrm{Fix}(P) \Rightarrow$ in some cone the coordinate x is monotone along the orbits of P $\Rightarrow$ no periodic orbits $\Rightarrow$ no cycles ($\mathcal{C}_n \cap \Omega = \emptyset$)

(2) If $\hat{C} \subset \mathrm{Fix}(P) \Rightarrow \hat{C}$ analytic and $\hat{S} = S$ is a continuum of cycles. Moreover, if $\hat{C} = C = (y=0)$, either x or y is monotone along the orbits of P outside C. $\Rightarrow \mathcal{C}_n \cap \Omega = S$

Conclusion



Analytic invariant surfaces corresponding to curves of fixed points of Poincaré maps

Central-center surfaces