



Multilayer canard cycles

The impact of layers and their connections

F. Dumortier
Joint work with P. De Maesschalck and R. Roussarie

Mallorca, 06.02.2023

Definition (Slow-fast family of vector fields)

(In this talk) a slow-fast family of vector fields on the plane, with singular parameter ε , is a smooth family of vector fields $X_{\varepsilon,\lambda}$, with:

$$X_{\varepsilon,\lambda} = X_{0,\lambda} + \varepsilon Q_{\lambda} + O(\varepsilon^2), \quad (1)$$

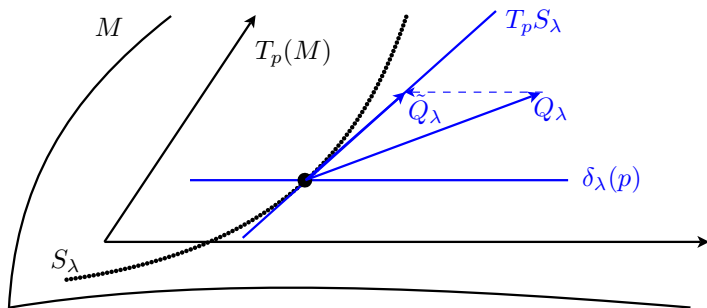
and

$$X_{0,\lambda} = F_{\lambda} Z_{\lambda},$$

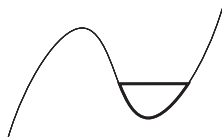
for some smooth family of functions F_{λ} and a smooth family of vector fields Z_{λ} , where for each λ the following properties hold:

- 1 Z_{λ} is a vector field without singularities,
- 2 F_{λ} is a function with a regular set of zeros S_{λ} : $dF_{\lambda}(p) \neq 0$ for $p \in \{F_{\lambda} = 0\}$.

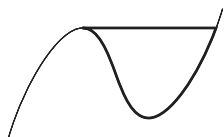
Slow vector field on $S_\lambda \setminus C_\lambda$



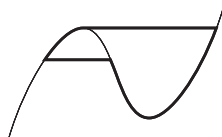
Dodging layer (D) and terminal layer (T)



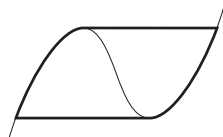
(1)



(2)

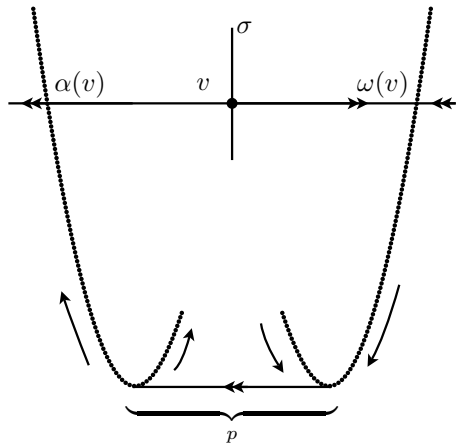
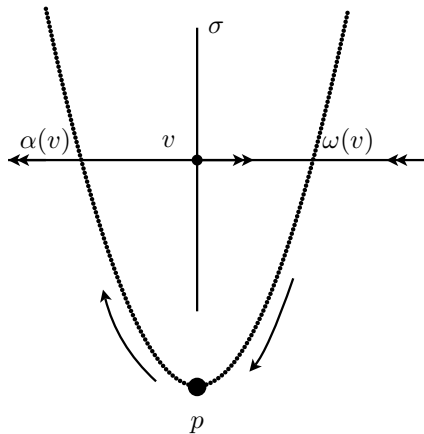


(3)

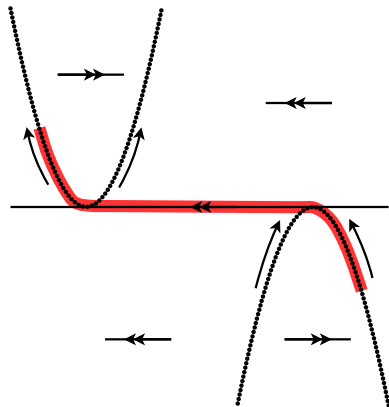
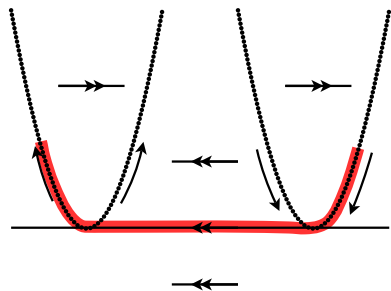


(4)

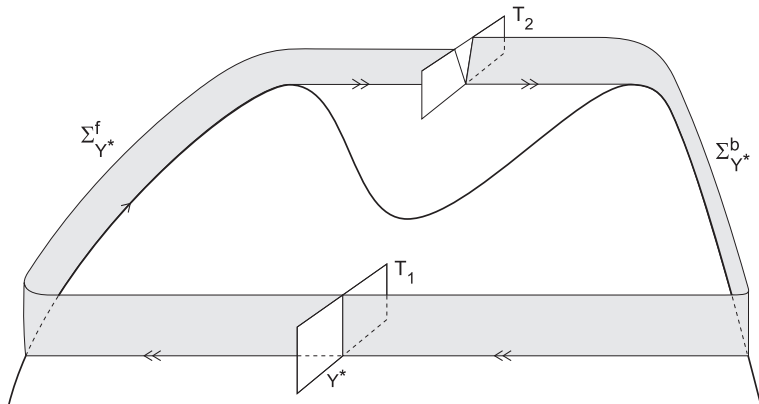
Breaking mechanisms: Hopf (H) and Jump (J)



Jump breaking mechanism



Use of sections: layer section and target section



Rescaled breaking parameter in Hopf case (1)

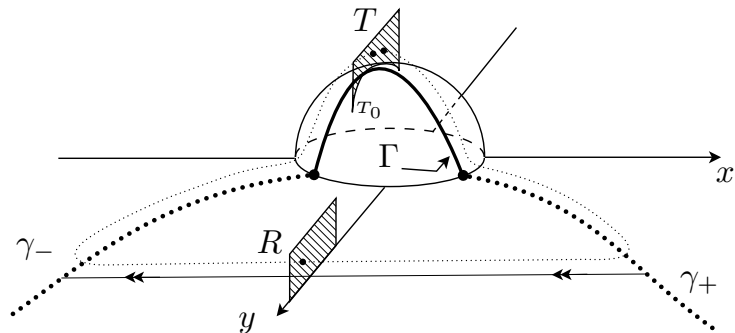
Normal form:

$$\begin{cases} \dot{x} &= y - x^2 + x^3 h_1(x, \lambda) \\ \dot{y} &= \varepsilon(a - x + x^2 h_2(x, \varepsilon, \lambda) + y h_3(x, y, \varepsilon, \lambda)). \end{cases} \quad (2)$$

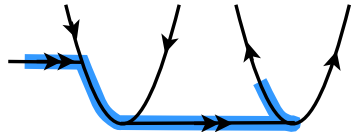
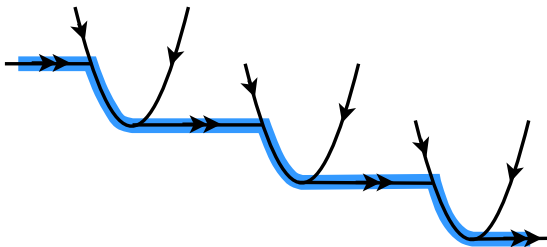
Rescaled breaking parameter:

$$a = \sqrt{\varepsilon} \bar{a}.$$

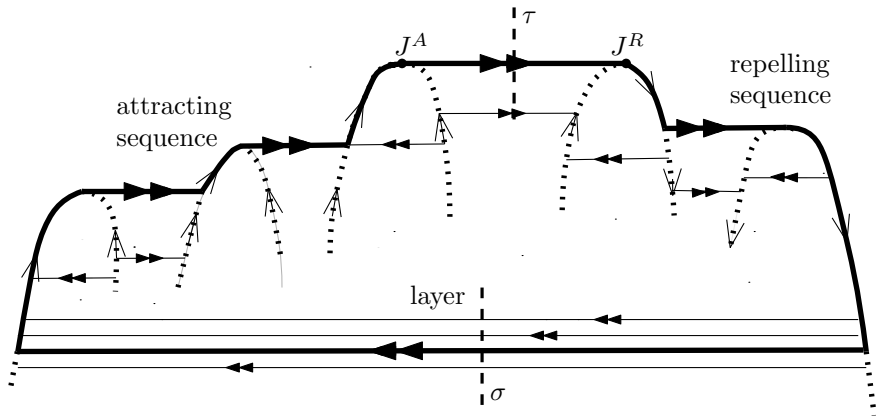
Rescaled breaking parameter in Hopf case (2)



Attracting sequence



Attracting sequence and repelling sequence



Transition map from layer section to target section

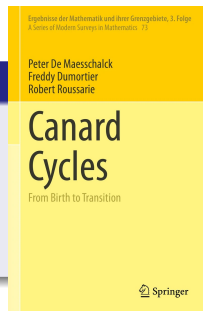
Technical elaboration in study of transition map

- Blow-up of contact points
- Center manifolds
- Normal forms
- Smooth functions on Admissible Monomials

Transition map from layer section to target section

Technical elaboration in study of transition map

- Blow-up of contact points
- Center manifolds
- Normal forms
- Smooth functions on Admissible Monomials



Canard cycles

- Part II Technical tools, pp. 97–212
- Ch 12: Local transition maps, pp. 215–265

Slow divergence integral (SDI)

Consider a closed interval $\gamma_\lambda \subset S_\lambda \setminus C_\lambda$. Parameterize it by r and write the slow vector field \tilde{Q}_λ as $q_\lambda(r) \frac{\partial}{\partial r}$, for some smooth family of functions q_λ . The differential equation of \tilde{Q}_λ in the coordinate r is

$$\frac{dr}{ds} = q_\lambda(r).$$

Suppose now that γ_λ does not contain zeros of \tilde{Q}_λ , then the slow divergence integral along γ_λ is defined as

$$I(\gamma_\lambda) = \int_{\gamma_\lambda} V_\lambda ds, \quad (3)$$

where V_λ at a point $p \in S_\lambda$ is the trace of $D(X_{0,\lambda})(p)$.

Structure of transition map

$$F_{\varepsilon, \lambda}^J(v) = z^A(\lambda) + \varepsilon^{\frac{2}{3}} \Psi^A(\lambda, \varepsilon) + \theta_{\Sigma_A} \exp\left(\frac{I^A(v, \lambda) + O(\varepsilon^{\frac{1}{3}})}{\varepsilon}\right), \quad (4)$$

where I^A is the SDI of the attracting sequence A . The function Ψ^A is smooth in $(\varepsilon^{\frac{1}{3}}, \varepsilon^{\frac{1}{3}} \ln \varepsilon, \lambda)$; the term $O(\varepsilon^{\frac{1}{3}})$ is smooth in $(v, \varepsilon^{\frac{1}{3}}, \varepsilon^{\frac{1}{3}} \ln \varepsilon, \lambda)$ and $\theta_{\Sigma_A} = \pm 1$ (depending on the chosen orientation at the target section.)

Structure of transition map

$$F_{\varepsilon,\lambda}^J(v) = z^A(\lambda) + \varepsilon^{\frac{2}{3}} \Psi^A(\lambda, \varepsilon) + \theta_{\Sigma_A} \exp\left(\frac{I^A(v, \lambda) + O(\varepsilon^{\frac{1}{3}})}{\varepsilon}\right), \quad (4)$$

where I^A is the SDI of the attracting sequence A . The function Ψ^A is smooth in $(\varepsilon^{\frac{1}{3}}, \varepsilon^{\frac{1}{3}} \ln \varepsilon, \lambda)$; the term $O(\varepsilon^{\frac{1}{3}})$ is smooth in $(v, \varepsilon^{\frac{1}{3}}, \varepsilon^{\frac{1}{3}} \ln \varepsilon, \lambda)$ and $\theta_{\Sigma_A} = \pm 1$ (depending on the chosen orientation at the target section.)

$$F_{\varepsilon,\lambda}^H(v) = z^A(\lambda) + \varepsilon^{\frac{1}{2}} \Psi^A(\lambda, \varepsilon) + \theta_{\Sigma_A} \exp\left(\frac{I^A(v, \lambda) + O(\varepsilon^{\frac{1}{3}})}{\varepsilon}\right), \quad (5)$$

where I^A is the SDI of the attracting sequence A and $\lambda = (\bar{a}, \mu)$. The function Ψ^A is smooth in $(\varepsilon^{\frac{1}{2}}, \lambda)$, while $O(\varepsilon^{\frac{1}{3}})$ is smooth in $(v, \varepsilon^{\frac{1}{2}}, \varepsilon^{\frac{1}{3}}, \varepsilon^{\frac{1}{3}} \ln \varepsilon, \lambda)$ and $\theta_{\Sigma_A} = \pm 1$ (depending on the chosen orientation at the target section.)

Common expression for difference function

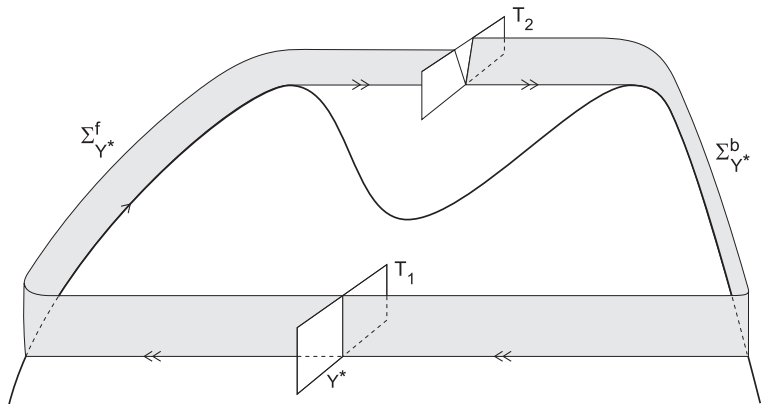
$$\Delta_{\varepsilon, \lambda}(v) = \alpha(\lambda, \varepsilon) + e^{\frac{I^A(v, \lambda) + o_\varepsilon(1)}{\varepsilon}} - \theta_C e^{\frac{I^{\bar{R}}(v, \lambda) + o_\varepsilon(1)}{\varepsilon}} \quad (6)$$

where α and the remainder terms $o_\varepsilon(1)$ are smooth in an appropriate set Ω of ε -monomials and in the other variables (v, λ) . $\theta_C = \pm 1$, depending on θ_{Σ_A} and $\theta_{\Sigma_{\bar{R}}}$ (and an appropriate choice of orientation at the target section.)

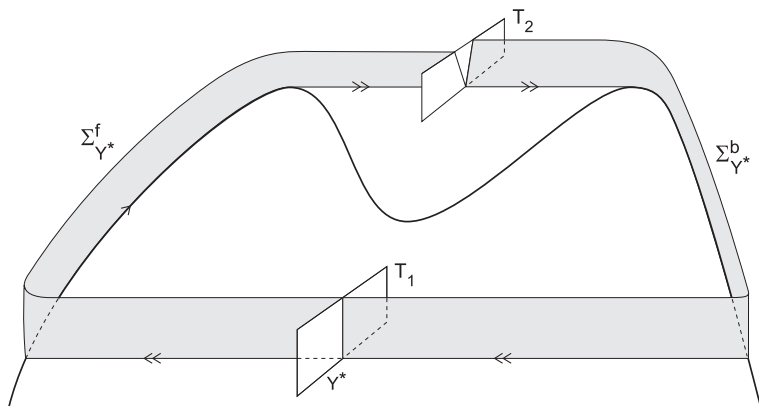
We can take $\lambda = (b, \mu)$ so that the function α verifies

$$\alpha((b, \mu), 0) = b.$$

Intrinsic meaning of θ_C

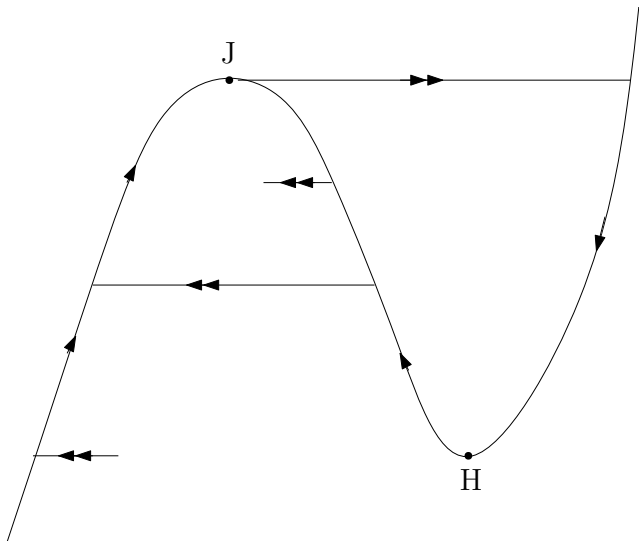


Intrinsic meaning of θ_C



- Side preserving connection (P) : $\theta_C = 1$
- Side reversing connection (R) : $\theta_C = -1$

Single dodging layer



Single dodging layer: looking for limit cycles

The connection is side reversing. Limit cycles are given by zeroes of

$$\Delta_{\varepsilon, \lambda}(v) = \alpha(\lambda, \varepsilon) + e^{\frac{I^A(v, \lambda) + o_{\varepsilon}(1)}{\varepsilon}} + e^{\frac{I^{\bar{R}}(v, \lambda) + o_{\varepsilon}(1)}{\varepsilon}} = 0 \quad (7)$$

Single dodging layer: looking for limit cycles

The connection is side reversing. Limit cycles are given by zeroes of

$$\Delta_{\varepsilon, \lambda}(v) = \alpha(\lambda, \varepsilon) + e^{\frac{I^A(v, \lambda) + o_{\varepsilon}(1)}{\varepsilon}} + e^{\frac{I^{\bar{R}}(v, \lambda) + o_{\varepsilon}(1)}{\varepsilon}} = 0 \quad (7)$$

Important remark:

Since $\frac{\partial I^A}{\partial v}(v, \lambda)$ and $\frac{\partial I^{\bar{R}}}{\partial v}(v, \lambda)$ have opposite signs we get:

$$\frac{\partial(I^A - I^{\bar{R}})}{\partial v}(v, \lambda) \neq 0,$$

so that $I^A - I^{\bar{R}} = I^A + I^R$ has at most one zero, which is simple.

Main aim of his talk

Studying multilayer canard cycles, as much as possible in the context of polynomial (generalized) Liénard equations

$$\begin{cases} \dot{x} &= y - F_\lambda(x) \\ \dot{y} &= \varepsilon g_\lambda(x) \end{cases}$$

Here $\lambda = (\lambda_1, \dots, \lambda_n)$ are parameters that are used for the individual canard breaking mechanisms.

Benefits of Liénard setting: easy basic understanding

- The critical curve is a graph $y = F_\lambda(x)$
- The fast dynamics, for $\varepsilon = 0$, is trivial:

$$\begin{cases} \dot{x} &= y - F_\lambda(x) \\ \dot{y} &= 0 \end{cases}$$

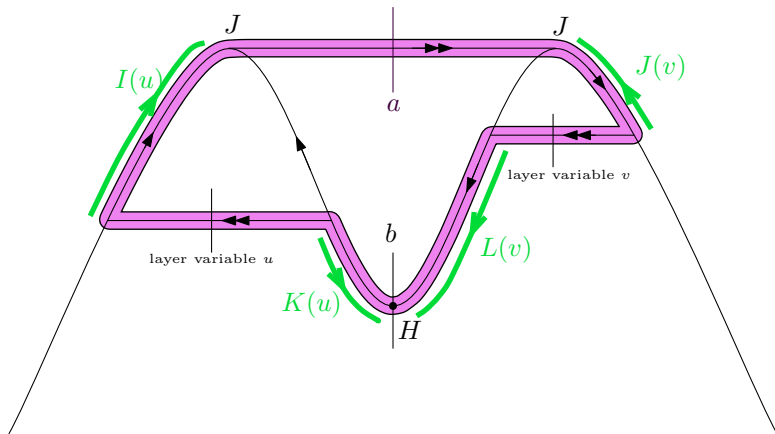
- The slow vector field can be expressed as:

$$y' = g_\lambda(x)|_{x=F_\lambda^{-1}(y)}$$

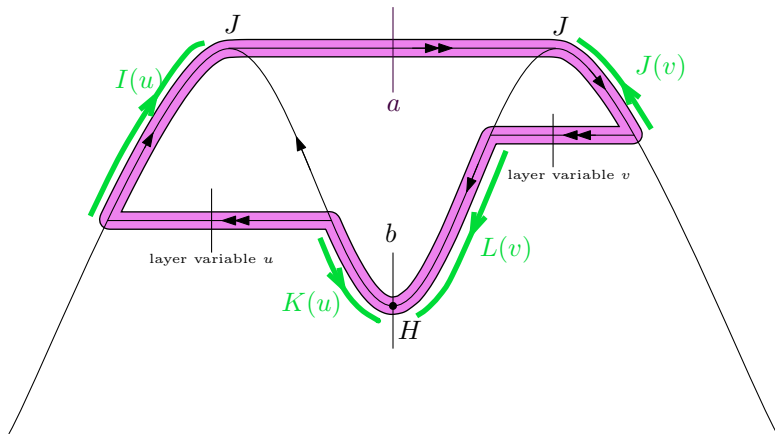
(and $x' = y'/F'_\lambda(x)$)

- The contact points are found at the local extremes of $F_\lambda(x)$. The singular ones are those at which $g_\lambda = 0$

2-layer cycle: DPDP -case



2-layer cycle: DPDP -case



How to generalize?

Signs θ play a role

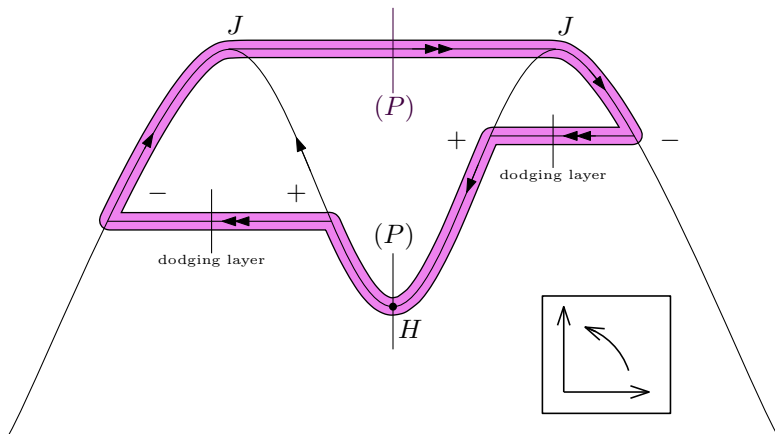
Canard cycles p.256:

We define a coefficient θ_A for the whole sequence A by

$$\theta_A = \prod_{i=2}^k \theta_{J_i} \prod_{j=1}^{k-1} \theta_f^j. \quad (8)$$

Moreover, independently of the previous coefficients associated to A itself, we will use the θ_{Σ_1} associated to the position of the starting section Σ in relation to γ_S^1 .

DPDP -cycle



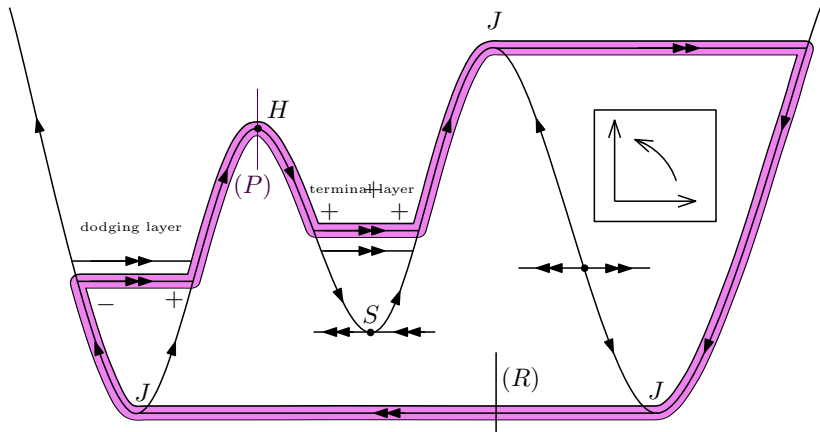
Easy check on the type of connection

Choose an orientation along the canard cycle under consideration. In a plane take e.g. $((1, 0), (0, 1))$ as positive, and represent it counterclockwise. Check whether the passages from slow to fast, when entering a layer, and those from fast to slow, when leaving a layer, are positive or negative and put a $+$ or a $-$.

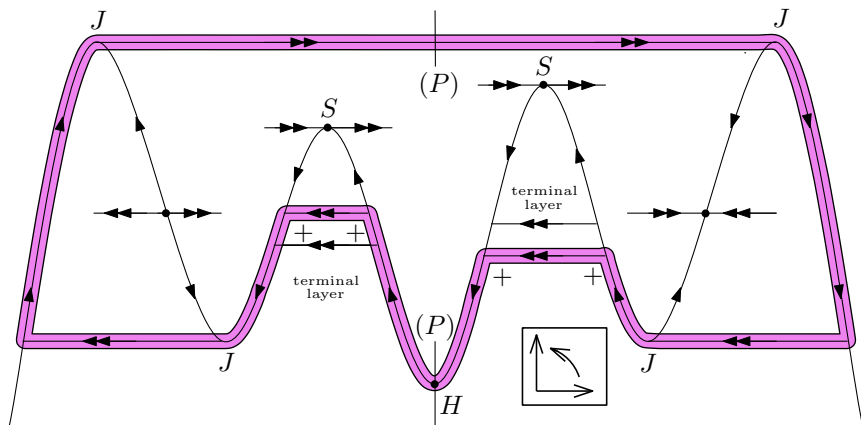
A connection between two $+$ or between two $-$ is side preserving.

A connection between $+$ and $-$, or $-$ and $+$ is side reversing.

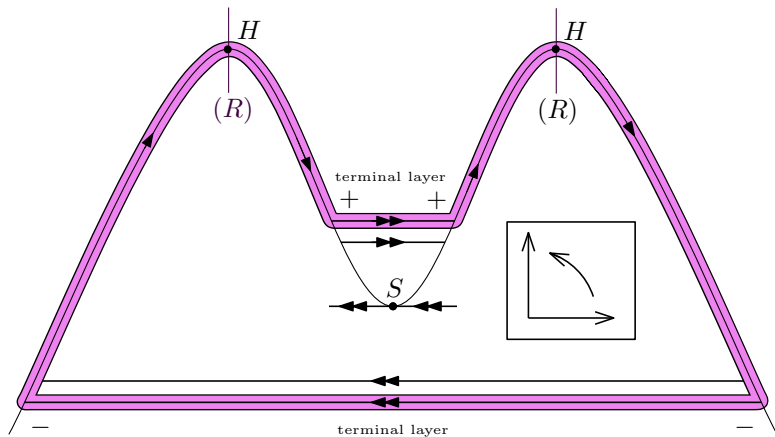
DPTR-cycle



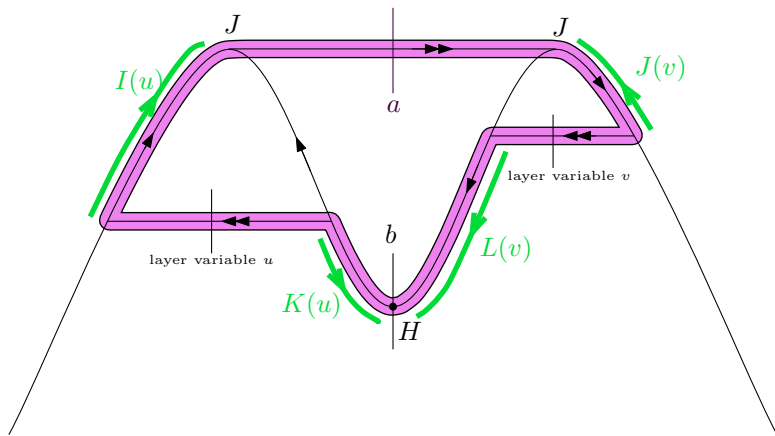
TPTP-cycle



TRTR-cycle



Results on DPDP-cycle



Theorem for DPDP-cycle

Let $D(u, v) = I(u) - J(v) + L(v) - K(u)$ be the (total) SDI of Γ_{uv} and choose some $(u, v) = (u_0, v_0)$.

Theorem

- 1 If $D(u_0, v_0) \neq 0$, then a hyperbolic relaxation oscillation bifurcates from $\Gamma_{u_0v_0}$.
- 2 If $D(u_0, v_0) = 0$ and $I(u_0) - J(v_0) \neq 0$, then a codimension 1 semi-stable relaxation oscillation bifurcates from $\Gamma_{u_0v_0}$. This semi-stable limit cycle is generically unfolded by the parameter (a, b) , for $\varepsilon > 0$ small enough, producing a pair of hyperbolic limit cycles.
- 3 If $D(u_0, v_0) = 0$ and $I(u_0) - J(v_0) = 0$ and $I'(u_0)L'(v_0) - K'(u_0)J'(v_0) \neq 0$, then a codimension 2 relaxation oscillation bifurcates from $\Gamma_{u_0v_0}$. This degenerate limit cycle is generically unfolded by the parameter (a, b) , for $\varepsilon > 0$ small enough, producing systems having three hyperbolic limit cycles in the vicinity of $\Gamma_{u_0v_0}$.

Connections in the general 2-layer case

We have to deal with

$$\begin{cases} \exp\left(\frac{\tilde{I}(u, \lambda, \varepsilon)}{\varepsilon}\right) - \theta_1 \exp\left(\frac{\tilde{J}(v, \lambda, \varepsilon)}{\varepsilon}\right) = a \cdot F(\lambda, \varepsilon) \\ \exp\left(\frac{\tilde{K}(u, \lambda, \varepsilon)}{\varepsilon}\right) - \theta_2 \exp\left(\frac{\tilde{L}(v, \lambda, \varepsilon)}{\varepsilon}\right) = b \cdot G(\lambda, \varepsilon). \end{cases} \quad (9)$$

with $F(\lambda_0, 0) \neq 0$ and $G(\lambda_0, 0) \neq 0$. This can be written as :

$$\begin{cases} \exp\left(\frac{\tilde{I}(u, \lambda, \varepsilon)}{\varepsilon}\right) - \theta_1 \exp\left(\frac{\tilde{J}(v, \lambda, \varepsilon)}{\varepsilon}\right) = a \\ \exp\left(\frac{\tilde{K}(u, \lambda, \varepsilon)}{\varepsilon}\right) - \theta_2 \exp\left(\frac{\tilde{L}(v, \lambda, \varepsilon)}{\varepsilon}\right) = b. \end{cases} \quad (10)$$

with new functions \tilde{I} , \tilde{J} , \tilde{K} and \tilde{L} , which differ from the previous ones by terms of order $o_\varepsilon(1)$.

The equations can be changed to a similar system in which $\lambda = (a, b)$ disappears from the left hand side of the equations.

Introducing a map from the plane to the plane

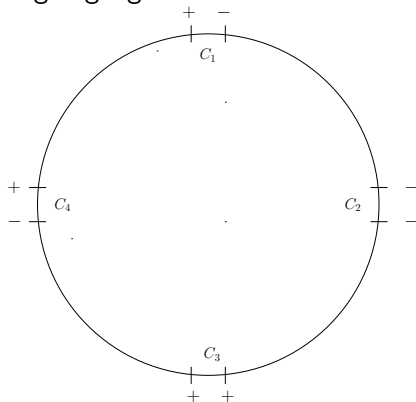
Let us consider the mapping

$$\Phi_\varepsilon : \begin{cases} a = \exp\left(\frac{\tilde{I}(u,v,\varepsilon)}{\varepsilon}\right) - \theta_1 \exp\left(\frac{\tilde{J}(u,v,\varepsilon)}{\varepsilon}\right) \\ b = \exp\left(\frac{\tilde{K}(u,v,\varepsilon)}{\varepsilon}\right) - \theta_2 \exp\left(\frac{\tilde{L}(u,v,\varepsilon)}{\varepsilon}\right). \end{cases} \quad (11)$$

We can see Φ_ε as a family of maps from a plane with coordinates $(u, v) \in [\alpha, \beta]^2$, to another plane with coordinates (a, b) near (u_0, v_0) depending on a parameter ε . For each (a, b) in the image of Φ_ε , with $\varepsilon > 0$ small enough, each counter-image (u, v) corresponds to a limit cycle Γ_{uv}^ε for the value ε cutting the layer sections in resp. u and v .

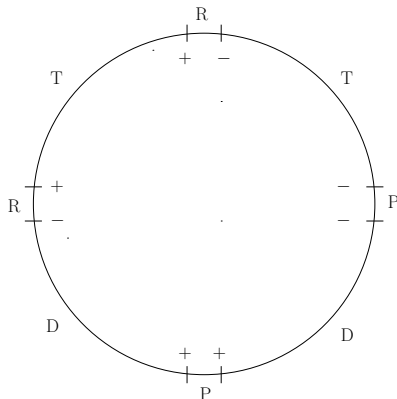
Connection diagram (1)

Primary data: represent on a circle (in cyclic order) the connections (C_1, \dots, C_n) by small segments and add, at the end points of each segment, the incoming and outgoing signs near the connection.

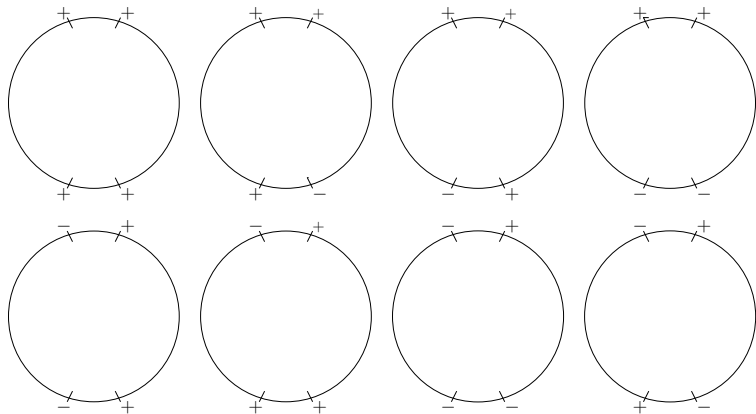


Connection diagram (2)

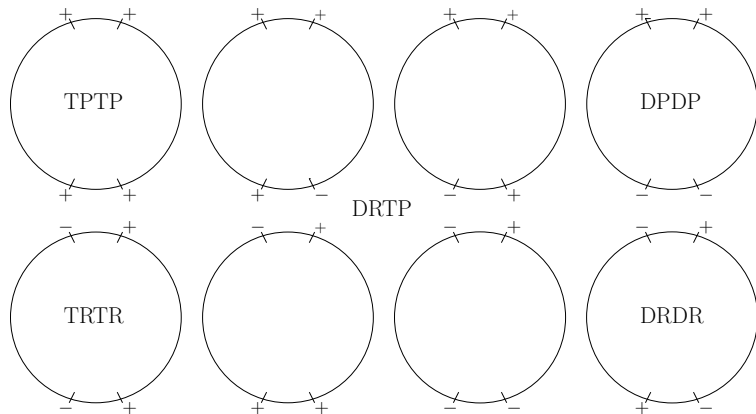
Complete the connection diagram by indicating the kind of connections (P or R) and the kind of layers (D or T) we encounter between the connections.



Connection diagram with 2 layers

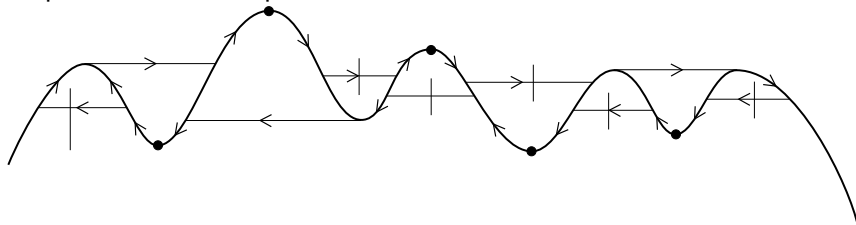


Connection diagram with 2 layers



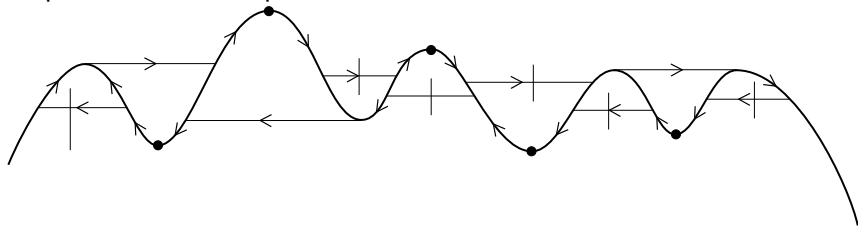
Realisation of connection diagrams in general

Example in a Liénard equation:



Realisation of connection diagrams in general

Example in a Liénard equation:



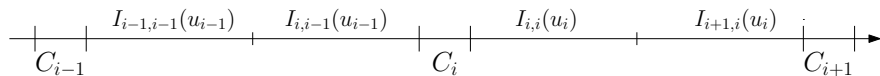
Question 1: can all connection diagrams be realized in Liénard equations?

Question 2: can all connection diagrams be realized by a polynomial vector field?

Adding the SDI's to the connection diagram

We introduce a layer variable u_i for the layer L_i coming after C_i . We hence have u_{i-1} for the layer L_{i-1} preceding C_i . We take $i \in \mathbb{Z}/n\mathbb{Z}$.

We suppose to have canard cycles $\Gamma_{u,0}$, given by $u = (u_1, \dots, u_n)$ and $\lambda = (a_1, \dots, a_n) = (0, \dots, 0)$. In each layer L_i we consider (and calculate) the SDI's $I_{i,i}(u_i)$, linked to C_i , and $I_{i+1,i}(u_i)$ linked to C_{i+1} , of the canard cycle $\Gamma_{u,0}$.



System of equations to be solved

$$\left\{ \begin{array}{l} \exp\left(\frac{\tilde{I}_{1,n}(u_n, \lambda, \varepsilon)}{\varepsilon}\right) - \theta_1 \exp\left(\frac{\tilde{I}_{1,1}(u_1, \lambda, \varepsilon)}{\varepsilon}\right) = a_1 \\ \vdots \\ \exp\left(\frac{\tilde{I}_{i,i-1}(u_{i-1}, \lambda, \varepsilon)}{\varepsilon}\right) - \theta_i \exp\left(\frac{\tilde{I}_{i,i}(u_i, \lambda, \varepsilon)}{\varepsilon}\right) = a_i \\ \vdots \\ \exp\left(\frac{\tilde{I}_{n,n-1}(u_{n-1}, \lambda, \varepsilon)}{\varepsilon}\right) - \theta_n \exp\left(\frac{\tilde{I}_{n,n}(u_n, \lambda, \varepsilon)}{\varepsilon}\right) = a_n. \end{array} \right. \quad (12)$$

System of equations to be solved

$$\left\{ \begin{array}{l} \exp\left(\frac{\tilde{l}_{1,n}(u_n, \lambda, \varepsilon)}{\varepsilon}\right) - \theta_1 \exp\left(\frac{\tilde{l}_{1,1}(u_1, \lambda, \varepsilon)}{\varepsilon}\right) = a_1 \\ \vdots \\ \exp\left(\frac{\tilde{l}_{i,i-1}(u_{i-1}, \lambda, \varepsilon)}{\varepsilon}\right) - \theta_i \exp\left(\frac{\tilde{l}_{i,i}(u_i, \lambda, \varepsilon)}{\varepsilon}\right) = a_i \\ \vdots \\ \exp\left(\frac{\tilde{l}_{n,n-1}(u_{n-1}, \lambda, \varepsilon)}{\varepsilon}\right) - \theta_n \exp\left(\frac{\tilde{l}_{n,n}(u_n, \lambda, \varepsilon)}{\varepsilon}\right) = a_n. \end{array} \right. \quad (12)$$

The system can be adapted to eliminate λ from the $\tilde{l}_{i,j}(u_j, \lambda, \varepsilon)$, in a way that $\tilde{l}_{i,j}(u, 0) = l_{i,j}(u_i)$.

The a_i can be supposed to be the original (rescaled) breaking parameters by choosing in a good way the orientation on the target sections.

Adding the signs of the derivatives

There is still a freedom to choose an orientation on the different layer sections. We can e.g. suppose that, for all i :

$$\frac{\partial l_{i,i}}{\partial u_i}(u_i) > 0.$$

It implies

- for a dodging layer:

$$\frac{\partial l_{i+1,i}}{\partial u_i}(u_i) < 0,$$

- for a terminal layer:

$$\frac{\partial l_{i+1,i}}{\partial u_i}(u_i) > 0.$$

Introducing a map from \mathbb{R}^n to \mathbb{R}^n

Like in the 2-layer case the system of equations defines a map Φ_ε from the (u_1, \dots, u_n) -space to the (a_1, \dots, a_n) -space. For $\varepsilon > 0$, counter-images of (a_1, \dots, a_n) correspond to limit cycles.

To find the singularities of Φ_ε , for $\varepsilon > 0$ but small, we consider $d\Phi_\varepsilon$ and calculate $\Delta(u, \varepsilon) = \det d\Phi_\varepsilon(u)$, with $u = (u_1, \dots, u_n)$.

The rows of the Jacobian matrix of $d\Phi_\varepsilon$ (essentially) contain two entries:

$$-\frac{\theta_i}{\varepsilon} \tilde{l}'_{i,i} \exp \frac{\tilde{l}_{i,i}}{\varepsilon}$$

on the diagonal, and

$$\frac{1}{\varepsilon} \tilde{l}'_{i,i-1} \exp \frac{\tilde{l}_{i,i-1}}{\varepsilon}$$

on the sub diagonal (except for $i = 1$, in which case it is situated in the right upper corner).

Matrix of $\det d\Phi_\varepsilon(u)$

$$\begin{pmatrix} c_{1,1} & 0 & 0 & \cdots & 0 & 0 & c_{1,n} \\ c_{2,1} & c_{2,2} & 0 & \cdots & 0 & 0 & 0 \\ 0 & c_{3,2} & c_{3,3} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & c_{n-2,n-2} & 0 & 0 \\ 0 & 0 & 0 & \cdots & c_{n-1,n-2} & c_{n-1,n-1} & 0 \\ 0 & 0 & 0 & \cdots & 0 & c_{n,n-1} & c_{n,n} \end{pmatrix}$$

Matrix of $\det d\Phi_\varepsilon(u)$

$$\begin{pmatrix} c_{1,1} & 0 & 0 & \cdots & 0 & 0 & c_{1,n} \\ c_{2,1} & c_{2,2} & 0 & \cdots & 0 & 0 & 0 \\ 0 & c_{3,2} & c_{3,3} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & c_{n-2,n-2} & 0 & 0 \\ 0 & 0 & 0 & \cdots & c_{n-1,n-2} & c_{n-1,n-1} & 0 \\ 0 & 0 & 0 & \cdots & 0 & c_{n,n-1} & c_{n,n} \end{pmatrix}$$

$$(c_{i,i-1}, c_{i,i}) = \left(\frac{1}{\varepsilon} \tilde{l}'_{i,i-1} \exp \frac{\tilde{l}_{i,i-1}}{\varepsilon}, -\frac{\theta_i}{\varepsilon} \tilde{l}'_{i,i} \exp \frac{\tilde{l}_{i,i}}{\varepsilon} \right)$$

Calculating $\Delta(u, \varepsilon) = \det d\Phi_\varepsilon(u)$

Seen the structure of the matrix, it is easy to get:

$$\begin{aligned}(-1)^n \varepsilon^n \Delta(u, \varepsilon) &= (-1)^n \varepsilon^n \det d\Phi_\varepsilon(u) = \\ &= \prod_i^n \theta_i \tilde{l}'_{i,i} \exp \frac{\tilde{l}_{i,i}}{\varepsilon} - \prod_i^n \tilde{l}'_{i,i-1} \exp \frac{\tilde{l}_{i,i-1}}{\varepsilon} = \\ &= \left(\prod_i^n \theta_i \tilde{l}'_{i,i} \right) \cdot \exp \frac{\sum_i^n \tilde{l}_{i,i}}{\varepsilon} - \left(\prod_i^n \tilde{l}'_{i,i-1} \right) \cdot \exp \frac{\sum_i^n \tilde{l}_{i,i-1}}{\varepsilon} =\end{aligned}$$

Calculating $\Delta(u, \varepsilon) = \det d\Phi_\varepsilon(u)$

Seen the structure of the matrix, it is easy to get:

$$\begin{aligned}(-1)^n \varepsilon^n \Delta(u, \varepsilon) &= (-1)^n \varepsilon^n \det d\Phi_\varepsilon(u) = \\ &= \prod_i^n \theta_i \tilde{l}'_{i,i} \exp \frac{\tilde{l}_{i,i}}{\varepsilon} - \prod_i^n \tilde{l}'_{i,i-1} \exp \frac{\tilde{l}_{i,i-1}}{\varepsilon} = \\ &= \left(\prod_i^n \theta_i \tilde{l}'_{i,i} \right) \cdot \exp \frac{\sum_i^n \tilde{l}_{i,i}}{\varepsilon} - \left(\prod_i^n \tilde{l}'_{i,i-1} \right) \cdot \exp \frac{\sum_i^n \tilde{l}_{i,i-1}}{\varepsilon} = \\ &= \left(\prod_i^n \theta_i \tilde{l}'_{i,i} \right) \cdot \exp \frac{\sum_i^n \tilde{l}_{i,i-1}}{\varepsilon} \cdot \left(\exp \frac{\tilde{D}}{\varepsilon} - \prod_i^n \frac{\tilde{l}'_{i,i-1}}{\theta_i \tilde{l}'_{i,i}} \right),\end{aligned}$$

with

$$\tilde{D} = \sum_i^n (\tilde{l}_{i,i} - \tilde{l}_{i,i-1}).$$

Reducing the equation $\Delta = 0$

We get that $\Delta(u, \varepsilon) = 0$ if and only if

$$\exp \frac{\tilde{D}}{\varepsilon} - \prod_i^n \frac{\tilde{l}'_{i,i-1}}{\theta_i \tilde{l}'_{i,i}} = 0,$$

where

$$\tilde{D}(u, \varepsilon) = \sum_i^n (\tilde{l}_{i,j} - \tilde{l}_{i,j-1})$$

and

$$D(u) = \tilde{D}(u, 0) = \sum_i^n (l_{i,j} - l_{i,j-1}),$$

expresses the total SDI of the canard cycles Γ_u .

Regularizing the equation $\Delta = 0$

We will now show that

$$\prod_i^n \frac{\tilde{l}'_{i,i-1}}{\theta_i \tilde{l}'_{i,j}} > 0,$$

implying that we can regularize the equation $\Delta(u, \varepsilon) = 0$ to

$$\tilde{D}(u, \varepsilon) - \varepsilon \log \prod_i^n \frac{\tilde{l}'_{i,i-1}}{\theta_i \tilde{l}'_{i,j}} = 0,$$

i.e.

$$D(u) + o_\varepsilon(1) = 0,$$

with $o_\varepsilon(1)$ smooth in (u, Ω, λ) , and $\Omega = (\varepsilon^{\frac{1}{2}}, \varepsilon^{\frac{1}{3}}, \varepsilon^{\frac{1}{3}} \ln \varepsilon)$.

Checking that the product is positive

Consider

$$\prod_i^n \frac{\tilde{l}'_{i,i-1}}{\theta_i \tilde{l}'_{i,i}} = \prod_i^n \frac{\tilde{l}'_{i+1,i}}{\theta_i \tilde{l}'_{i,i}}.$$

Checking that the product is positive

Consider

$$\prod_i^n \frac{\tilde{l}'_{i,i-1}}{\theta_i \tilde{l}'_{i,i}} = \prod_i^n \frac{\tilde{l}'_{i+1,i}}{\theta_i \tilde{l}'_{i,i}}.$$

Proving that the product is positive amounts to prove that, for each connection diagram, we have

$$n_D + n_R = 0 \pmod{2},$$

where n_D is the number of dodging layers and n_R is the number of side-reversing connections.

$n_D + n_R$ is even

The connection diagram is divided in $2n$ segments and at each of the $2n$ endpoints there is a sign (+ or -).

The connection diagram is divided in $2n$ segments and at each of the $2n$ endpoints there is a sign (+ or -).

Between a succession of two equal signs there is either a terminal layer or a side preserving connection. Between a succession of two different signs there is either a side reversing connection or a dodging layer.

The connection diagram is divided in $2n$ segments and at each of the $2n$ endpoints there is a sign (+ or -).

Between a succession of two equal signs there is either a terminal layer or a side preserving connection. Between a succession of two different signs there is either a side reversing connection or a dodging layer.

Starting at some point (e.g. with a +) we come back to that point after a full turn, implying that the number of switches has to be even.

Canard cycles Γ_u with $D(u) \neq 0$.

At canard cycles where the total SDI $D(u)$ is different from zero we already know that Hausdorff-close to the canard cycle there can be at most one limit cycle and it is necessarily hyperbolic.

P. De Maesschalck, F. Dumortier, R. Roussarie: Cyclicity of common slow-fast cycles. *Indag. Math. (N.S.)* 22 (2011), no. 3-4, 165–206.

Canard cycles Γ_u with $D(u) \neq 0$.

At canard cycles where the total SDI $D(u)$ is different from zero we already know that Hausdorff-close to the canard cycle there can be at most one limit cycle and it is necessarily hyperbolic.

P. De Maesschalck, F. Dumortier, R. Roussarie: Cyclicity of common slow-fast cycles. *Indag. Math. (N.S.)* 22 (2011), no. 3-4, 165–206.

Thanks to the breaking parameters we see, with the help of Φ_ε , that there is a limit cycle.

Canard cycles Γ_u with $D(u) = 0$.

We can rely on the theory of singularities of maps from n -space to n -space. Like in the 2-layer case we work at points of

$$S = \{\Delta(u, \varepsilon) = 0\},$$

and analyse the relative position between $T_p S$ and $\text{Ker}(d\Phi_\varepsilon)$. It can easily be seen that $\text{Ker}(d\Phi_\varepsilon)$ is 1-dimensional.

The equations are given by:

$$u_i = \theta_i \frac{\tilde{l}'_{i,i-1}}{\tilde{l}'_{i,i}} \exp\left(\frac{\tilde{l}_{i,i-1} - \tilde{l}_{i,i}}{\varepsilon}\right) u_{i-1}.$$

SN-bifurcation of limit cycles

We get a saddle-node bifurcation of limit cycles at the points where Φ_ε has a fold. This happens under the condition that for all i :

- $l_{i,i-1} - l_{i,i} \neq 0$, and
- $\tilde{l}'_{i+1,i} - \tilde{l}'_{i,i} \neq 0$.

The first conditions are related to the connections, the second ones to the layers.

SN-bifurcation of limit cycles

We get a saddle-node bifurcation of limit cycles at the points where Φ_ε has a fold. This happens under the condition that for all i :

- $l_{i,i-1} - l_{i,i} \neq 0$, and
- $\tilde{l}'_{i+1,i} - \tilde{l}'_{i,i} \neq 0$.

The first conditions are related to the connections, the second ones to the layers.

The conditions on the derivatives surely hold in the dodging layers.

SN-bifurcation of limit cycles

We get a saddle-node bifurcation of limit cycles at the points where Φ_ε has a fold. This happens under the condition that for all i :

- $l_{i,i-1} - l_{i,i} \neq 0$, and
- $\tilde{l}'_{i+1,i} - \tilde{l}'_{i,i} \neq 0$.

The first conditions are related to the connections, the second ones to the layers.

The conditions on the derivatives surely hold in the dodging layers.

We can violate up to $n - 1$ conditions, seen that we have n parameters.

Elementary catastrophes of limit cycles

The only stable singularities of the map Φ_ε are singularities of type $\Sigma^{1,\dots,1,0} = \Sigma^{1r,0}$.

Elementary catastrophes of limit cycles

The only stable singularities of the map Φ_ε are singularities of type $\Sigma^{1,\dots,1,0} = \Sigma^{1_r,0}$.

Up to smooth right-left equivalence they have a normal form:

$$\begin{cases} X_1 = x_1 \\ \vdots \\ X_{n-1} = x_{n-1} \\ Y = y^{r+1} + x_1 \cdot y^{r-1} + x_2 \cdot y^{r-2} + \dots + x_{r-1} \cdot y, \end{cases} \quad (13)$$

with $1 \leq r \leq n$.

Elementary catastrophes of limit cycles

The only stable singularities of the map Φ_ε are singularities of type $\Sigma^{1,\dots,1,0} = \Sigma^{1r,0}$.

Up to smooth right-left equivalence they have a normal form:

$$\begin{cases} X_1 = x_1 \\ \vdots \\ X_{n-1} = x_{n-1} \\ Y = y^{r+1} + x_1 \cdot y^{r-1} + x_2 \cdot y^{r-2} + \dots + x_{r-1} \cdot y, \end{cases} \quad (13)$$

with $1 \leq r \leq n$.

The codimension is r , leading to elementary catastrophes of limit cycles and phase portraits with $r + 1$ limit cycles. Most degenerate is $r = n$.

Balanced canard cycles

A canard cycle $\Gamma_{u_0,0}$ is called (totally) balanced if, for all i :

$$l_{i,i} = l_{i,i-1}$$

Under the generic condition that

$$\prod_{i=1}^n \frac{\tilde{l}'_{i,i-1}}{\tilde{l}'_{i,i}} \neq 1,$$

these canard cycles have been studied for the cases where all connections are side-preserving.

Balanced canard cycles

A canard cycle $\Gamma_{u_0,0}$ is called (totally) balanced if, for all i :

$$l_{i,i} = l_{i,i-1}$$

Under the generic condition that

$$\prod_{i=1}^n \frac{\tilde{l}_{i,i-1}}{\tilde{l}_{i,i}} \neq 1,$$

these canard cycles have been studied for the cases where all connections are side-preserving.

F. Dumortier & R. Roussarie, Multi-layer canard cycles and translated power functions. J. Differential Equations 244 (2008), no. 6, 1329–1358.

Balanced canard cycles: curve of maximal degeneracy

If the canard cycle Γ_{u_0} is balanced and generic, then there exists an application $\varepsilon \rightarrow u(\varepsilon) = (u_1(\varepsilon), \dots, u_n(\varepsilon))$, with $u(0) = u_0$ such that

$$\tilde{l}_{i,j}(u_i(\varepsilon), \varepsilon) = \tilde{l}_{i,j-1}(u_{i-1}(\varepsilon), \varepsilon), \quad (14)$$

for all $\varepsilon > 0$ small enough. The application is smooth in (u, Ω, λ) .

We write:

$$l_i^0(\varepsilon) = \tilde{l}_{i,i}(u_i(\varepsilon), \varepsilon) = \tilde{l}_{i,i-1}(u_{i-1}(\varepsilon), \varepsilon)$$

Balanced canard cycles: rescaling

We consider the rescaled layer variables U_i , defined by

$$u_i = u_i(\varepsilon) + \varepsilon U_i.$$

where $U_i \in [-K_i, K_i]$, with $K_i > 0$. The constants K_i can be chosen arbitrarily large, but must verify some compatibility conditions.

We introduce $V_i = \exp U_i$ and $\tilde{a}_i = a_i \exp(-\frac{I_i^0(\varepsilon)}{\varepsilon})$ and continue working in a rescaled layer with these new parameters \tilde{a}_i as exponentially rescaled parameters.

Balanced canard cycles: rescaled equations

The equations to solve take the form :

$$L_i : \tilde{a}_i = V_i^{\tau_i(\varepsilon)}(1 + \varepsilon\psi_{i,i}(V_i, \varepsilon)) - V_{i-1}^{\nu_{i-1}(\varepsilon)}(1 + \varepsilon\psi_{i,i-1}(V_{i-1}, \varepsilon)) \quad (15)$$

where the functions $\psi_{i,i}(V_i, \varepsilon)$ and $\psi_{i,i-1}(V_{i-1}, \varepsilon)$ are smooth in (V, Ω, \tilde{a}_i) , with $V = (V_1, \dots, V_n)$.

If ε is small enough one can solve the equation L_i as a \tilde{a}_i -parametrized map $\varphi_\varepsilon^i(V_{i-1}, \tilde{a}_i)$:

$$\varphi_{\varepsilon, \tilde{a}_i}^i(\cdot) = \varphi_\varepsilon^i(\cdot, \tilde{a}_i) : V_{i-1} \rightarrow V_i = (\tilde{a}_i + V_{i-1}^{\nu_{i-1}})^{\frac{1}{\tau_i}} + \varepsilon \theta_{i-1}(V_{i-1}, \tilde{a}_i, \varepsilon) \quad (16)$$

defined on an appropriate domain and with $\theta_{i-1}(V_{i-1}, \tilde{a}_i, \varepsilon)$ smooth in $(V_{i-1}, \Omega, \tilde{a}_i)$.

The composition of these maps, for $\varepsilon = 0$ is:

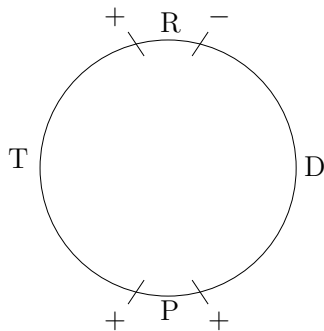
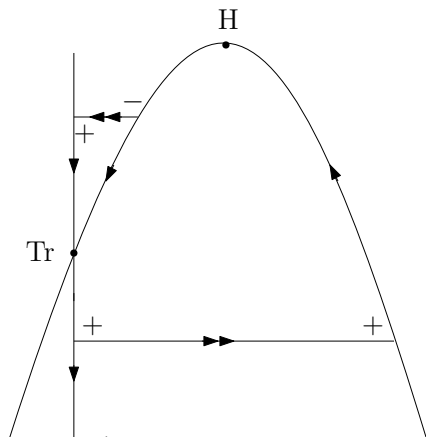
$$\varphi_{0, \tilde{a}} : V_n \rightarrow (\tilde{a}_n + (\tilde{a}_{n-1} + (\tilde{a}_{n-2} + \dots + (\tilde{a}_1 + V_n^{\nu_n})^{\frac{\nu_1}{\tau_1}}) \dots)^{\frac{\nu_{n-1}}{\tau_{n-1}}})^{\frac{1}{\tau_n}}.$$

1. Other breaking mechanisms , besides the Hopf- and jump breaking mechanisms, can also be accepted. An example is the transcritical breaking mechanism.

Extending the field of application

1. Other breaking mechanisms , besides the Hopf- and jump breaking mechanisms, can also be accepted. An example is the transcritical breaking mechanism.
2. To the attracting and repelling sequences can be added passages along some persistent slow curves with contact points.

Transcritical breaking mechanism





Thank you for your attention.

What are the most important results about slow-fast systems?

Question to ChatGPT

What are the most important results about slow-fast systems?

Answer of ChatGPT:

Slow-fast systems refer to dynamical systems that have both slow and fast variables. Some important results about slow-fast systems include:

1. The existence of multiple time scales in the system can lead to the formation of "canards," which are special solutions that behave differently from typical solutions.

2. Slow-fast systems can exhibit bifurcations, or qualitative changes in behavior, as a parameter in the system is varied. This can lead to the appearance of new types of behavior, such as periodic orbits or chaos.
3. The presence of a slow variable can lead to the emergence of a reduced or "averaged" system, which captures the essential dynamics of the full system but is simpler to analyze.
4. In some cases, slow-fast systems can exhibit "geometric singular perturbations," where the fast variables behave like a perturbation of the slow variables, leading to the development of geometric and topological methods to study such systems.
5. Slow-fast systems can exhibit a phenomenon known as "relaxation oscillations", which is characterized by slow, large amplitude oscillations that are superimposed on a fast, small amplitude oscillation.
6. In the context of neuroscience, slow-fast systems are used to model the dynamics of neural circuits, and have been used to explain phenomena such as the generation of rhythmic patterns of activity in the brain.