# Dynamics of a flow with constant slope on the torus and the Klein bottle

JAUME LLIBRE
Universitat Autònoma de Barcelona

Work with L. Barreira, C. Valls and A. Bakhshalizadeh

Advances in Qualitative Theory of Differential Equations
Soller, February 6–10, 2023

A flow on a manifold **M** is a group action of the additive group of the real numbers $\mathbb{R}$ on **M**, A flow on a manifold **M** is a group action of the additive group of the real numbers $\mathbb{R}$ on **M**, i.e. a flow is a mapping $\varphi\colon M \times \mathbb{R} \to M$ such that letting $\varphi_t = \varphi(\cdot, t)$ we have

$$\varphi_0 = \mathrm{id} \quad \text{and} \quad \varphi_s \circ \varphi_t = \varphi_{s+t}$$

for all $s, t \in \mathbb{R}$.

A flow on a manifold **M** is a group action of the additive group of the real numbers $\mathbb{R}$ on **M**, i.e. a flow is a mapping $\varphi\colon M \times \mathbb{R} \to M$ such that letting $\varphi_t = \varphi(\cdot, t)$ we have

$$\varphi_0 = \mathrm{id} \quad \text{and} \quad \varphi_s \circ \varphi_t = \varphi_{s+t}$$

for all $s, t \in \mathbb{R}$.

We recall that the orbit of a point $x \in \mathbf{M}$ is the set of points $\varphi_t(x)$ for $t \in \mathbb{R}$.

An orbit is periodic of period $T > 0$ if $\varphi_T(x) = x$ and $\varphi_t(x) \neq x$ for $t \in (0, T)$.

An orbit is periodic of period $T > 0$ if $\varphi_T(x) = x$ and $\varphi_t(x) \neq x$ for $t \in (0, T)$.

The orbit of $x$ is dense if for each $\varepsilon > 0$ and $\bar{x} \in \mathbf{M}$ there exists $t \in \mathbb{R}$ such that

$$d(\varphi_t(x), \bar{x}) < \varepsilon,$$

where $d$ is the distance on $\mathbf{M}$.

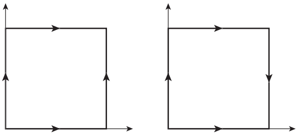


Figure: Identifications for the 2-dimensional torus (on the left) and for the Klein bottle (on the right).

Let **Q** be the closed square formed by the points $(x, y)$ with $(x, y) \in [0, 1]$.

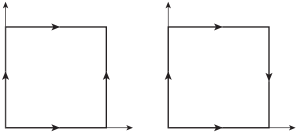


Figure: Identifications for the 2-dimensional torus (on the left) and for the Klein bottle (on the right).

Let **Q** be the closed square formed by the points $(x, y)$ with $(x, y) \in [0, 1]$.

We obtain the 2-dimensional torus $\mathbb{T}^2$ identifying the point $(x, 0)$ with the point $(x, 1)$ for all $x \in [0, 1]$, and the point $(0, y)$ with the point $(1, y)$ for all $y \in [0, 1]$ (see the image on the left of the figure).

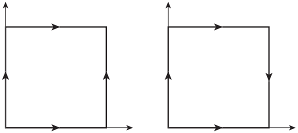


Figure: Identifications for the 2-dimensional torus (on the left) and for the Klein bottle (on the right).

Let **Q** be the closed square formed by the points $(x, y)$ with $(x, y) \in [0, 1]$.

We obtain the 2-dimensional torus $\mathbb{T}^2$ identifying the point $(x, 0)$ with the point $(x, 1)$ for all $x \in [0, 1]$, and the point $(0, y)$ with the point $(1, y)$ for all $y \in [0, 1]$ (see the image on the left of the figure). Note that the four vertices of **Q** are identified.

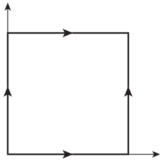


Figure: Identifications for the 2-dimensional torus (on the left) and for the Klein bottle (on the right).

We obtain the Klein bottle $\mathbb{K}$ identifying the point $(x, 0)$ with the point $(x, 1)$ for all $x \in [0, 1]$, and the point $(0, y)$ with the point $(1, 1 - y)$ for all $y \in [0, 1]$ (see the image on the right of the figure).

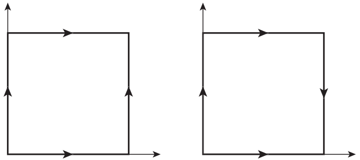
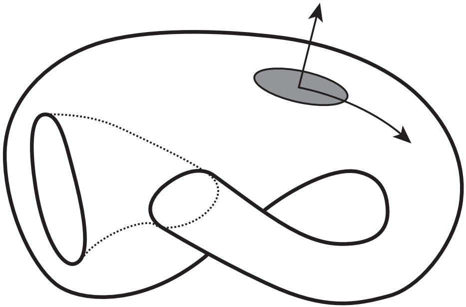


Figure: Identifications for the 2-dimensional torus (on the left) and for the Klein bottle (on the right).

We obtain the Klein bottle $\mathbb{K}$ identifying the point $(x, 0)$ with the point $(x, 1)$ for all $x \in [0, 1]$, and the point $(0, y)$ with the point $(1, 1 - y)$ for all $y \in [0, 1]$ (see the image on the right of the figure). Note that the four vertices of $\mathbf{Q}$ are identified.



Take a vector $(u, v) \in \mathbb{R}^2$ with $u^2 + v^2 = 1$.

Take a vector $(u, v) \in \mathbb{R}^2$ with $u^2 + v^2 = 1$.

The flow on the torus $\mathbb{T}^2$ with direction $(u, v)$ is defined as follows.

Take a vector $(u, v) \in \mathbb{R}^2$ with $u^2 + v^2 = 1$.

The flow on the torus $\mathbb{T}^2$ with direction $(u, v)$ is defined as follows. Take a point $(x_0, y_0) \in \mathbb{T}^2$ in the interior of the square **Q**.

Take a vector $(u, v) \in \mathbb{R}^2$ with $u^2 + v^2 = 1$.

The flow on the torus $\mathbb{T}^2$ with direction $(u, v)$ is defined as follows. Take a point $(x_0, y_0) \in \mathbb{T}^2$ in the interior of the square **Q**. Then the flow through the point $(x_0, y_0)$ is defined by

$$\varphi_t(x_0, y_0) = (x_0 + ut, y_0 + vt)$$

for $t$ in a neighborhood of $t = 0$ such that $\varphi_t(x_0, y_0)$ remains in the interior of the square **Q**.

Take a vector $(u, v) \in \mathbb{R}^2$ with $u^2 + v^2 = 1$.

The flow on the torus $\mathbb{T}^2$ with direction $(u, v)$ is defined as follows. Take a point $(x_0, y_0) \in \mathbb{T}^2$ in the interior of the square **Q**. Then the flow through the point $(x_0, y_0)$ is defined by

$$\varphi_t(x_0, y_0) = (x_0 + ut, y_0 + vt)$$

for $t$ in a neighborhood of $t = 0$ such that $\varphi_t(x_0, y_0)$ remains in the interior of the square **Q**.

This means that the orbit travels on the straight line

$$v(x - x_0) - u(y - y_0) = 0$$

through the point $(x_0, y_0)$.

If at time $t$ (either with $t > 0$ or $t < 0$) the point $(x_1, y_1) = \varphi_t(x_0, y_0)$ is at the boundary of the square **Q** for the first time, we consider a point $(x_2, y_2)$ identified with the point $(x_1, y_1)$.

If at time $t$ (either with $t > 0$ or $t < 0$) the point $(x_1, y_1) = \varphi_t(x_0, y_0)$ is at the boundary of the square $\mathbf{Q}$ for the first time, we consider a point $(x_2, y_2)$ identified with the point $(x_1, y_1)$. Then the flow continues on the straight line

$$v(x - x_2) - u(y - y_2) = 0$$

in the same direction (so that it remains inside the square), until it reaches again the boundary of the square $\mathbf{Q}$, and so on.

If at time $t$ (either with $t > 0$ or $t < 0$) the point $(x_1, y_1) = \varphi_t(x_0, y_0)$ is at the boundary of the square **Q** for the first time, we consider a point $(x_2, y_2)$ identified with the point $(x_1, y_1)$. Then the flow continues on the straight line

$$v(x - x_2) - u(y - y_2) = 0$$

in the same direction (so that it remains inside the square), until it reaches again the boundary of the square **Q**, and so on.

For a point $(x_0, y_0) \in \mathbb{T}^2$ at the boundary of **Q** the flow is defined as above for $(x_1, y_1)$.

Any flow on the torus $\mathbb{T}^2$ with direction $(u, v)$ for some vector $(u, v) \in \mathbb{R}^2$ with $u^2 + v^2 = 1$ is called a flow with constant slope on the torus $\mathbb{T}^2$.

Any flow on the torus $\mathbb{T}^2$ with direction $(u, v)$ for some vector $(u, v) \in \mathbb{R}^2$ with $u^2 + v^2 = 1$ is called a flow with constant slope on the torus $\mathbb{T}^2$.

Recall that $u$ and $v$ are said to be rationally dependent if there exist $p, q \in \mathbb{Z}$ not both zero such that $pu + qv = 0$, otherwise they are rationally independent.

Any flow on the torus $\mathbb{T}^2$ with direction $(u, v)$ for some vector $(u, v) \in \mathbb{R}^2$ with $u^2 + v^2 = 1$ is called a flow with constant slope on the torus $\mathbb{T}^2$.

Recall that $u$ and $v$ are said to be rationally dependent if there exist $p, q \in \mathbb{Z}$ not both zero such that $pu + qv = 0$, otherwise they are rationally independent.

The following result is well known.

THEOREM 1 Consider a flow with constant slope on the torus $\mathbb{T}^2$ with direction $(u, v) \in \mathbb{R}^2$ of norm $1$. Then the following statements hold.

THEOREM 1 Consider a flow with constant slope on the torus $\mathbb{T}^2$ with direction $(u, v) \in \mathbb{R}^2$ of norm $1$. Then the following statements hold.

(a) If $u$ and $v$ are rationally dependent, THEOREM 1 Consider a flow with constant slope on the torus $\mathbb{T}^2$ with direction $(u, v) \in \mathbb{R}^2$ of norm 1. Then the following statements hold.

(a) If $u$ and $v$ are rationally dependent, then all orbits are periodic of period
- (a.1) 1 if either $v = 0$ or $u = 0$;
- (a.2) $|q/u|$ if $pu + qv = 0$ with $p$ and $q$ relatively prime.

THEOREM 1 Consider a flow with constant slope on the torus $\mathbb{T}^2$ with direction $(u, v) \in \mathbb{R}^2$ of norm 1. Then the following statements hold.

(a) If $u$ and $v$ are rationally dependent, then all orbits are periodic of period
- (a.1) 1 if either $v = 0$ or $u = 0$;
- (a.2) $|q/u|$ if $pu + qv = 0$ with $p$ and $q$ relatively prime.

(b) If $u$ and $v$ are rationally independent, then all orbits are dense.

For a proof of the statement (a) of THEOREM 1 see for instance the appendix 1 of the book:

For a proof of the statement (a) of THEOREM 1 see for instance the appendix 1 of the book:

V. Arnold and A. Avez, Ergodic Problems of Classical Mechanics, W. A. Benjamin, Inc., New York-Amsterdam, 1968.

For a proof of the statement (a) of THEOREM 1 see for instance the appendix 1 of the book:

V. Arnold and A. Avez, Ergodic Problems of Classical Mechanics, W. A. Benjamin, Inc., New York-Amsterdam, 1968.

While for a proof of the statement (b) see the works:

A. Denjoy, Sur les courbes définies par les équations différentielles à la surface du tore, J. Math. Pures Appl. **11** (1932), 333–376.

C. Siegel, Note on differential equations on the torus, Ann. of Math. (2) **46** (1945), 423–428.

For a proof of the statement (a) of THEOREM 1 see for instance the appendix 1 of the book:

V. Arnold and A. Avez, Ergodic Problems of Classical Mechanics, W. A. Benjamin, Inc., New York-Amsterdam, 1968.

While for a proof of the statement (b) see the works:

A. Denjoy, Sur les courbes définies par les équations différentielles à la surface du tore, J. Math. Pures Appl. **11** (1932), 333–376.

C. Siegel, Note on differential equations on the torus, Ann. of Math. (2) **46** (1945), 423–428.

In fact Poincaré in

H. Poincaré, Oeuvres Complètes, vol. 1, 137–158.

was the first to describe these results without a rigorous proof for statement (b).

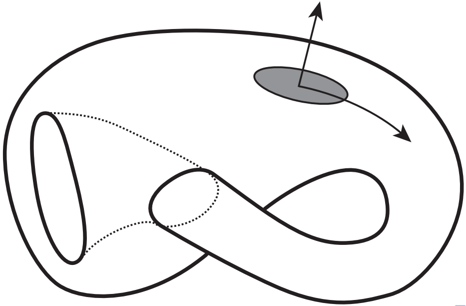
Take a vector $(u, v) \in \mathbb{R}^2$ with $u^2 + v^2 = 1$.

Take a vector $(u, v) \in \mathbb{R}^2$ with $u^2 + v^2 = 1$.

The flow on the Klein bottle $\mathbb{K}$ with direction $(u, v)$ is defined in a similar manner to that of the flow on $\mathbb{T}^2$, although the slope along which an orbit travels needs to change its sign when the flow reaches a vertical side of **Q**.

Take a vector $(u, v) \in \mathbb{R}^2$ with $u^2 + v^2 = 1$.

The flow on the Klein bottle $\mathbb{K}$ with direction $(u, v)$ is defined in a similar manner to that of the flow on $\mathbb{T}^2$, although the slope along which an orbit travels needs to change its sign when the flow reaches a vertical side of **Q**. This change will be determined by a normal vector to $\mathbb{K}$.



Now take a point $(x_0, y_0) \in \mathbb{K}$ in the interior of the square **Q** and a unit normal vector $w \in \{-1, 1\}$ (the sign of $w$ depends on whether it points to one or the other side of the surface).

Now take a point $(x_0, y_0) \in \mathbb{K}$ in the interior of the square **Q** and a unit normal vector $w \in \{-1, 1\}$ (the sign of $w$ depends on whether it points to one or the other side of the surface). Then the flow on $\mathbb{K}^* = \mathbb{K} \times \{-1, 1\}$ with direction $(u, v)$ through the point $(x_0, y_0, w)$ is defined by

$$\varphi_t(x_0, y_0, w) = (x_0 + ut, y_0 + vt, w)$$

for $t$ in a neighborhood of $t = 0$ such that $(x_0 + ut, y_0 + vt)$ remains in the interior of the square **Q**, taking always the same normal vector $w$.

If at time $t$ (either with $t > 0$ or $t < 0$) the point $(x_1, y_1)$ with $(x_1, y_1, w) = \varphi_t(x_0, y_0, w)$ is at the boundary of the square **Q** for the first time, we consider a point $(x_2, y_2)$ identified with $(x_1, y_1)$.

If at time $t$ (either with $t > 0$ or $t < 0$) the point $(x_1, y_1)$ with $(x_1, y_1, w) = \varphi_t(x_0, y_0, w)$ is at the boundary of the square **Q** for the first time, we consider a point $(x_2, y_2)$ identified with $(x_1, y_1)$.

If these two points are on the horizontal sides of **Q** (but not at vertices), then the flow continues as

$$\varphi_t(x_2, y_2, w) = (x_2 + ut, y_2 + vt, w),$$

If at time $t$ (either with $t > 0$ or $t < 0$) the point $(x_1, y_1)$ with $(x_1, y_1, w) = \varphi_t(x_0, y_0, w)$ is at the boundary of the square **Q** for the first time, we consider a point $(x_2, y_2)$ identified with $(x_1, y_1)$.

If these two points are on the horizontal sides of **Q** (but not at vertices), then the flow continues as

$$\varphi_t(x_2, y_2, w) = (x_2 + ut, y_2 + vt, w),$$

while if they are on the vertical sides of **Q**, then the flow continues as

$$\varphi_t(x_2, y_2, w) = (x_2 + ut, y_2 - vt, -w),$$

so that $(x_2 + ut, y_2 + vt)$ and $(x_2 + ut, y_2 - vt)$, respectively, are inside the square for small values of $t$, until they reach again the boundary of **Q**, and so on.

If at time $t$ (either with $t > 0$ or $t < 0$) the point $(x_1, y_1)$ with $(x_1, y_1, w) = \varphi_t(x_0, y_0, w)$ is at the boundary of the square **Q** for the first time, we consider a point $(x_2, y_2)$ identified with $(x_1, y_1)$.

If these two points are on the horizontal sides of **Q** (but not at vertices), then the flow continues as

$$\varphi_t(x_2, y_2, w) = (x_2 + ut, y_2 + vt, w),$$

while if they are on the vertical sides of **Q**, then the flow continues as

$$\varphi_t(x_2, y_2, w) = (x_2 + ut, y_2 - vt, -w),$$

so that $(x_2 + ut, y_2 + vt)$ and $(x_2 + ut, y_2 - vt)$, respectively, are inside the square for small values of $t$, until they reach again the boundary of **Q**, and so on.

For a point $(x_0, y_0)$ in $\mathbb{K}$ at the boundary of **Q** the flow is defined as above for $(x_1, y_1)$.

The reason for the need of changing the sign of the slope.

The reason for the need of changing the sign of the slope.

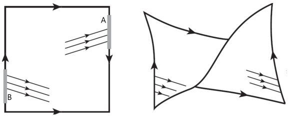



Figure: Change of slope after reaching the boundary $x = 1$.

When the orbits reach the boundary $x = 1$ we need to identify the sets $A$ and $B$ in the figure, but with opposite orientation.

The reason for the need of changing the sign of the slope.

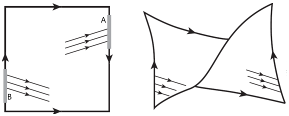



Figure: Change of slope after reaching the boundary $x = 1$.

When the orbits reach the boundary $x = 1$ we need to identify the sets $A$ and $B$ in the figure, but with opposite orientation.

In other words, when we flip the boundary $x = 1$ the order of the orbits as well as their slopes change, which causes that unless $v$ vanishes we need to change the sign of the slope.

We note that our constant flows on the Klein bottle $\mathbb{K}$ really are defined on $\mathbb{K}^* = \mathbb{K} \times \{-1, 1\}$, i.e. are defined on the unit normal bundle of the Klein bottle, but simply we call such a flow as a flow with constant slope on the Klein bottle $\mathbb{K}$.

We note that our constant flows on the Klein bottle $\mathbb{K}$ really are defined on $\mathbb{K}^* = \mathbb{K} \times \{-1, 1\}$, i.e. are defined on the unit normal bundle of the Klein bottle, but simply we call such a flow as a flow with constant slope on the Klein bottle $\mathbb{K}$.

The following result is a version of THEOREM 1 for the Klein bottle.

THEOREM 2 Consider a flow $\varphi_t$ with constant slope on the Klein bottle $\mathbb{K}$ with direction $(u, v) \in \mathbb{R}^2$ of norm $1$. Then the following statements hold.

THEOREM 2 Consider a flow $\varphi_t$ with constant slope on the Klein bottle $\mathbb{K}$ with direction $(u, v) \in \mathbb{R}^2$ of norm $1$. Then the following statements hold.

(a) If $u$ and $v$ are rationally dependent, then all orbits are periodic of period

- (a.1) $1$ if $u = 0$;
- (a.2) $2$ if $v = 0$;
- (a.3) $2|q/u|$ if $pu + qv = 0$ with $p$ and $q$ relatively prime and $q$ is odd;
- (a.4) $|q/u|$ if $pu + qv = 0$ with $p$ and $q$ relatively prime and $q$ is even.

THEOREM 2 Consider a flow $\varphi_t$ with constant slope on the Klein bottle $\mathbb{K}$ with direction $(u, v) \in \mathbb{R}^2$ of norm $1$. Then the following statements hold.

(a) If $u$ and $v$ are rationally dependent, then all orbits are periodic of period
  - (a.1) $1$ if $u = 0$;
  - (a.2) $2$ if $v = 0$;
  - (a.3) $2|q/u|$ if $pu + qv = 0$ with $p$ and $q$ relatively prime and $q$ is odd;
  - (a.4) $|q/u|$ if $pu + qv = 0$ with $p$ and $q$ relatively prime and $q$ is even.

(b) If $u$ and $v$ are rationally independent, then all orbits are dense.

We note that in the proof of THEOREM 2, with the exception of the constant flows with vector $(0, v)$, it suffices to consider the orbits which pass through the points $(0, y, w)$ with $y \in [0, 1]$ and $w \in \{-1, 1\}$ because all the orbits contain at least one of these points.

We note that in the proof of THEOREM 2, with the exception of the constant flows with vector $(0, v)$, it suffices to consider the orbits which pass through the points $(0, y, w)$ with $y \in [0, 1]$ and $w \in \{-1, 1\}$ because all the orbits contain at least one of these points.

The proof of statement (a.1) is easy. Indeed,

We note that in the proof of THEOREM 2, with the exception of the constant flows with vector $(0, v)$, it suffices to consider the orbits which pass through the points $(0, y, w)$ with $y \in [0, 1]$ and $w \in \{-1, 1\}$ because all the orbits contain at least one of these points.

The proof of statement (a.1) is easy. Indeed,

For the case $u = 0$ and consequently $v = \pm 1$ we have $\varphi_t(0, y, w) = (0, y + vt, w)$ for all $t \in \mathbb{R}$.

We note that in the proof of THEOREM 2, with the exception of the constant flows with vector $(0, v)$, it suffices to consider the orbits which pass through the points $(0, y, w)$ with $y \in [0, 1]$ and $w \in \{-1, 1\}$ because all the orbits contain at least one of these points.

The proof of statement (a.1) is easy. Indeed,

For the case $u = 0$ and consequently $v = \pm 1$ we have $\varphi_t(0, y, w) = (0, y + vt, w)$ for all $t \in \mathbb{R}$. Hence, the orbit of any point $(0, y, w)$ has period 1.

We first consider the case $v = 0$, and consequently $u = \pm 1$.

We first consider the case $v = 0$, and consequently $u = \pm 1$.

Then $\varphi_t(0, y, w) = (ut, y, w)$ for $t$ between $0$ and $1/u$.

We first consider the case $v = 0$, and consequently $u = \pm 1$.

Then $\varphi_t(0, y, w) = (ut, y, w)$ for $t$ between $0$ and $1/u$. Since the first component $ut$ takes different values for $t$ strictly between $0$ and $1/u$, to look for periodic orbits we need to take $|t| \geq 1$.

We first consider the case $v = 0$, and consequently $u = \pm 1$.

Then $\varphi_t(0, y, w) = (ut, y, w)$ for $t$ between $0$ and $1/u$. Since the first component $ut$ takes different values for $t$ strictly between $0$ and $1/u$, to look for periodic orbits we need to take $|t| \geq 1$. For $t = 1/u$ we have

$$\varphi_{1/u}(0, y, w) = (1, y, w) \equiv (0, 1 - y, -w).$$

We first consider the case $v = 0$, and consequently $u = \pm 1$.

Then $\varphi_t(0, y, w) = (ut, y, w)$ for $t$ between $0$ and $1/u$. Since the first component $ut$ takes different values for $t$ strictly between $0$ and $1/u$, to look for periodic orbits we need to take $|t| \geq 1$. For $t = 1/u$ we have

$$\varphi_{1/u}(0, y, w) = (1, y, w) \equiv (0, 1 - y, -w).$$

Similarly, for $t$ between $0$ and $1/u$ we have

$$\varphi_{t+1/u}(0, y, w) = \varphi_t(0, 1 - y, -w) = (ut, 1 - y, -w),$$

We first consider the case $v = 0$, and consequently $u = \pm 1$.

Then $\varphi_t(0, y, w) = (ut, y, w)$ for $t$ between $0$ and $1/u$. Since the first component $ut$ takes different values for $t$ strictly between $0$ and $1/u$, to look for periodic orbits we need to take $|t| \geq 1$. For $t = 1/u$ we have

$$\varphi_{1/u}(0, y, w) = (1, y, w) \equiv (0, 1 - y, -w).$$

Similarly, for $t$ between $0$ and $1/u$ we have

$$\varphi_{t+1/u}(0, y, w) = \varphi_t(0, 1 - y, -w) = (ut, 1 - y, -w),$$

and so to look for periodic orbits again we need to take $|t| \geq 1$. Since

$$\varphi_{2/u}(0, y, w) = (1, 1 - y, -w) \equiv (0, y, w),$$

the orbit of any point $(0, y, w)$ has period $2$.

Now we assume that $uv \neq 0$, and that $u$ and $v$ are rationally dependent, i.e. $pu + qv = 0$ with $p$ and $q$ relatively prime nonzero integer numbers,

Now we assume that $uv \neq 0$, and that $u$ and $v$ are rationally dependent, i.e. $pu + qv = 0$ with $p$ and $q$ relatively prime nonzero integer numbers, we have

$$\begin{aligned}\varphi_{2/u}(0, y, w) = \varphi_{1/u}(1, y + v/u, w) &\equiv \varphi_{1/u}(0, 1 - y - v/u, -w) \\ = (1, 1 - y - 2v/u, -w) &\equiv (0, y + 2v/u, w).\end{aligned}$$

Now we assume that $uv \neq 0$, and that $u$ and $v$ are rationally dependent, i.e. $pu + qv = 0$ with $p$ and $q$ relatively prime nonzero integer numbers, we have

$$\begin{aligned}\varphi_{2/u}(0, y, w) &= \varphi_{1/u}(1, y + v/u, w) \equiv \varphi_{1/u}(0, 1 - y - v/u, -w) \\ &= (1, 1 - y - 2v/u, -w) \equiv (0, y + 2v/u, w).\end{aligned}$$

Writing $v/u = -p/q$, it follows from the former identity that the orbit of any point $(0, y, w)$ is periodic of period $2|q/u|$ when $q$ is odd, and is periodic of period $|q/u|$ when $q$ is even.

Now we assume that $uv \neq 0$, and that $u$ and $v$ are rationally dependent, i.e. $pu + qv = 0$ with $p$ and $q$ relatively prime nonzero integer numbers, we have

$$\begin{aligned}\varphi_{2/u}(0, y, w) &= \varphi_{1/u}(1, y + v/u, w) \equiv \varphi_{1/u}(0, 1 - y - v/u, -w) \\ &= (1, 1 - y - 2v/u, -w) \equiv (0, y + 2v/u, w).\end{aligned}$$

Writing $v/u = -p/q$, it follows from the former identity that the orbit of any point $(0, y, w)$ is periodic of period $2|q/u|$ when $q$ is odd, and is periodic of period $|q/u|$ when $q$ is even. This completes the proof of statement (a).

Finally, we assume that $u$ and $v$ are rationally independent (then $v/u$ is an irrational number).

Finally, we assume that $u$ and $v$ are rationally independent (then $v/u$ is an irrational number). Again we have

$$\varphi_{2/u}(0, y, w) \equiv (0, y + 2v/u, w).$$

Finally, we assume that $u$ and $v$ are rationally independent (then $v/u$ is an irrational number). Again we have

$$\varphi_{2/u}(0, y, w) \equiv (0, y + 2v/u, w).$$

The map $(0, y) \mapsto (0, y + 2v/u)$ is an irrational rotation of the circle and so all its orbits $(0, y + 2nv/u)$ for $n \in \mathbb{Z}$ are dense on the circle (see for instance Denjoy).

Finally, we assume that $u$ and $v$ are rationally independent (then $v/u$ is an irrational number). Again we have

$$\varphi_{2/u}(0, y, w) \equiv (0, y + 2v/u, w).$$

The map $(0, y) \mapsto (0, y + 2v/u)$ is an irrational rotation of the circle and so all its orbits $(0, y + 2nv/u)$ for $n \in \mathbb{Z}$ are dense on the circle (see for instance Denjoy). Therefore, for any point $(0, y, w)$ the collection of segments

$$\varphi_{t+2n/u}(0, y, w) \equiv (0, y + 2nv/u, w) + (u, v, 0)t$$

for $n \in \mathbb{N}$ and $t \in (0, 1/|u|)$ of its orbit are dense on $Q \times \{w\}$,

Finally, we assume that $u$ and $v$ are rationally independent (then $v/u$ is an irrational number). Again we have

$$\varphi_{2/u}(0, y, w) \equiv (0, y + 2v/u, w).$$

The map $(0, y) \mapsto (0, y + 2v/u)$ is an irrational rotation of the circle and so all its orbits $(0, y + 2nv/u)$ for $n \in \mathbb{Z}$ are dense on the circle (see for instance Denjoy). Therefore, for any point $(0, y, w)$ the collection of segments

$$\varphi_{t+2n/u}(0, y, w) \equiv (0, y + 2nv/u, w) + (u, v, 0)t$$

for $n \in \mathbb{N}$ and $t \in (0, 1/|u|)$ of its orbit are dense on $Q \times \{w\}$, while the collection of segments

$$\varphi_{t+2n/u}(0, y, w) \equiv (0, 1 - y - (2n+1)v/u, -w) + (u, -v, 0)(t - 1/|u|)$$

for $n \in \mathbb{N}$ and $t \in (1/|u|, 2/|u|)$ of its orbit are dense on $Q \times \{-w\}$.

Finally, we assume that $u$ and $v$ are rationally independent (then $v/u$ is an irrational number). Again we have

$$\varphi_{2/u}(0, y, w) \equiv (0, y + 2v/u, w).$$

The map $(0, y) \mapsto (0, y + 2v/u)$ is an irrational rotation of the circle and so all its orbits $(0, y + 2nv/u)$ for $n \in \mathbb{Z}$ are dense on the circle (see for instance Denjoy). Therefore, for any point $(0, y, w)$ the collection of segments

$$\varphi_{t+2n/u}(0, y, w) \equiv (0, y + 2nv/u, w) + (u, v, 0)t$$

for $n \in \mathbb{N}$ and $t \in (0, 1/|u|)$ of its orbit are dense on $Q \times \{w\}$, while the collection of segments

$$\varphi_{t+2n/u}(0, y, w) \equiv (0, 1 - y - (2n + 1)v/u, -w) + (u, -v, 0)(t - 1/|u|)$$

for $n \in \mathbb{N}$ and $t \in (1/|u|, 2/|u|)$ of its orbit are dense on $Q \times \{-w\}$. This establishes statement (b), which completes the proof of the theorem.

Until now we have studied constant flows on the Klein bottle $\mathbb{K}$ which are defined on $\mathbb{K}^* = \mathbb{K} \times \{-1, 1\}$, i.e. are defined on the unit normal bundle of the Klein bottle.

Until now we have studied constant flows on the Klein bottle $\mathbb{K}$ which are defined on $\mathbb{K}^* = \mathbb{K} \times \{-1, 1\}$, i.e. are defined on the unit normal bundle of the Klein bottle.

Now we shall study a continuous constant flow on the Klein bottle $\mathbb{K}$ but not in unit normal bundle of the Klein bottle.

Take a vector $(u, v) \in \mathbb{R}^2$ with $u^2 + v^2 = 1$, and take a point $(x_0, y_0) \in \mathbb{K}$ in the interior of the square **Q**, Take a vector $(u, v) \in \mathbb{R}^2$ with $u^2 + v^2 = 1$, and take a point $(x_0, y_0) \in \mathbb{K}$ in the interior of the square **Q**, then the flow on $\mathbb{K}$ with direction $(u, v)$ through the point $(x_0, y_0)$ is defined by

$$\varphi((x_0, y_0), t) = (x_0 + ut, y_0 + vt)$$

for $t$ in a neighborhood of $t = 0$ such that $(x_0 + ut, y_0 + vt)$ remains in the interior of the square **Q**.

If at time $t$ (either with $t > 0$ or $t < 0$) the point $(x_1, y_1)$ with $(x_1, y_1) = \varphi((x_0, y_0), t)$ is at the boundary of the square **Q** for the first time, we consider a point $(x_2, y_2)$ identified with $(x_1, y_1)$.

If at time $t$ (either with $t > 0$ or $t < 0$) the point $(x_1, y_1)$ with $(x_1, y_1) = \varphi((x_0, y_0), t)$ is at the boundary of the square **Q** for the first time, we consider a point $(x_2, y_2)$ identified with $(x_1, y_1)$. Then the flow continues as

$\varphi((x_2, y_2), t) = (x_2 + ut, y_2 + vt)$,

so that $(x_2 + ut, y_2 + vt)$ is inside the square for small values of $t$, until they reach again the boundary of **Q**, and so on.

If at time $t$ (either with $t > 0$ or $t < 0$) the point $(x_1, y_1)$ with $(x_1, y_1) = \varphi((x_0, y_0), t)$ is at the boundary of the square $\mathbf{Q}$ for the first time, we consider a point $(x_2, y_2)$ identified with $(x_1, y_1)$. Then the flow continues as

$\varphi((x_2, y_2), t) = (x_2 + ut, y_2 + vt)$,

so that $(x_2 + ut, y_2 + vt)$ is inside the square for small values of $t$, until they reach again the boundary of $\mathbf{Q}$, and so on.

For a point $(x_0, y_0)$ in $\mathbb{K}$ at the boundary of $\mathbf{Q}$ the flow is defined as above for $(x_1, y_1)$.

If at time $t$ (either with $t > 0$ or $t < 0$) the point $(x_1, y_1)$ with $(x_1, y_1) = \varphi((x_0, y_0), t)$ is at the boundary of the square **Q** for the first time, we consider a point $(x_2, y_2)$ identified with $(x_1, y_1)$. Then the flow continues as
$\varphi((x_2, y_2), t) = (x_2 + ut, y_2 + vt)$,
so that $(x_2 + ut, y_2 + vt)$ is inside the square for small values of $t$, until they reach again the boundary of **Q**, and so on.

For a point $(x_0, y_0)$ in $\mathbb{K}$ at the boundary of **Q** the flow is defined as above for $(x_1, y_1)$.

Note that the flow is smooth in the whole square **Q**, except on the vertical sides of **Q** the flow is only continuous if $v \neq 0$.

THEOREM 3 Consider the discontinuous flow with constant slope on the Klein bottle $\mathbb{K}$ with direction $(u, v) \in \mathbb{R}^2$ of norm 1 previously defined.

THEOREM 3 Consider the discontinuous flow with constant slope on the Klein bottle $\mathbb{K}$ with direction $(u, v) \in \mathbb{R}^2$ of norm 1 previously defined. Then all its orbits are periodic of period

(a) 1 if $u = 0$;

(b) $2/u$ if $u \neq 0$, except the orbits of the points $(0, 1/2 - v/(2u))$ and $(0, -v/(2u))$ which have period $1/u$.

We note that in the proof of THEOREM 3 with the exception of the constant flows with vector $(0, \pm 1)$, it suffices to consider the orbits which pass through the points $(0, y_0)$ with $y_0 \in [0, 1]$, because all the orbits contain at least one of these points.

Assume that $u = 0$ and $v = \pm 1$.

Assume that $u = 0$ and $v = \pm 1$. Then flow $\varphi((x_0, y_0), t) = (x_0, y_0 \pm t)$ satisfies that $\varphi_1((x_0, y_0)) = (x_0, y_0 \pm 1) = (x_0, y_0)$ and that $\varphi_t((x_0, y_0)) = (x_0, y_0 \pm t) \neq (x_0, y_0)$ for all $t \in (0, 1)$.

Assume that $u = 0$ and $v = \pm 1$. Then flow
$\varphi((x_0, y_0), t) = (x_0, y_0 \pm t)$ satisfies that
$\varphi_1((x_0, y_0)) = (x_0, y_0 \pm 1) = (x_0, y_0)$
and that $\varphi_t((x_0, y_0)) = (x_0, y_0 \pm t) \neq (x_0, y_0)$
for all $t \in (0, 1)$.

Therefore the orbit through the point $(x_0, y_0)$ is periodic of period 1. Hence statement (a) of THEOREM 3 is proved.

Now we assume that $u \neq 0$, then we have

$$\varphi_{1/u}(0, y_0) = \left(1, y_0 + \frac{v}{u}\right) \equiv \left(0, 1 - y_0 - \frac{v}{u}\right).$$

Now we assume that $u \neq 0$, then we have

$$\varphi_{1/u}(0, y_0) = \left(1, y_0 + \frac{v}{u}\right) \equiv \left(0, 1 - y_0 - \frac{v}{u}\right).$$

When $y_0 = 1 - y_0 - v/u\,(mod 1)$ the orbit of the point $(0, y_0)$ has period $1/u$.

Now we assume that $u \neq 0$, then we have

$$\varphi_{1/u}(0, y_0) = \left(1, y_0 + \frac{v}{u}\right) \equiv \left(0, 1 - y_0 - \frac{v}{u}\right).$$

When $y_0 = 1 - y_0 - v/u\,(mod 1)$ the orbit of the point $(0, y_0)$ has period $1/u$.

Also we have

$$\varphi_{1/u}\left(0, -\frac{v}{2u}\right) = \left(1, \frac{v}{2u}\right) \equiv \left(0, 1 - \frac{v}{2u}\right) = \left(0, -\frac{v}{2u}\right).$$

Recall that $\varphi_{1/u}(0, y_0) = \left(0, 1 - y_0 - \dfrac{v}{u}\right)$. If $y_0$ does not satisfy $y_0 = 1 - y_0 - v/u\,(mod 1)$ and $y_0 \neq -v/(2u)$, then

Recall that $\varphi_{1/u}(0, y_0) = \left(0, 1 - y_0 - \dfrac{v}{u}\right)$. If $y_0$ does not satisfy $y_0 = 1 - y_0 - v/u\,(mod 1)$ and $y_0 \neq -v/(2u)$, then

$$\psi_{2/u}(0, y_0) \equiv \psi_{1/u}\left(0, 1 - y_0 - \frac{v}{u}\right) = (1, 1 - y_0) \equiv (0, y_0).$$

Recall that $\varphi_{1/u}(0, y_0) = \left(0, 1 - y_0 - \dfrac{v}{u}\right)$. If $y_0$ does not satisfy $y_0 = 1 - y_0 - v/u\,(mod 1)$ and $y_0 \neq -v/(2u)$, then

$$\psi_{2/u}(0, y_0) \equiv \psi_{1/u}\left(0, 1 - y_0 - \frac{v}{u}\right) = (1, 1 - y_0) \equiv (0, y_0).$$

So the orbit of the point $(0, y_0)$ has period $2/u$. This proves statement (b) of THEOREM 3.

A continuous linear differential system on the torus $\mathbb{T}^2$ is of the form $\dot{x} = a + bx + cy, \qquad \dot{y} = A + Bx + Cy$, satisfying

$$\begin{aligned} \dot{x}|_{x=0} - \dot{x}|_{x=1} &= -b = 0, \qquad & \dot{y}|_{x=0} - \dot{y}|_{x=1} &= -B = 0, \\ \dot{x}|_{y=0} - \dot{x}|_{y=1} &= -c = 0, \qquad & \dot{y}|_{y=0} - \dot{y}|_{y=1} &= -C = 0. \end{aligned}$$

A continuous linear differential system on the torus $\mathbb{T}^2$ is of the form $\dot{x} = a + bx + cy, \qquad \dot{y} = A + Bx + Cy$, satisfying

$$\begin{array}{ll} \dot{x}|_{x=0} - \dot{x}|_{x=1} = -b = 0, & \dot{y}|_{x=0} - \dot{y}|_{x=1} = -B = 0, \\ \dot{x}|_{y=0} - \dot{x}|_{y=1} = -c = 0, & \dot{y}|_{y=0} - \dot{y}|_{y=1} = -C = 0. \end{array}$$

Then the continuous linear differential systems on the torus $\mathbb{T}^2$ are

$$\dot{x} = a, \qquad \dot{y} = A,$$

In fact these differential systems define a flow with constant slow on the torus $\mathbb{T}^2$.

A continuous quadratic differential system on the torus $\mathbb{T}^2$ is of the form

$$
\begin{aligned}
\dot{x} &= a_0 + a_1 x + a_2 y + a_3 x^2 + a_4 xy + a_5 y^2, \\
\dot{y} &= b_0 + b_1 x + b_2 y + b_3 x^2 + b_4 xy + b_5 y^2,
\end{aligned}
$$

satisfying

$$
\begin{aligned}
\dot{x}|_{x=0} - \dot{x}|_{x=1} &= -a_1 - a_3 - a_4 y = 0, \qquad & \dot{y}|_{x=0} - \dot{y}|_{x=1} &= -b_1 - b_3 - \\
\dot{x}|_{y=0} - \dot{x}|_{y=1} &= -a_2 - a_5 - a_4 x = 0, \qquad & \dot{y}|_{y=0} - \dot{y}|_{y=1} &= -b_2 - b_5 -
\end{aligned}
$$

A continuous quadratic differential system on the torus $\mathbb{T}^2$ is of the form

$$\begin{aligned}\dot{x} &= a_0 + a_1 x + a_2 y + a_3 x^2 + a_4 xy + a_5 y^2,\\ \dot{y} &= b_0 + b_1 x + b_2 y + b_3 x^2 + b_4 xy + b_5 y^2,\end{aligned}$$

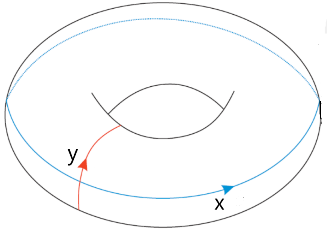
satisfying

$$\begin{aligned}\dot{x}|_{x=0} - \dot{x}|_{x=1} &= -a_1 - a_3 - a_4 y = 0, \qquad & \dot{y}|_{x=0} - \dot{y}|_{x=1} &= -b_1 - b_3\\ \dot{x}|_{y=0} - \dot{x}|_{y=1} &= -a_2 - a_5 - a_4 x = 0, \qquad & \dot{y}|_{y=0} - \dot{y}|_{y=1} &= -b_2 - b_5\end{aligned}$$

Then the continuous quadratic differential systems on the torus $\mathbb{T}^2$ are

$$\begin{aligned}\dot{x} &= a_0 + a_3 x(x-1) + a_5 y(y-1),\\ \dot{y} &= b_0 + b_3 x(x-1) + b_5 y(y-1).\end{aligned}$$

In summary on the red and blue circles in the torus the quadratic system is only continuous in the rest it is analytic.



Renaming the parameters the continuous quadratic differential systems on the torus $\mathbb{T}^2$ are

$$\dot{x} = a + bx(x-1) + cy(y-1),$$
$$\dot{y} = A + Bx(x-1) + Cy(y-1).$$

In what follows these quadratic differential systems are denotes simply by quadratic systems or QS.

Renaming the parameters the continuous quadratic differential systems on the torus $\mathbb{T}^2$ are

$$\dot{x} = a + bx(x-1) + cy(y-1),$$
$$\dot{y} = A + Bx(x-1) + Cy(y-1).$$

In what follows these quadratic differential systems are denotes simply by quadratic systems or QS.

These quadratic systems on the torus $\mathbb{T}^2$ depend on 6 parameters, while the quadratic differential systems on the plane $\mathbb{R}^2$ depend on 12 parameters.

Renaming the parameters the continuous quadratic differential systems on the torus $\mathbb{T}^2$ are

$$\dot{x} = a + bx(x-1) + cy(y-1),$$
$$\dot{y} = A + Bx(x-1) + Cy(y-1).$$

In what follows these quadratic differential systems are denotes simply by quadratic systems or QS.

These quadratic systems on the torus $\mathbb{T}^2$ depend on 6 parameters, while the quadratic differential systems on the plane $\mathbb{R}^2$ depend on 12 parameters.

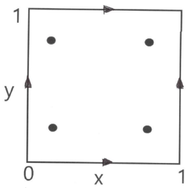
We do not consider QS in the torus $\mathbb{T}^2$ with infinitely many equilibria.

Assume that $Bc - bC \neq 0$ and that

$$(aC - Ac)(Ab - aB)(1 + 4\frac{aC - Ac}{Bc - bC})(1 + 4\frac{Ab - aB}{Bc - bC}) \neq 0.$$

Then the QS have the following 4 equilibria

$$\left(\frac{1}{2} \pm \frac{1}{2}\sqrt{1 + 4\frac{aC - Ac}{Bc - bC}}, \frac{1}{2} \pm \frac{1}{2}\sqrt{1 + 4\frac{Ab - aB}{Bc - bC}}\right),$$



BERLINSKII THEOREM. Assume that a quadratic system

$$\dot{x} = a_0 + a_1 x + a_2 y + a_3 x^2 + a_4 xy + a_5 y^2,$$
$$\dot{y} = b_0 + b_1 x + b_2 y + b_3 x^2 + b_4 xy + b_5 y^2,$$

in the plane $\mathbb{R}^2$ has four equilibria at the vertices of a convex quadrilateral. Then two opposite equilibria are saddles (index $-1$) and the other two are antisaddles (index $1$).

BERLINSKII THEOREM. Assume that a quadratic system

$$\dot{x} = a_0 + a_1 x + a_2 y + a_3 x^2 + a_4 xy + a_5 y^2,$$
$$\dot{y} = b_0 + b_1 x + b_2 y + b_3 x^2 + b_4 xy + b_5 y^2,$$

in the plane $\mathbb{R}^2$ has four equilibria at the vertices of a convex quadrilateral. Then two opposite equilibria are saddles (index $-1$) and the other two are antisaddles (index $1$).

A. N. BERLINSKII, On the behavior of the integral curves of a differential equation, Izv. Vyssh. Uchebn. Zaved. Mat. **2** (1960), 3–18.

Berlinskii Theorem for quadratic systems on the torus can be improved as follows.

Berlinskii Theorem for quadratic systems on the torus can be improved as follows.

THEOREM. Assume that a quadratic system

$$\dot{x} = a + bx(x-1) + cy(y-1),$$
$$\dot{y} = a + Bx(x-1) + Cy(y-1).$$

in the torus $\mathbb{T}^2$ has four equilibria. Then they are localized at the vertices of a rectangle with center at the point $(1/2, 1/2)$. Two opposite equilibria are saddles (index $-1$) and the other two are antisaddles (index $1$). The two antisaddles are both either nodes, or foci, or centers, these three possibilities are realizable.

The four equilibria are

$$\left(\frac{1}{2} \pm \frac{1}{2}\sqrt{1+4\frac{aC-Ac}{Bc-bC}}, \frac{1}{2} \pm \frac{1}{2}\sqrt{1+4\frac{Ab-aB}{Bc-bC}}\right) = \left(\frac{1}{2} \pm K, \frac{1}{2} \pm L\right)$$

The four equilibria are

$$\left(\frac{1}{2} \pm \frac{1}{2}\sqrt{1+4\frac{aC-Ac}{Bc-bC}}, \frac{1}{2} \pm \frac{1}{2}\sqrt{1+4\frac{Ab-aB}{Bc-bC}}\right) = \left(\frac{1}{2} \pm K, \frac{1}{2} \pm L\right)$$

They exist if $K > 0$, $L > 0$ and $(aC - Ac)(Ab - aB) \neq 0$.

THEOREM. The following statements hold.

THEOREM. The following statements hold.

If $(bC - Bc)KL < 0$ the equilibrium is a saddle.

THEOREM. The following statements hold.

If $(bC - Bc)KL < 0$ the equilibrium is a saddle.

If $(bC - Bc)KL > 0$ and $4(Bc - bC)KL + (bK + CL)^2 > 0$ the equilibrium is a node.

THEOREM. The following statements hold.

If $(bC - Bc)KL < 0$ the equilibrium is a saddle.

If $(bC - Bc)KL > 0$ and $4(Bc - bC)KL + (bK + CL)^2 > 0$ the equilibrium is a node.

If $(bC - Bc)KL > 0$, $4(Bc - bC)KL + (bK + CL)^2 < 0$ and $bK + CL \neq 0$ the equilibrium is a strong focus.

THEOREM. The following statements hold.

If $(bC - Bc)KL < 0$ the equilibrium is a saddle.

If $(bC - Bc)KL > 0$ and $4(Bc - bC)KL + (bK + CL)^2 > 0$ the equilibrium is a node.

If $(bC - Bc)KL > 0$, $4(Bc - bC)KL + (bK + CL)^2 < 0$ and $bK + CL \neq 0$ the equilibrium is a strong focus.

If $(bC - Bc)KL > 0$, $4(Bc - bC)KL + (bK + CL)^2 < 0$, $bK + CL = 0$ and $b^3c - BC^3 \neq 0$ the equilibrium point is a weak focus.

THEOREM. The following statements hold.

If $(bC - Bc)KL < 0$ the equilibrium is a saddle.

If $(bC - Bc)KL > 0$ and $4(Bc - bC)KL + (bK + CL)^2 > 0$ the equilibrium is a node.

If $(bC - Bc)KL > 0$, $4(Bc - bC)KL + (bK + CL)^2 < 0$ and $bK + CL \neq 0$ the equilibrium is a strong focus.

If $(bC - Bc)KL > 0$, $4(Bc - bC)KL + (bK + CL)^2 < 0$, $bK + CL = 0$ and $b^3c - BC^3 \neq 0$ the equilibrium point is a weak focus.

If $(bC - Bc)KL > 0$, $4(Bc - bC)KL + (bK + CL)^2 < 0$, $bK + CL = 0$ and $b^3c - BC^3 = 0$ the equilibrium point is a center.

A QS has 2 equilibria in the following four cases:

A QS has 2 equilibria in the following four cases:

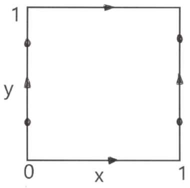
1) If $bC - Bc \neq 0$, $aC - Ac = 0$ and $(aB - Ab)L \neq 0$, then the two equilibria are $(0, 1/2 + L) = (1, 1/2 + L)$ and $(0, 1/2 - L) = (1, 1/2 - L)$.

A QS has 2 equilibria in the following four cases:

1) If $bC - Bc \neq 0$, $aC - Ac = 0$ and $(aB - Ab)L \neq 0$, then the two equilibria are $(0, 1/2 + L) = (1, 1/2 + L)$ and $(0, 1/2 - L) = (1, 1/2 - L)$. In these two equilibria the system is not $C^1$.

A QS has 2 equilibria in the following four cases:

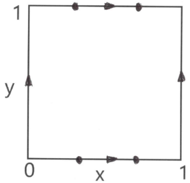
1) If $bC - Bc \neq 0$, $aC - Ac = 0$ and $(aB - Ab)L \neq 0$, then the two equilibria are $(0, 1/2 + L) = (1, 1/2 + L)$ and $(0, 1/2 - L) = (1, 1/2 - L)$. In these two equilibria the system is not $C^1$. The local phase portraits at the four points in the plane $(0, 1/2 + L)$, $(1, 1/2 + L)$, $(0, 1/2 - L)$ and $(1, 1/2 - L)$ satisfy the Berlinskii Theorem.



2) If $bC - Bc \neq 0$, $Ab - aB = 0$ and $(aC - Ac)K \neq 0$, then the two equilibria are $(1/2 + K, 0) = (1/2 + K, 1)$ and $(1/2 - K, 0) = (1/2 - K, 1)$.

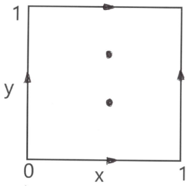
2) If $bC - Bc \neq 0$, $Ab - aB = 0$ and $(aC - Ac)K \neq 0$, then the two equilibria are $(1/2 + K, 0) = (1/2 + K, 1)$ and $(1/2 - K, 0) = (1/2 - K, 1)$. In these two equilibria the system is not $C^1$.

2) If $bC - Bc \neq 0$, $Ab - aB = 0$ and $(aC - Ac)K \neq 0$, then the two equilibria are $(1/2 + K, 0) = (1/2 + K, 1)$ and $(1/2 - K, 0) = (1/2 - K, 1)$. In these two equilibria the system is not $C^1$. The local phase portraits at the four points in the plane $(1/2 + K, 0)$, $(1/2 + K, 1)$, $(1/2 - K, 0)$ and $(1/2 - K, 1)$ satisfy the Berlinskii Theorem.



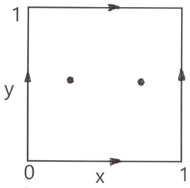
3) If $bC - Bc \neq 0$, $K = 0$ and $(aB - Bb)L \neq 0$, the two equilibria are $(1/2, 1/2 + L)$ and $(1/2, 1/2 - L)$.

3) If $bC - Bc \neq 0$, $K = 0$ and $(aB - Bb)L \neq 0$, the two equilibria are $(1/2, 1/2 + L)$ and $(1/2, 1/2 - L)$. Moreover both equilibria are saddle-nodes.



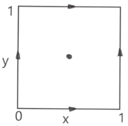
4) If $bC - Bc \neq 0$, $(aC - Ac)K \neq 0$ and $L = 0$, the two equilibria are $(1/2 + K, 1/2)$ and $(1/2 - K, 1/2)$.

4) If $bC - Bc \neq 0$, $(aC - Ac)K \neq 0$ and $L = 0$, the two equilibria are $(1/2 + K, 1/2)$ and $(1/2 - K, 1/2)$. Moreover both equilibria are saddle-nodes.

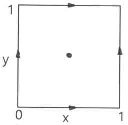


A QS has 1 equilibrium point in the following four cases:

A QS has 1 equilibrium point in the following four cases:

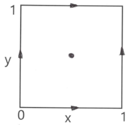


A QS has 1 equilibrium point in the following four cases:




1) If $bC - Bc \neq 0$ and $K = L = 0$, then the QS has the equilibrium $(1/2, 1/2)$.

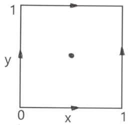
A QS has 1 equilibrium point in the following four cases:




1) If $bC - Bc \neq 0$ and $K = L = 0$, then the QS has the equilibrium $(1/2, 1/2)$.

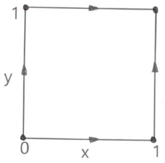
If we translate this equilibrium to the origin of coordinates, the quadratic system becomes the homogeneous quadratic system $\dot{x} = bx^2 + cy^2$, $\dot{y} = Bx^2 + Cy^2$.

A QS has 1 equilibrium point in the following four cases:

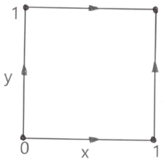



1) If $bC - Bc \neq 0$ and $K = L = 0$, then the QS has the equilibrium $(1/2, 1/2)$.

If we translate this equilibrium to the origin of coordinates, the quadratic system becomes the homogeneous quadratic system $\dot{x} = bx^2 + cy^2$, $\dot{y} = Bx^2 + Cy^2$. And all the homogeneous quadratic systems have been classified.

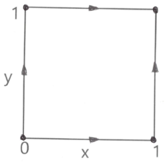



2) If $bC - Bc \neq 0$ and $aC - Ac = Ab - aB = 0$ then the QS has the equilibrium $(0,0) = (1,0) = (0,1) = (1,1)$.




2) If $bC - Bc \neq 0$ and $aC - Ac = Ab - aB = 0$ then the QS has the equilibrium $(0,0) = (1,0) = (0,1) = (1,1)$.

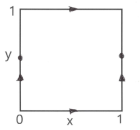
But in this equilibrium the system is not $C^1$.



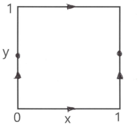

2) If $bC - Bc \neq 0$ and $aC - Ac = Ab - aB = 0$ then the QS has the equilibrium $(0,0) = (1,0) = (0,1) = (1,1)$.

But in this equilibrium the system is not $C^1$.

The local phase portraits at the four points in the plane $(0,0)$, $(1,0)$, $(0,1)$ and $(1,1)$ satisfies the Berlinskii Theorem.

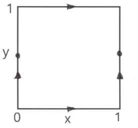



3) If $bC - Bc \neq 0$, $Ab - aB \neq 0$ and $aC - Ac = L = 0$, then the system has the equilibrium point $(0, 1/2) = (1, 1/2)$.




3) If $bC - Bc \neq 0$, $Ab - aB \neq 0$ and $aC - Ac = L = 0$, then the system has the equilibrium point $(0, 1/2) = (1, 1/2)$.

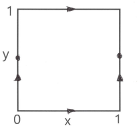
But in this equilibrium the system is not $C^1$,




3) If $bC - Bc \neq 0$, $Ab - aB \neq 0$ and $aC - Ac = L = 0$, then the system has the equilibrium point $(0, 1/2) = (1, 1/2)$.

But in this equilibrium the system is not $C^1$,

If $b \neq 0$ the "two equilibria" are semi-hyperbolic saddle-nodes.

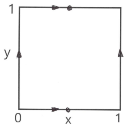



3) If $bC - Bc \neq 0$, $Ab - aB \neq 0$ and $aC - Ac = L = 0$, then the system has the equilibrium point $(0, 1/2) = (1, 1/2)$.

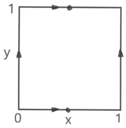
But in this equilibrium the system is not $C^1$,

If $b \neq 0$ the “two equilibria” are semi-hyperbolic saddle-nodes.

If $b = 0$ the “two equilibria” are nilpotent cusps.

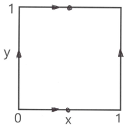



4) If $bC - Bc \neq 0$, $aC - Ac \neq 0$ and $aB - Ab = L = 0$, then the system has the equilibrium point $(1/2, 0) = (1/2, 1)$.




4) If $bC - Bc \neq 0$, $aC - Ac \neq 0$ and $aB - Ab = L = 0$, then the system has the equilibrium point $(1/2, 0) = (1/2, 1)$.

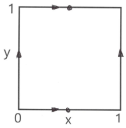
But in this equilibrium the system is not $C^1$.




4) If $bC - Bc \neq 0$, $aC - Ac \neq 0$ and $aB - Ab = L = 0$, then the system has the equilibrium point $(1/2, 0) = (1/2, 1)$.

But in this equilibrium the system is not $C^1$.

If $C \neq 0$ the "two equilibria" are semi-hyperbolic saddle-nodes.




4) If $bC - Bc \neq 0$, $aC - Ac \neq 0$ and $aB - Ab = L = 0$, then the system has the equilibrium point $(1/2, 0) = (1/2, 1)$.

But in this equilibrium the system is not $C^1$.

If $C \neq 0$ the "two equilibria" are semi-hyperbolic saddle-nodes.

If $C = 0$ the "two equilibria" are nilpotent cusps.

The QS has infinitely many equilibria under the following conditions:

The QS has infinitely many equilibria under the following conditions:

$bC - Bc = 0$ and either $A = B = C = 0$, or $a = b = c = 0$, or $aB - Ab = 0$, or $aC - Ac = 0$.

The following results obtained for the quadratic systems in the plane $\mathbb{R}^2$ also hold for the continuous quadratic differential systems on the 2-dimensional torus

The following results obtained for the quadratic systems in the plane $\mathbb{R}^2$ also hold for the continuous quadratic differential systems on the 2-dimensional torus if the closed curve defined by a periodic orbit can be deformed in a continuous way to a point on the surface of the torus. With other words if the closed curve defined by a periodic orbit is contractible on the surface of the torus.

The following results obtained for the quadratic systems in the plane $\mathbb{R}^2$ also hold for the continuous quadratic differential systems on the 2-dimensional torus if the closed curve defined by a periodic orbit can be deformed in a continuous way to a point on the surface of the torus. With other words if the closed curve defined by a periodic orbit is contractible on the surface of the torus.

1) There exists a unique equilibrium point in the interior of the region homeomorphic to a disc limited by a periodic orbit.

The following results obtained for the quadratic systems in the plane $\mathbb{R}^2$ also hold for the continuous quadratic differential systems on the 2-dimensional torus if the closed curve defined by a periodic orbit can be deformed in a continuous way to a point on the surface of the torus. With other words if the closed curve defined by a periodic orbit is contractible on the surface of the torus.

1) There exists a unique equilibrium point in the interior of the region homeomorphic to a disc limited by a periodic orbit. If the periodic orbit is a limit cycle this equilibrium is a focus,

The following results obtained for the quadratic systems in the plane $\mathbb{R}^2$ also hold for the continuous quadratic differential systems on the 2-dimensional torus if the closed curve defined by a periodic orbit can be deformed in a continuous way to a point on the surface of the torus. With other words if the closed curve defined by a periodic orbit is contractible on the surface of the torus.

1) There exists a unique equilibrium point in the interior of the region homeomorphic to a disc limited by a periodic orbit. If the periodic orbit is a limit cycle this equilibrium is a focus, and if the periodic orbit is not a limit cycle this equilibrium is a center.

The following results obtained for the quadratic systems in the plane $\mathbb{R}^2$ also hold for the continuous quadratic differential systems on the 2-dimensional torus if the closed curve defined by a periodic orbit can be deformed in a continuous way to a point on the surface of the torus. With other words if the closed curve defined by a periodic orbit is contractible on the surface of the torus.

1) There exists a unique equilibrium point in the interior of the region homeomorphic to a disc limited by a periodic orbit. If the periodic orbit is a limit cycle this equilibrium is a focus, and if the periodic orbit is not a limit cycle this equilibrium is a center.

2) Two periodic orbits are oppositely oriented if the regions homeomorphic to a disc limited by them have no common point.

3) If the differential system has two equilibrium points which are either foci, or centers, then they are oppositely oriented.

3) If the differential system has two equilibrium points which are either foci, or centers, then they are oppositely oriented.

For a proof of all these properties see the paper:

W.A. Coppel, A Survey of Quadratic Systems, J. Differential Equations **2** (1966), 293–304.

THEOREM. (A) For the continuous QS on the 2-dimensional torus from a Hopf bifurcation at most bifurcates one limit cycle.

THEOREM. (A) For the continuous QS on the 2-dimensional torus from a Hopf bifurcation at most bifurcates one limit cycle.

(B) The next configurations of contractible limit cycles to a point are the unique that the continuous QS on the 2-dimensional torus can exhibit.

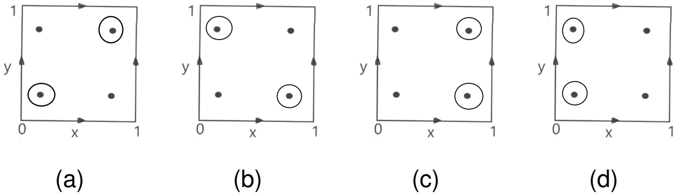



Figure: All the configurations of the contractible limit cycles of the continuous quadratic differential systems. If $[x]$ denotes the integer part function, then figure (a) takes place when $[K] < 1/2$ and $[L] < 1/2$, figure (b) takes place when $1/2 < [K]$ and $1/2 < [L]$, figure (c) takes place when $[K] < 1/2$ and $1/2 < [L]$, and figure (d) takes place when $1/2 < [K]$ and $[L] < 1/2$.

In the proof of the previous THEOREM play a main role the following result:

In the proof of the previous THEOREM play a main role the following result:

For the quadratic systems having four equilibria, if a focus is surrounded by one limit cycle, then there can be at most one limit cycle surrounding the other focus.

In the proof of the previous THEOREM play a main role the following result:

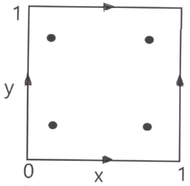
For the quadratic systems having four equilibria, if a focus is surrounded by one limit cycle, then there can be at most one limit cycle surrounding the other focus.

A. Zegeling and R.E. Kooij, The Distribution of limit cycles in quadratic systems with four finite singularities, J. Differential Equations **151** (1999), 373–385.

For the differential system

$$\dot{x} = bx(x-1), \qquad \dot{y} = A + Bx(x-1) + Cy(y-1), \quad \text{with } Ab \neq 0,$$

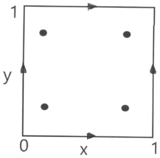
on the 2-dimensional torus has the circle $x = 0$, or equivalently the circle $x = 1$ as a non-contractible limit cycle.



For the differential system

$$\dot{x} = a + bx(x-1) + cy(y-1), \qquad \dot{y} = Cy(y-1), \quad \text{with } aC \neq 0,$$

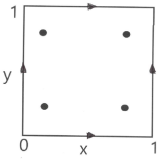
on the 2-dimensional torus has the circle $y = 0$, or equivalently the circle $y = 1$ as a non-contractible limit cycle.



For the differential system

$$\dot{x} = a + bx(x-1) + cy(y-1), \qquad \dot{y} = Cy(y-1), \quad \text{with } aC \neq 0,$$

on the 2-dimensional torus has the circle $y = 0$, or equivalently the circle $y = 1$ as a non-contractible limit cycle.




We conjecture that these configurations are all the configurations of the limit cycles for the continuous quadratic differential systems on the torus $\mathbb{T}^2$

# The end

THANK YOU VERY MUCH FOR YOUR ATTENTION