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Studying the dynamics of some Lagrangian systems by nonlocal constants of motion

This is a joint work with Gianluca Gorni
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Theorem in the variational case

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- ***For smooth scalar valued Lagrangian function*** $L(t,q,\dot{q})$, $t \in \mathbb{R}$, $q,\dot{q} \in \mathbb{R}^n$, ***Euler-Lagrange equation***

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- ***Theorem.*** *Let* $q(t)$ *be a sol. to the Euler-Lagrange eq. and let* $q_\lambda(t)$, $\lambda \in \mathbb{R}$, *be a smooth family of perturbed motions, such that* $q_0(t) \equiv q(t)$.
Then the following function of t *is constant*

$$\boxed{\partial_{\dot q} L\big(t, q(t), \dot q(t)\big) \cdot \partial_\lambda q_\lambda(t)\big|_{\lambda=0} - \int_{t_0}^{t} \frac{\partial}{\partial\lambda} L\big(s, q_\lambda(s), \dot q_\lambda(s)\big)\big|_{\lambda=0} ds\,.}$$

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- ***The constant of motion can be unuseful or trivial. In general it is nonlocal, i.e. its value at a time*** t ***depends not only on the state*** $(q(t), \dot{q}(t))$ ***at time*** t***, but also on the whole history between*** t_0 ***and*** t***.***

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  - Proof. Taking the time derivative we have

$$\frac{d}{dt}\Big(\partial_{\dot q} L\big(t, q(t), \dot q(t)\big) \cdot \partial_\lambda q_\lambda(t)\big|_{\lambda=0}\Big) - \frac{\partial}{\partial\lambda} L\big(t, q_\lambda(t), \dot q_\lambda(t)\big)\Big|_{\lambda=0} =$$
$$= \frac{d}{dt}\partial_{\dot q} L\big(t, q(t), \dot q(t)\big) \cdot \partial_\lambda q_\lambda(t)\big|_{\lambda=0} + \partial_{\dot q} L\big(t, q(t), \dot q(t)\big) \cdot \frac{d}{dt}\partial_\lambda q_\lambda(t)\big|_{\lambda=0} +$$
$$-\partial_q L\big(t, q(t), \dot q(t)\big) \cdot \partial_\lambda q_\lambda(t)\big|_{\lambda=0} - \partial_{\dot q} L\big(t, q(t), \dot q(t)\big) \cdot \partial_\lambda \dot q_\lambda(t)\big|_{\lambda=0} = 0$$

since the sum of the red terms vanishes by means of the Euler-Lagrange equation and the blu terms are equal by reversing the derivation order. q.e.d.

Angular momentum

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- The perturbed motions $q_\lambda(t)$ were originally inspired by the mechanism that Noether's theorem uses to deduce conservation laws whenever the Lagrangian function L enjoys certain invariance properties. A simple classical example, particle of mass m in the plane that is driven by a central force field

$$L(t, q, \dot{q}) := \frac{1}{2} m |\dot{q}|^2 - U\big(t, |q|\big), \quad q = (q_1, q_2) \in \mathbb{R}^2.$$

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To exploit the rotational symmetry of L it is natural to take the rotated family

$$q_\lambda(t) := \begin{pmatrix} \cos\lambda & -\sin\lambda \\ \sin\lambda & \cos\lambda \end{pmatrix} \begin{pmatrix} q_1(t) \\ q_2(t) \end{pmatrix}, \qquad \partial_\lambda q_\lambda(t)\big|_{\lambda=0} = \big(-q_2(t), q_1(t)\big).$$

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It is clear that $L(t, q_\lambda(t), \dot{q}_\lambda(t))$ does not depend on λ. The constant of motion associated to the rotation family reduces to Noether's theorem and gives the angular momentum as constant of motion:

$$\partial_{\dot{q}} L \cdot \partial_\lambda q_\lambda\big|_{\lambda=0} = m\dot{q} \cdot (-q_2, q_1) = m(q_1\dot{q}_2 - q_2\dot{q}_1).$$

Energy conservation

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- Next, we revisit another classical example, from our point of view. For time indep. $L(t,q,\dot{q}) = \mathcal{L}(q,\dot{q})$, $q \in \mathbb{R}^n$, and the time-shift family $q_\lambda(t) = q(t+\lambda)$:

$$\partial_\lambda L\big(t, q_\lambda(t), \dot{q}_\lambda(t)\big)\Big|_{\lambda=0} = \partial_q \mathcal{L} \cdot \dot{q}(t) + \partial_{\dot{q}} \mathcal{L} \cdot \ddot{q}(t) = \frac{d}{dt}\mathcal{L}(q(t), \dot{q}(t)).$$

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The constant of motion is

$$\partial_{\dot{q}} \mathcal{L} \cdot \dot{q}(t) - \int_{t_0}^{t} \frac{d}{ds}\mathcal{L}(q(s), \dot{q}(s))ds =$$

$$= \partial_{\dot{q}} \mathcal{L}\left(q(t), \dot{q}(t)\right) \cdot \dot{q}(t) - \mathcal{L}\left(q(t), \dot{q}(t)\right) + \mathcal{L}\left(q(t_0), \dot{q}(t_0)\right) =$$

$$= E\left(q(t), \dot{q}(t)\right) + \mathcal{L}\left(q(t_0), \dot{q}(t_0)\right).$$

Energy $E(q, \dot{q}) = \partial_{\dot{q}} \mathcal{L}\left(q, \dot{q}\right) \cdot \dot{q} - \mathcal{L}\left(q, \dot{q}\right)$ up to a trivial additive const.

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Energy $E(q,\dot q) = \partial_{\dot q}\mathcal{L}\left(q,\dot q\right)\cdot \dot q - \mathcal{L}\left(q,\dot q\right)$ up to a trivial additive const.

For instance $\mathcal{L}(q,\dot q) = \frac{1}{2}m|\dot q|^2 - U(q)$ gives $E(q,\dot q) = \frac{1}{2}m|\dot q|^2 + U(q)$

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Well known that: if $q(t)$ is solution to Euler-Lagrange eq. $\ddot q = -\nabla U(q)$ then

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$$m\dot q(t)\cdot q(t) + t\left(\tfrac{\alpha}{2}-1\right)E(q(t),\dot q(t)) - \left(\tfrac{\alpha}{2}+1\right)\int_{t_0}^t \mathcal{L}(q(s),\dot q(s))ds.$$

with $E := \frac{1}{2}m|\dot q|^2 + U(q)$, the energy conserved too.

In the special case $\alpha = -2$ we get a time-dependent first integral in the usual sense

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- Examples central $U(q) = -k/|q|^2$ and Calogero's

$$U(q_1, \dots, q_n) = g^2 \sum_{1 \le j < k \le n} (q_j - q_k)^{-2},$$

for $q_j \in \mathbb{R}$, $q_j \neq q_k$ when $j \neq k$.

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- Notice that for $\alpha = -2$ the integrand in the formula of the theorem does not vanish.

Homogeneous potentials

- Take the antiderivative in time of $0 = mq(t) \cdot \dot{q}(t) - 2tE - F$ and obtain one more time-dependent constant of motion

$$F_1 = \frac{1}{2} m |q(t)|^2 - t^2 E - tF \, .$$

We can also solve for $|q(t)|$:

$$|q(t)| = \tfrac{2}{m}\sqrt{t^2 E + tF + F_1} \, .$$

This formula gives the time-dependence of distance from the origin even though we don't know the shape of the orbit. So we generalized a formula known in the central case.

- ***Consider*** $k > 0$, ***a smooth*** $U : \mathbb{R}^n \to \mathbb{R}$,

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  - Energy first integral for $k = 0$

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$$\dot{E} = m\dot{q} \cdot \frac{1}{m}\Big(-k\dot{q} - \nabla U(q)\Big) + \nabla U(q) \cdot \dot{q} = -k\left|\dot{q}\right|^2 \leq 0.$$

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- ***In the sequel*** U ***bounded from below, say*** $U \geq 0$. ***For a solution to the o.d.e.*** $\dot{q}(t)$ ***is bounded in the future:***

$$\tfrac{1}{2}m\big|\dot{q}(t)\big|^2 \leq \tfrac{1}{2}m\big|\dot{q}(t)\big|^2 + U(q(t)) \leq \tfrac{1}{2}m\big|\dot{q}(t_0)\big|^2 + U\big(q(t_0)\big), \qquad t \geq t_0,$$

so $q(t)$ is bounded for bounded t and we get global existence in the future. What about the past?

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- Then

$$\frac{\partial}{\partial\lambda}L\big(t,q_\lambda(t),\dot q_\lambda(t)\big)\Big|_{\lambda=0} =$$

$$= \frac{d}{dt}\Big(-2e^{(a+\frac{k}{m})t}U\left(q\left(t\right)\right)\Big) + e^{(a+\frac{k}{m})t}\Big(\big(a-\frac{k}{m}\big)m\left|\dot q(t)\right|^2 + 2\big(a+\frac{k}{m}\big)U\left(q\left(t\right)\right)\Big).$$

(we eliminated $\ddot q(t)$ using the diff. eq.).

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$$= e^{2kt/m}\Big(m|\dot q(t)|^2 + 2U\big(q(t)\big)\Big) + 4\,\frac{k}{m}\int_{t}^{t_0} e^{2ks/m}U\big(q(s)\big)ds.$$

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Since $U \geq 0$, the blue integral decreases for $t \leq t_0$ and $t \mapsto e^{2kt/m}\left(m\left|\dot{q}(t)\right|^2 + 2U(q(t))\right)$ increases with t for all $t \leq t_0$.

Finally, we have the estimate for $t \leq t_0$***:***

$$\boxed{m\left|\dot{q}(t)\right|^2 \leq e^{2k(t_0-t)/m}\left(m\left|\dot{q}(t_0)\right|^2 + 2U\left(q(t_0)\right)\right)}$$

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In a bounded interval $(t_1, t_0]$ ***the velocity*** $\dot{q}(t)$ ***is bounded, so also*** $q(t)$ ***and we have global existence of solutions.***

Conservative Maxwell-Bloch

- *Maxwell-Bloch eq. model laser dynamics (Arecchi and Bonifacio 1965)*

$$\dot{x}_1 = y_1, \qquad \dot{y}_1 = x_1 z,$$
$$\dot{x}_2 = y_2, \qquad \dot{y}_2 = x_2 z,$$
$$\dot{z} = -(x_1 y_1 + x_2 y_2).$$

Physical meaning: $(x_1 + i x_2)/2$ *complex* ***amplitude of the electric field;*** $(y_1 + i y_2)/2$ ***polarization of the atomic medium;*** z *real* ***population inversion.***

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By

$$q_1 = x_1, \;\; q_2 = x_2, \;\; \dot{q}_3 = z,$$

Maxwell-Bloch 5-dim. is embedded into the 6-dim.

Conservative Maxwell-Bloch

variational ***Lagrangian system***

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- ***3 known first integrals***

$$E = \frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2), \qquad B = \dot{q}_3 + \frac{1}{2}(q_1^2 + q_2^2), \qquad J = q_1\dot{q}_2 - q_2\dot{q}_1.$$

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Invariance under t ***transl. gives*** E, ***under*** q_3 ***transl. gives*** B, ***and under rotations in the*** (q_1, q_2) ***plane gives*** J.

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- ***Non-uniform scaling family***

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    The choice $a = -2$ simplifies the formula:

    $$\frac{\partial}{\partial\lambda} L\left(t, q_\lambda(t), \dot{q}_\lambda(t)\right)\Big|_{\lambda=0} = \dot{q}_1(t)^2 + \dot{q}_2(t)^2 - 2z(t)^2 = 2E - 3\dot{q}_3(t)^2.$$

Using B, the associated constant of motion is

$$-\ddot{q}_3(t) - 2Bq_3(t) - 2Et + 3\int_{t_0}^{t} \dot{q}_3(s)^2 ds$$

with only q_3. By derivation we get a diff. eq. of order 2 for $z(t) = \dot{q}_3(t)$

$\ddot{z}(t) + 2Bz(t) + 2E - 3z^2 = 0$, the so called **fish**.

Its energy constant of motion

$\frac{1}{2}\dot{z}^2 + 2Ez + Bz^2 - z^3 = 2BE - J^2/2$

is solved for z by quadratures. Using Euler-Lagrange eq. we have

$$K = 2BE - J^2/2$$

Given initial data $q_1(0)$, $\dot{q}_1(0)$, $q_2(0)$, $\dot{q}_2(0)$, $\dot{q}_3(0)$ we calculate E, B, J, and these determine the particular level set to which $(z(t), \dot{z}(t)) = (\dot{q}_3(t), \ddot{q}_3(t))$ belongs for all t.

Conservative Maxwell-Bloch

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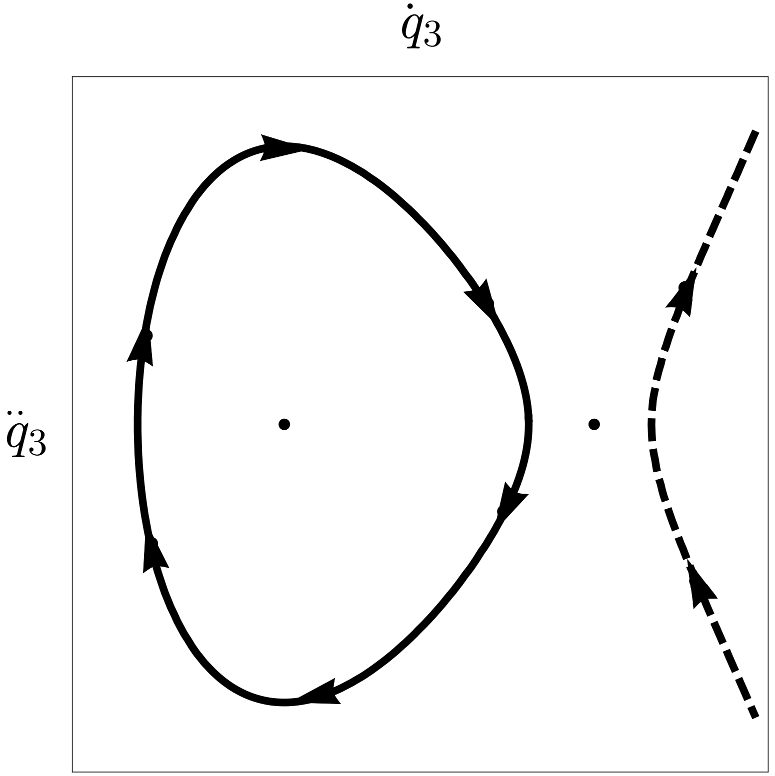
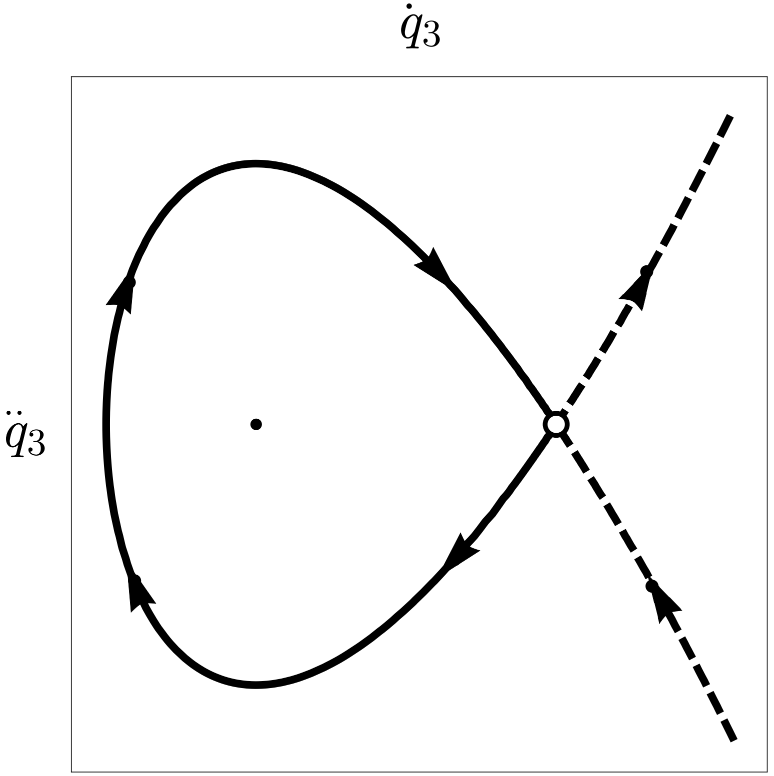
Parts are inaccessible, outside the stripe $|z| \leq (2E)^{1/2}$, a region forbitten by energy conservation



a. Generic case



b. Homoclinic case

Conservative Maxwell-Bloch

We do not only separate z since eliminating $\dot{q}_3$ by the first 2 Lagrange eq.

$$\ddot{q}_1 = q_1\dot{q}_3, \quad \ddot{q}_2 = q_2\dot{q}_3,$$

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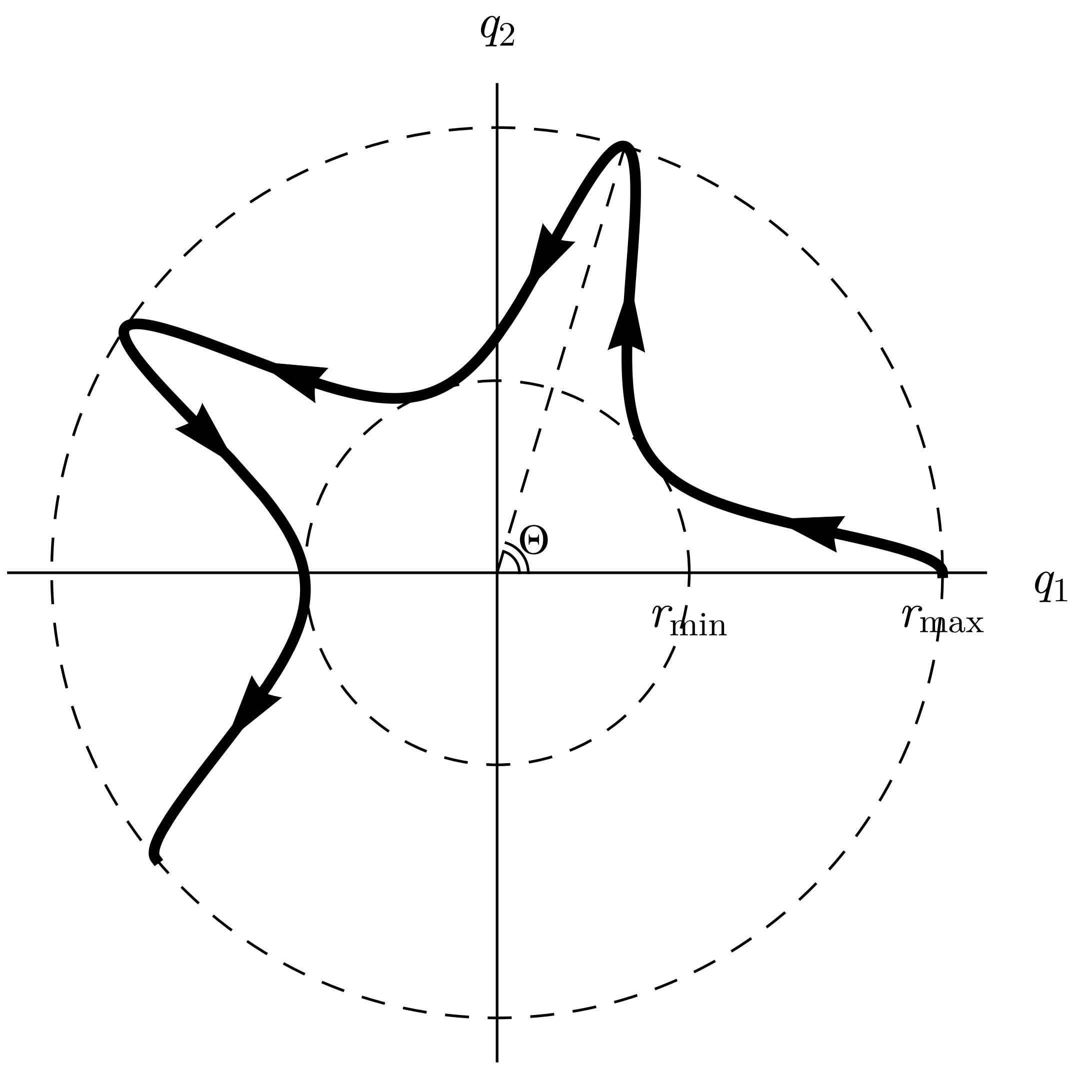
we have the separated central force dynamics

$$\ddot{\vec{r}} = -\frac{1}{2}\left(4B - r^2\right)\,\vec{r},$$

$\vec{r} = (q_1, q_2)$, $r = |\vec{r}|$. Here is an orbit of (q_1, q_2):

Conservative Maxwell-Bloch



Theorem for nonvariational eq.

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- *For smooth $L(t, q, \dot{q})$, $Q(t, q, \dot{q})$, $t \in \mathbb{R}$, $q, \dot{q} \in \mathbb{R}^n$, nonvariational* ***Lagrange equation***

$$\boxed{\frac{d}{dt}\partial_{\dot{q}} L\big(t, q(t), \dot{q}(t)\big) - \partial_q L\big(t, q(t), \dot{q}(t)\big) = Q\big(t, q(t), \dot{q}(t)\big)}$$

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- ***Theorem.*** ***Let*** $q_\lambda(t)$ ***be a smooth family of perturbed motions of the solution*** $q(t)$ ***to the Lagrange equation. Then the following func. of*** t ***is constant***

$$\boxed{\partial_{\dot q}L\big(t,q(t),\dot q(t)\big)\cdot\partial_\lambda q_\lambda(t)\big|_{\lambda=0}-\int_{t_0}^{t}\frac{\partial}{\partial\lambda}L\big(s,q_\lambda(s),\dot q_\lambda(s)\big)\big|_{\lambda=0}ds}$$

$$\boxed{-\int_{t_0}^{t}Q\big(s,q(s),\dot q(s)\big)\cdot\partial_\lambda q_\lambda(s)\big|_{\lambda=0}ds\,.}$$

Explosion in the past for hydraulic resist.

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- **Hydraulic resistance in a bounded potential field:**

$$m\ddot{q} = -k|\dot{q}|\dot{q} - \nabla U\big(q(t)\big), \qquad q \in \mathbb{R}^n,$$

$m, k > 0$ ***parameters, and the smooth potential is bounded:***

$$0 \le U(q) \le U_{\sup} < +\infty \qquad \forall q \in \mathbb{R}^n.$$

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Scalar example $m\ddot{q} = -k|\dot{q}|\dot{q}$, $q \in \mathbb{R}$, ***nonconstant solutions*** $q(t) = \frac{m}{k}\log(\omega(t - t_0))$, ***parameters*** $\omega > 0$, $t_0 \in \mathbb{R}$, ***defined for*** $t > t_0$, ***non-global in the past.***

Explosion in the past for hydraulic resist.

- Lagrange nonvariational formulation with

$$L(t, q, \dot{q}) := \frac{1}{2} m |\dot{q}|^2 - U(q), \qquad Q(t, q, \dot{q}) := -k \dot{q} |\dot{q}|.$$

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Family $q_\lambda(t) := q(t+\lambda e^{-at})$, with $a > 0$ parameter, from the theorem, and integration by parts we get

$$\frac{m}{2}e^{-at}|\dot q(t)|^2 + e^{-at}U\big(q(t)\big) - e^{-at_0}U\big(q(t_0)\big) + \\ + \frac{1}{2}\int_{t_0}^{t} e^{-as}\Big(2k|\dot q(s)|^3 + am|\dot q(s)|^2 + 2aU\big(q(s)\big)\Big)ds \equiv \frac{m}{2}e^{-at_0}|\dot q(t_0)|^2.$$

This fact, and $0 \le U(q) \le U_{\sup} < +\infty$, give the following inequality

Explosion in the past for hydraulic resist.

- for $t < t_0$:

$$\frac{m}{2}e^{-at_0}|\dot{q}(t_0)|^2 \leq \frac{m}{2}e^{-at}|\dot{q}(t)|^2 + e^{-at}U_{\text{sup}} + \frac{1}{2}\int_{t_0}^{t} e^{-as}\left(2k|\dot{q}(s)|^3 + am|\dot{q}(s)|^2\right)ds$$

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An a priori estimate, for all values of the parameter $a > 0$, gives

Conclusion: All the solutions such that the initial kinetic energy satisfies

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We also applied the nonvariational theorem to the (more complicated) dissipative Maxwell-Bloch eq.

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Thank you
for your attention