The number of limit cycles from a cubic center by the Melnikov function of any order

P. Yang and J. Yu

School of Mathematical Sciences, Shanghai Jiaotong University, China, AQTDE-2019

Introduction

For the weaken Hilbert's 16th problem, many researchers studied the perturbations of the integrable non-Hamiltonian system:

$$\begin{cases} \dot{x} = yh(x,\, y) + \epsilon p(x,\, y), \\ \dot{y} = -xh(x,\, y) + \epsilon q(x,\, y), \end{cases} \tag{0.1}$$

where $h(x, y)$ is a polynomial in x and y with $h(0, 0) \neq 0$, $p(x, y)$ and $q(x, y)$ are polynomials of degree n. We remark σ the maximum number of limit cycles bifurcating from the periodic orbits of the unperturbed system (0.1).

Giacomini et al. [3] obtained $\sigma = [\frac{n-1}{2}]$ for (0.1) $h(x, y) \equiv 1$ up to first order in ϵ. Llibre et al. [5, 6] studied system (0.1) with $h(x) = 1+x$ and obtained n limit cycles by using the averaging theory of first order, and obtained at most $2n-1$ limit cycles by using the averaging theory of second order. Xiang and Han [7] studied system (0.1) with $h(x) = (1-x)^m$ and they obtained the upper bound of the number of limit cycles is $n+m-1$ by using the Melnikov function of first order, and when $m = 1$, the upper bound is reached. Gasull et al. [4] studied system (0.1) with $h(y) = (1-y)^m$ and obtained $\sigma \leq [\frac{m+n}{2}]-1$ when $n < m-1$ and $\sigma \leq n$ when $n \geq m-1$ by using the Melnikov function of first order.

There are many works about (0.1), eg. Buică Coll and libre et al. studied the case with $h(x, y) = y^2+ax^2+bx+c$ or $(x+a)(y+b)(x+c)$ etc. by using the averaging theory or Melnikov function of first order. Because of the complexity of calculating and studying the Melnikov function of high order, there are few papers studying the perturbations of integrable systems by the Melnikov function of any high orders. Francoise [2] provided a algorithm to calculate the Melnikov function of high order, and Iliev gave a method for calculating the Melnikov function of second order with elliptic or hyperellipitic Hamiltonians under polynomial perturbations, and it also gave an explicit expression of $M_2(h)$. Recently Gavrilov and Iliev studied a cubic system in the plane which is a polynomial Hamiltonian system of degree 4 under cubic polynomial perturbations by using the Melnikov function of high order.

At the same time, there are also few papers that studied system (0.1) by using the Melnikov function of high order. Buică et al. [1] studied system (0.1) with $h(x, y) = 1+x$ under quadratic perturbations and they proved the upper bound of the number of limit cycles is 3 by the Melnikov function of the first three order.

1. Main results

In this paper, we consider system (0.1) with $h(x, y) = (1+ax+by)^2$ with $b \neq 0$. Doing some linear changes of variables, system could be written as,

$$\begin{cases} \dot{x} = y(1+x)^2 - \epsilon P(x,\, y), \\ \dot{y} = -x(1+x)^2 + \epsilon Q(x,\, y), \end{cases} \tag{0.2}$$

where $\epsilon > 0$ is a small parameter and

$$P(x,\, y) = a_{00} + a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + a_{02}y^2,$$

$$Q(x,\, y) = b_{00} + b_{10}x + b_{01}y + b_{20}x^2 + b_{11}xy + b_{02}y^2.$$

The unperturbed system has the first integral $H(x, y) = \frac{1}{2}(x^2+y^2) = h$ with $h_0 \in (0, 1/2)$ in the region $0 < x^2+y^2 < 1$. If we fix a transversal segment to the flow in (0.2) and use the energy level h to parameterise it, we can get the corresponding displacement function,

$$d(h,\, \epsilon) = \epsilon M_1(h) + \epsilon^2 M_2(h) + \epsilon^3 M_3(h) + \cdots,\, h \in (0, \frac{1}{2}), \tag{0.3}$$

where $M_k(h)$ is called the Melnikov function of $k-$th order of system (0.2). It's well known that the Melnikov function of first order of system(0.2) has the form

$$M_1(h) = \oint_{\Gamma(h)} \frac{Q(x,\, y)}{(1+x)^2}dx + \frac{P(x,\, y)}{(1+x)^2}dy,$$

Theorem 0.1 *For $k \geq 1$, let $M_k(h)$ be the Melnikov functions associated to system (0.2). Then $M_1(h)$ has at most 2 zeros, taking into account their multiplicities. If $M_1(h) \equiv 0$, then M_2 has also at most 2 zeros, taking into account their multiplicities. If $M_1(h) \equiv M_2(h) \equiv 0$, then M_3 has also at most 2 zeros, taking into account their multiplicities. If $M_1(h) \equiv M_2(h) \equiv M_3(h) \equiv 0$, then M_4 has at most 3 zeros, taking into account their multiplicities. If $M_1(h) \equiv M_2(h) \equiv M_3(h) \equiv M_4(h) \equiv 0$, then $M_k(h) \equiv 0$ for all $k \geq 5$, namely, the perturbation is integrable. Therefore, taking into account the expansion of the displacement map (0.3) of any order in ϵ, the system (0.2) has at most three limit cycles which bifurcate from the period annulus of the unperturbed system and this upper bound of the number of limit cycles is reached.*

Theorem 0.2 *The perturbation (0.2) is integrable if and only if at least one set of the following conditions is satisfied:*
1) $a_{00} = a_{02} = a_{10} = a_{20} = b_{01} = b_{11} = 0$;
2) $a_{00} = a_{10} = -b_{01}, \quad a_{01} = a_{11} = -2b_{02}, \quad a_{02} = a_{20} = b_{11} = 0$;
3) $a_{00} = a_{20} - b_{01}, \quad a_{01} = a_{11}, \quad a_{10} = 2a_{20} - b_{01}, \quad b_{02} = -\frac{1}{2}a_{11}, \quad a_{02} = b_{11} = 0$;
4) $a_{00} = a_{10} = a_{20} = b_{00} = b_{01} = b_{02} = 0, \quad a_{02} = b_{11}, \quad b_{10} = a_{01}, \quad b_{20} = a_{11}$.

Remark 0.1 *Consider the system (0.1) with $h(x) = (1+x)^m$, Buică et al. [1] studied case $m = 1$ and the perturbation polynomials of degree $n = 2$ by using the Melnikov functions up to three order. And they obtained three limit cycles by the Melnikov function of third order. They stopped at the Melnikov function of fourth order because of its complexity. With regard to the results of the two articles, we can clearly observe the differences between them in the following*

	$M_1(h)$	$M_2(h)$	$M_3(h)$	$M_4(h)$	$M_5(h)$
$(1+x)$	2	2	3	–	–
$(1+x)^2$	2	2	2	3	0

Remark 0.2 *Gasull et al. [4] considered system (0.1) with $h(x, y) = (1+x)^m$ under perturbations of polynomials in degree n and obtained the upper bound of the number of limit cycles is n when $n \geq m-1$ by the Melnikov function of first order. That means the value m in $(1+x)^m$ has no influence on the number of limit cycles by the Melnikov function of first order when $n \geq m-1$. While by using the Melnikov function of high order, the upper bound of the number of limit cycles is related to the degree m. For $m = 2$ and $n = 2$, in the present paper we find the upper bound of the number of limit cycles is 3 by the Melnikov function of the 4th high order.*

2. Preliminaries

The algorithm of calculating $M_k(h)$ is given by Francoise [2] and Iliev.

Lemma 0.1 *Assume Γ_h are the period annulus defined by $H(x, y) = h$, the polynomial function $H(x, y)$ and the $1-$form ω satisfy $\oint_{\Gamma_h}\omega \equiv 0$, if and only if there are two analytic functions $q(x, y)$ and $Q(x, y)$ in a neighborhood of Γ_h such that*

$$\omega = qdH + dQ.$$

□

Rewrite ω as

$$\omega = \frac{Q(x,\, y)}{(1+x)^2}dx + \frac{P(x,\, y)}{(1+x)^2}dy,$$

then the system (0.2) can be written in a Pfaffian form $dH = \epsilon\omega$. Following Francoise's algorithm [2] and Lemma 0.1, we give the following lemma:

Lemma 0.2 *The Melnikov function of first order is given by $M_1(h) = \oint_{\Gamma_h}\omega$, rewrite ω in the form $\omega = \bar{q}_0 dH + d\bar{Q}_0 + N_0$, then $M_1(h) = \oint_{H=h}\omega = \oint_{H=h} N_0$;*
If $M_1(h) \equiv 0$, that is $\oint_{\Gamma_h} N_0 \equiv 0$, according to Lemma 0.1, we have $N_0 = \tilde{q}_0 dH + d\tilde{Q}_0$. Denote by $q_0 = \bar{q}_0 + \tilde{q}_0$, $Q_0 = \bar{Q}_0 + \tilde{Q}_0$. In this way, $\omega = q_0 dH + dQ_0$, write $q_0\omega$ in the decomposed form $q_0\omega = \bar{q}_1 dH + d\bar{Q}_1 + N_1$, then $M_2(h) = \oint_{\Gamma_h} q_0\omega = \oint_{\Gamma_h} N_1$;
if $M_1(h) = M_2(h) = \cdots = M_{i-1}(h) \equiv 0$, according to Lemma 0.1, $q_j\omega$, $j \leq i-2$ can be written as $q_j\omega = q_{j+1}dH + dQ_{j+1}$, and when $j = -1$, we assume $q_{-1} = 1$. Then we have $M_i(h) = \oint_{\Gamma_h} q_{i-2}\omega = \oint_{\Gamma_h} N_{i-1}$, where $q_{i-2}\omega = \bar{q}_{i-1}dH + d\bar{Q}_{i-1} + N_{i-1}$. □

Denote by

$$\omega_{ij}^k = \frac{x^i y^j}{(1+x)^k}dx, \quad \delta_{ij}^k = \frac{x^i y^j}{(1+x)^k}dy, \quad 0 \leq i+j \leq k, \quad k \geq 1$$

and

$$J_k(h) = \oint_{H(x,\, y)=h} \delta_{00}^k, \qquad k \geq 1. \tag{0.4}$$

Then we claim that $J_k(h)$ are the generators of $M_i(h)$. In the following we decompose ω_{ij}^k, δ_{ij}^k into the combination of δ_{00}^k, $h_{ij}^k(x, H)dH$ and $dQ_{ij}^k(x, H)$, where dQ_{ij}^k are the forms of perfect differential of functions in variable of x and H.

Lemma 0.3 *All the 1-forms ω_{ij} and δ_{ij} can be expressed as follows:*
(i) For $0 \leq i+j \leq 2$ with $k = 2$, we have

$$\omega_{00}^2 = d(-\frac{1}{1+x}), \qquad \omega_{01}^2 = d(-\frac{y}{1+x}) + \delta_{00}^1,$$
$$\omega_{02}^2 = \frac{2}{1+x}dH + d(-\frac{2H}{1+x} - x + \frac{1}{1+x} + 2ln(1+x)),$$
$$\omega_{11}^2 = \frac{y}{(1+x)^2}dH - 2H\delta_{00}^2 + \delta_{00}^2 - 2\delta_{00}^1 + dy,$$
$$\omega_{10}^2 = d(ln(1+x) - \frac{x}{1+x}), \qquad \omega_{20}^2 = d(x - \frac{1}{1+x} - 2ln(1+x)),$$
$$\delta_{01}^2 = \frac{1}{(1+x)^2}dH - d(ln(1+x) - \frac{x}{1+x}), \qquad \delta_{10}^2 = \delta_{00}^1 - \delta_{00}^2,$$
$$\delta_{11}^2 = \frac{x}{(1+x)^2}dH - d(x - \frac{1}{1+x} - 2ln(1+x)),$$
$$\delta_{20}^2 = \delta_{00}^2 - 2\delta_{00}^1 + dy, \qquad \delta_{02}^2 = 2H\delta_{00}^2 - \delta_{00}^2 + 2\delta_{00}^1 - dy.$$

(ii) For $0 \leq i+j < k$ with $k > 2$, we have
if j is even,

$$\omega_{ij}^k = h_{ij}^k(x,\, H)dH + d(Q_{ij}^k(x,\, H)),$$

$$\delta_{ij}^k = \sum_{r=0}^{j/2}(2H)^r\mathfrak{L}(\delta_{00}^{k-i-j+2r}, \cdots, \delta_{00}^k),$$

if j is odd, $\cdots$
where h_{ij}^k and Q_{ij}^k are functions of x and H, and $\mathfrak{L}(\delta_{00}^1, \cdots, \delta_{00}^k)$ is the linear combination of $\delta_{00}^l (l = 1, 2, \cdots, k)$ in $\mathbb{Z}$.

3. Outline of Proof

With the above notations, ω can be written as the following form,

$$\omega = a_{00}\delta_{00}^2 + a_{10}\delta_{10}^2 + a_{01}\delta_{01}^2 + a_{20}\delta_{20}^2 + a_{11}\delta_{11}^2 + a_{02}\delta_{02}^2 + b_{00}\omega_{00}^2 + b_{10}\omega_{10}^2 + b_{01}\omega_{01}^2 + b_{20}\omega_{20}^2 + b_{11}\omega_{11}^2 + b_{02}\omega_{02}^2.$$

The first Melnikov function is

$$M_1(h) = \oint_{H(x,\, y)} N_0 = A_1 J_1(h) + A_2 J_2(h) + A_3 h J_2(h),$$

where J_1, J_2, hJ_2 are independent,

$$A_1 = b_{01} - 2b_{11} + a_{10} - 2a_{20} + 2a_{02}$$
$$A_2 = b_{11} + a_{00} - a_{10} + a_{20} - a_{02}$$
$$A_3 = 2a_{02} - 2b_{11}.$$

Theorem 0.3 *$M_1(h)$ has at most two zeros, namely, system (0.2) has at most two limit cycles by the Melnikov function of first order, which can be reached.*

If $M_1(h) \equiv 0$, then $A_1 = A_2 = A_3 = 0$, that is,

$$a_{00} = a_{20} - b_{01}, \qquad a_{10} = 2a_{20} - b_{01}, \qquad a_{02} = b_{11}.$$

From Lemma 0.2, we have

$$M_2(h) = \oint_{H(x,\, y)=h} N_1 = B_1 J_1(h) + B_2 J_2(h) + B_3 J_3(h) + B_4 h J_3(h),$$

Theorem 0.4 *If $M_1(h) \equiv 0$, $M_2(h)$ has at most two zeros, namely, system (0.2) has at most two limit cycles by the Melnikov function of second order, which can be reached.*

If $M_1(h) = M_2(h) \equiv 0$, we have

$$-B_1 = B_1 - B_2 + \frac{3}{4}B_4 = B_2 - \frac{3}{2}B_3 - \frac{3}{2}B_4 = \frac{3}{2}B_3 + \frac{3}{4}B_4 = 0.$$

that is, one of (1)-(3) in Theorem 0.2 and the following case A holds

$$b_{11} \neq 0, \quad a_{00} = a_{20} - b_{01}, \quad a_{02} = b_{11}, \quad a_{10} = 2a_{20} - b_{01},$$
$$b_{00}b_{11} = -2a_{01}a_{20} + 2a_{01}b_{01} + a_{11}a_{20} - a_{11}b_{01} - 2a_{20}b_{02} + 2b_{01}b_{02},$$
$$b_{10}b_{11} = -2a_{01}a_{20} + a_{01}b_{11} + a_{11}b_{01} - 4a_{20}b_{02} + 2b_{01}b_{02} + b_{02}b_{11},$$
$$b_{20}b_{11} = -a_{11}a_{20} + a_{11}b_{11} - 2a_{20}b_{02} + b_{02}b_{11}.$$

Theorem 0.5 *If one of cases (1)-(3) holds, then $M_k(h) \equiv 0, \quad k \geq 3$, that is the perturbation (0.2) is integrable.*

Theorem 0.6 *If $M_1(h) = M_2(h) \equiv 0$, and case A holds, $M_3(h)$ has at most two zeros, namely, system (0.2) has at most two limit cycles by the Melnikov function of third order, which can be reached.*

In fact, we obtain

$$M_3 = C_1 J_2(h) + C_2 J_3(h) + C_3 J_4(h) + C_4 h J_4(h) + C_5 J_5(h) + C_6 h J_5(h), \\ = -\frac{\pi}{(\sqrt{1-2h})^9} h(2h-1)(c_2 h^2 + c_1 h + c_0).$$

Now if $M_1(h) = M_2(h) = M_3(h) \equiv 0$, we get the case (4) in Theorem 0.2 and other three cases (4b), (4c) and (4d).

Theorem 0.7 *Assume that (4) in Theorem 0.2 holds, which implies $M_1(h) = M_2(h) = M_3(h) \equiv 0$, we have $M_k(h) \equiv 0$, $k \geq 4$, that is the perturbation (0.2) is integrable.*

We obtain the fourth Melnikov function M_4 from Lemma 0.3

$$D_1 J_3 + D_2 J_4 + D_3 J_5 + D_4 h J_5 + D_5 J_6 + D_6 h J_6 + D_7 J_7 + D_8 h J_7 + D_9 h^2 J_7.$$

Theorem 0.8 *Assume $M_1(h) = M_2(h) = M_3(h) \equiv 0$, then*
(i) if case 4b holds, $M_4(h)$ has at most one zero,
(ii) if case 4c holds, $M_4(h)$ has at most two zeros,
(iii) if case 4d holds, $M_4(h)$ has at most three zeros.
Moreover, the zeros of $M_4(h)$ can be reached, and $M_4(h) \not\equiv 0$.

In fact, for the case (4d), we have

$$M_4(h) = *(\bar{d}_3 h^3 + \bar{d}_2 h^2 + \bar{d}_1 h + \bar{d}_0).$$

Its coefficients are depend on two parameters. Its three zeros are obtained by asymptotic analysis.

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