A soft quasi-invariant of Fuchsian equations on the complex projective line

How many roots has a non-polynomial?

Sergei Yakovenko

WEIZMANN INSTITUTE OF SCIENCE
REHOVOT, ISRAEL

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Roots and equations (Advertisement. Can be skipped in a motivated audience)

- Functions (of one variable) often have roots. Information on these roots (including their finiteness, number and location) is often valuable. This value is often pecuniary and quite large. Ever heard of the Riemann ζ?
- Functions can be very different: polynomials are the simplest, algebraic functions are more complicated, next come solutions of differential equations, infinite sums/integrals and their analytic continuation (remember ζ!). Sky is the limit.
- Counting roots of polynomials is an easy task (locating them a different story). *Ditto* for algebraic functions $y = f(x)$ implicitly defined by polynomial equations $P(x, y) = 0$.
- The simplest class of functions for which counting roots is nontrivial, is solutions of *ordinary differential equations* $P(x, y, y', \ldots, y^{(n)}) = 0$. Among such equations, the simplest are linear homogeneous.
- Tremendously many so called special functions (selected for their role in various applications) are solutions of such equations.

Homogeneous Linear ODE's: description of the problem

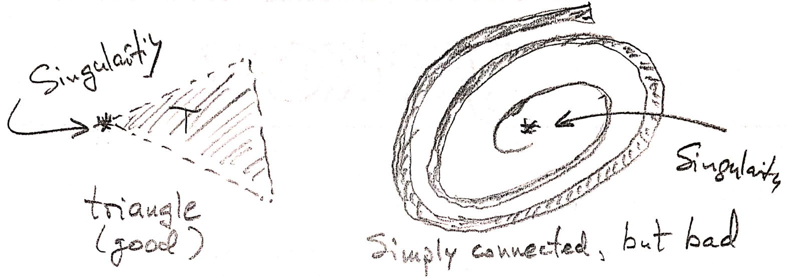
Equation: $A_0(t)\,y^{(n)} + A_1(t)\,y^{(n-1)} + \cdots + A_{n-1}(t)\,y' + A_n(t)\,y = 0, \quad t \in \mathbb{C}.$
Assumptions: $A_0, A_1, \dots, A_n \in \mathbb{C}[\,t\,]$ polynomials. Order $n \geqslant 0$.
Conventions: $\deg A_i \leqslant d, \quad A_0 \not\equiv 0, \quad \gcd\{A_0, A_1, \dots, A_n\} = 1.$

General facts.

- Roots of the leading coefficient $\Sigma = \{t : A_0(t) = 0\}$ are *singular points*. Infinity $t = \infty$ is included in Σ. Equations of order $\leqslant n$ with at most d singularities form a *complex projective space*.
- Solutions of the equation are analytic multivalued functions on $\mathbb{P} = \overline{\mathbb{C}}$, holomorphic outside Σ. All solutions except $y(t) \equiv 0$ have only isolated zeros, which may sometimes accumulate to Σ.
- Counting zeros of multivalued functions is tricky: one has to choose simply connected subsets of $\mathbb{P} \setminus \Sigma$ and indicate branches of solutions.

Definition. A *finite* number $N < +\infty$ is said to be a *global roots bound* for the HLODE, if any branch of any its solution has no more than N isolated roots in any *open* triangle $T \subseteq \mathbb{P} \setminus \Sigma$.

Why triangles?



Solutions are multivalued, the total number of roots on all branches may well be infinite even in the tame cases.

We have to count roots on a single branch. Spirals stretch between branches.

First examples

A. Equations with constant coefficients. WLOG $A_0 \equiv 1$ hence $\Sigma = \{+\infty\}$.

1. $n = 1$. Solutions are exponents, $N = 0$.
2. $n = 2$, $y'' - \lambda y = 0$. If $\lambda = 0$, $N = 1$, otherwise $N = +\infty$.

Example

If $\lambda = 1$, then $y = \mathrm{e}^z + \mathrm{e}^{-z}$ has roots $z_k = \pi \mathrm{i} k$, $k \in \mathbb{Z}$ on the imaginary axis.
If $\lambda = -1$, then $y = \mathrm{e}^{\mathrm{i}z} + \mathrm{e}^{-\mathrm{i}z} = 2\cos z$ has infinitely many real roots.

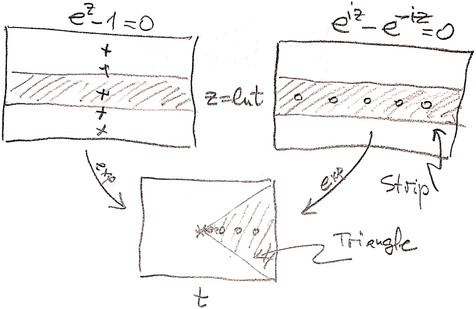
B. Euler equation. In the operator form using the Euler operator $\epsilon = t\frac{\mathrm{d}}{\mathrm{d}t}$, it is

$$Ly = 0, \qquad L = \epsilon^n + c_1\,\epsilon^{n-1} + \cdots + c_{n-1}\,\epsilon + c_n$$

with constant coefficients $1 = c_0, \quad c_1, \dots, c_n \in \mathbb{C}$ and $\Sigma = \{0, \infty\}$.
Substitution $z = \ln t$ brings EU to an equation with constant coefficients.

However, roots should be counted in simply connected triangles!

Euler equation: the complete answer



Roots of linear combinations of exponentials $\sum c_i e^{\lambda_i z}$ occur along "lines".

Preimage of a triangle is a horizontal strip.

Theorem (executive summary of many classical and recent results)

If all roots of the *characteristic polynomial* $\lambda^n + c_1\lambda^{n-1} + \cdots + c_{n-1}\lambda + c_n = 0$ are real, then $N < +\infty$ and can be explicitly estimated in terms of n and $R = \sum_{i=1}^n |c_i|$.

Otherwise $N = +\infty$.

General philosophy: operators as perturbations

- Operator of derivation $u \longmapsto \partial u = u'$ is an "unbounded" operator in any reasonable sense. Higher "powers" of ∂ strongly dominate lower powers.
- Multiplication by an analytic function is quite bounded, with a norm which is small if the function is small.
- The Euler operator $\epsilon = t\partial$ exhibits a trade-off near $t = 0$ (also unbounded, but much milder, since t is small). Should "behave better" in smaller neighborhoods of $t = 0$.

Heuristic principle

1. An operator of the form $L = \partial^n + \sum_{k=1}^n a_k(t)\partial^{n-k}$ with coefficients a_k holomorphic in a bounded domain $U \Subset \mathbb{C}$ is a perturbation of the leading term ∂^n.
2. An operator of the form $L = \epsilon^n + \sum_{k=1}^n b_k(t)\,\epsilon^{n-k}$ with coefficients b_k holomorphic in a bounded domain $U \ni 0$ is a perturbation of its Euler part $E = \epsilon^n + \sum c_k\,\epsilon^{n-k}$, $c_k = b_k(0) \in \mathbb{C}$.

Non-singular and Fuchsian equations

Classification of (non)-singularities

Let $U \Subset \mathbb{C}$ be a bounded domain containing the origin $0 \in \mathbb{C}$.

- An operator $L = \partial^n + \sum_{i=1}^n a_i(t)\partial^{n-i}$ is *nonsingular* in U, if all a_i are holomorphic in U.
- An operator $L = \epsilon^n + \sum_{i=1}^n b_i(t)\,\epsilon^{n-i}$ is *Fuchsian* in U (actually, at the only singularity at the origin $t = 0$), if all b_i are holomorphic in U.

The nonnegative number

$$R = \max_i \max_{t \in \overline{U}} |a_i(t)|, \qquad \text{resp.,} \qquad R' = \max_i \max_{t \in \overline{U}} |b_i(t) - b_i(0)|,$$

is the measure of proximity, the relative strength of non-leading terms.

What one can expect in general?

If the number of roots in U for "unperturbed" equation is finite, then it is finite also for the perturbed equation and explicit in terms of R, resp., R'.

Justification, nonsingular case

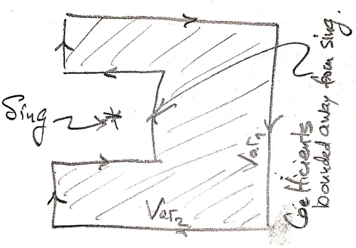
Theorem. Magnitude of perturbation is the only relevant factor:

Let $\gamma = [\alpha, \beta] \subseteq \mathbb{C}$ a finite line segment of length ℓ and

$$L = \partial^n + a_1(t)\,\partial^{n-1} + \cdots + a_{n-1}(t)\,\partial + a_n(t)$$

a monic differential operator with variable coefficients bounded on γ: $|a_k(t)| \leqslant R$ for all $t \in \gamma$ and $k = 1, \dots, n$.

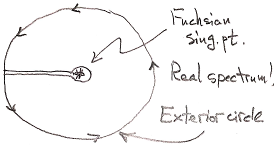
Then $|\mathrm{Var}_\gamma \mathrm{Arg}\, y(\cdot)| \leqslant (n-1) + O(1) \cdot n\ell R$. True also for circular arcs.



Opens the way to apply the *argument principle*, away from the singular points.

For non-singular operators the "magnitude of perturbation" determines the number of roots.

Justification, Fuchsian case



$$L = \partial^n + a_1(t)\,\partial^{n-1} + \cdots + a_{n-1}(t)\,\partial + a_n(t)$$

differential operator with coefficients bounded on the exterior circle $\varGamma = \{|t| = 1\}$ of radius 1:

$|a_k(t)| \leqslant R$ for all $t \in \varGamma$ and $k = 1, \ldots, n$.

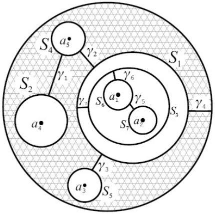
Theorem (M. Roitman - S.Y., 1998)

If L is Fuchsian at $t = 0$ (the center of the circle) and the spectrum is real, then N can be explicitly estimated in terms of n and R.

Bounds for slit disks of a different radius can be obtained by rescaling $t \longmapsto \mu t$ with a siutable $\mu > 0$.

Grand strategy

Let $L = A_0(t)\partial^n + A_1(t)\partial^{n-1} + \cdots + A_{n-1}(t)\partial + A_n(t)$ be an operator with *polynomial coefficients* $A_i \in \mathbb{C}[\,t\,]$ which has only Fuchsian singularities on $\mathbb{P}$, including $t = \infty$.



- Divide by A_0 and consider the rational fractions $a_k = A_k/A_0$. They are finite outside Σ and bounded along any curve which is distant enough from Σ.
- Draw a system of circular and polygonal slits which subdivides $\mathbb{P}$ into simply connected domains, of which only circles are allowed to contain singularities at their centers.
- Apply the argument principle and RY theorem.

The slits should be maximally distant from Σ (including the slit around infinity), i.e., the circles must be as large as possible. Yet the choice is constrained by the configuration (two close but distinct singularities should be separated).

Slope of linear equations: one number to rule them all

For $p = \sum_i c_i t^i \in \mathbb{C}[t]$ define the norm $\|p\| = \sum_i |c_i|$.
A polynomial $p(t)$, $\deg p = d$, admits lower bound away from its zero set $\Sigma = \{t : p(t) = 0\} \cup \{\infty\}$: $\quad p(t) \geqslant \|p\| \cdot 2^{-O(d)} \operatorname{dist}(t, \Sigma)^d$ (H. Cartan).

Definition

The *slope* of a differential operator $L = A_0 \partial^n + A_1 \partial^{n-1} + A_1 \partial + A_n$, $A_k \in \mathbb{C}[t]$, is the ratio $\angle L = \max_k \frac{\|A_k\|}{\|A_0\|}$.

Note. The *polynomial* coefficients $A_k \in \mathbb{C}[t]$ constrained by the assumption that $\gcd(A_0, \ldots, A_1) = 1$, are defined modulo a nonzero constant. Thus the slope is uniquely defined.

Claim.

If you know the slope $\angle L$, you can bound from above $\operatorname{Var}_\gamma \operatorname{Arg} y$ of any nonzero solution of $Ly = 0$ along any polyline γ in terms of $n = \operatorname{ord} L$, length $|\gamma|$, slope $\angle L$ and $\operatorname{dist}(\gamma, \Sigma)$.

Houston, we have a problem. . . ©

The slope $\angle L$ seems to be indeed a good measure of the "magnitude of perturbation": knowing it, one should be able to find N when it is finite. Unfortunately, $\angle L$ is not invariant by changes of the independent variable t. In particular, it may be changed by rescaling $t = cz$.

Example.

1. The slope of $y'' + y = 0$ is 1. After the change it becomes $\frac{\mathrm{d}^2}{\mathrm{d}z^2}y + c^2y = 0$, the slope is c^2, arbitrarily small or large as $c \to 0$ or $c \to \infty$.
2. The change doesn't affect $\epsilon = t\frac{\mathrm{d}}{\mathrm{d}t}$ hence preserves slope of Euler equations.
3. However, the shift $t = z + h$ changes the slope of the Euler equation.

More generally, linear equations with polynomial coefficients allow for Möbius transformations of the form $t \longmapsto z = \left(\dfrac{\alpha t + \beta}{\gamma t + \delta}\right)$, $\det\left(\begin{smallmatrix}\alpha & \beta \\ \gamma & \delta\end{smallmatrix}\right) \neq 0$.

What happens to the slope after such transformation? Can it grow to infinity?

After change of the independent variable (including the new derivation $\frac{\mathrm{d}}{\mathrm{d}z}$) one has to reduce the new rational coefficients to polynomials by cancelling the common denominator. The resulting mess is hard to describe or control.

Fuchsian equations revisited: moderate growth

Recall: $Ly = 0, \quad L = 1 \cdot \partial^n + \sum_1^n a_k(t)\partial^{n-k}, \quad a_k \in \mathbb{C}(t)$

Fuchsian condition at $t = \alpha$: a condition on *orders of poles* of a_k at α.

To verify it: expand L in powers of $\epsilon_\alpha = (t-\alpha)\partial$, get rid of the denominators and make sure that the leading term ϵ_α^n has non-vanishing at α coefficient.

Special provision for $\alpha = \infty$ (use $\epsilon_0 = -\epsilon_\infty$).

Equivalent description: moderate growth of solutions (L. Sauvage)

A singular point $t = \alpha$ is Fuchsian, *if and only if* all solutions of $Ly = 0$ grow moderately as $t \to \alpha$, i.e., no faster than a finite (negative) power $|t - \alpha|^{-r}$.

Don't forget to restrict on a triangle having α on the boundary, to deal with the multivaluedness!

Global Fuchsianity condition: all singular points on $\mathbb{P}$ are Fuchsian.

For a given number d of singular points on $\mathbb{P}$ and a given order n, globally Fuchsian equations form a *semialgebraic projective set of finite dimension*.

One-parametric holomorphic families of linear equations

Let $\lambda \in (\mathbb{C}^1, 0)$ be a complex (local) parameter. A *local family of equations* is of the form $L_\lambda y = 0$, where $L_\lambda = \sum_{k=0}^{n} p_k(t, \lambda)\partial^k$ with $p_k \in \mathbb{C}[\,t\,](\mathbb{C}, 0)$ holomorphically depending on λ.

Caveats.

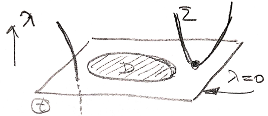
- It may well happen that $p_k(\cdot, 0) \equiv 0$ for all $k = 0, \dots, n$. However, we can always exclude this case, dividing by a proper power λ^s.
- The order of L_0 may drop down from $n = \operatorname{ord} L_\lambda$ (singular perturbation).
- Fuchsianity may also be destroyed in the limit as $\lambda \to 0$.

What can happen with the slope $\angle L_\lambda$ as $\lambda \to 0$ in an holomorphic family? Can it grow to infinity? In general, yes: e.g., $L_\lambda = \lambda^2\partial^2 + 1$.

Things change if solutions of the equations $L_\lambda y = 0$ exhibit moderate growth as $\lambda \to 0$ (a parametric analog of the Fuchsian condition).

Parametric moderate growth condition

For an OPF $\{L_\lambda\}$ one can find a disk $D \subseteq \mathbb{C}$ free of singularities of L_λ for all small $0 \neq \lambda \in (\mathbb{C}, 0)$ (recall that the case $a_0(\cdot, 0) \equiv 0$ is not excluded!).



Then one can find solutions $f_1(t, \lambda), \ldots, f_n(t, \lambda)$ of the equation $L_\lambda y = 0$ such that:

1. each $f_i(t, \lambda)$ is holomorphic in D in $\lambda \neq 0$,
2. $f_1(\cdot, \lambda), \ldots, f_n(\cdot, \lambda)$ are linear independent over $\mathbb{C}$ for $\lambda \neq 0$.

These functions may grow as $\lambda \to 0$, but even if they have uniform limits, these limits can become linear dependent.

Definition. Moderate parametric growth of solutions

OPF $\{L_\lambda\}$ exhibits moderate parametric growth, if one can choose D and $f_1, \ldots, f_n$ as before, such that for some finite $s \in \mathbb{N}$ the functions $\lambda^s f_i$ remain bounded as $\lambda \to 0$ (hence extend holomorphically at $\lambda = 0$).

In plain words, $f_i(\cdot, \lambda)$ grow no faster than polynomially in λ^{-1} as $\lambda \to 0$ in D.

Grigoriev theorem

Theorem (A. Grigoriev, Ph. D. thesis from 2001; S.Y., 2006)

If a OPF $\{L_\lambda\}$, $\lambda \in (\mathbb{C}, 0)$ exhibits moderate parametric growth of solutions, then the slope $\angle L_\lambda$ remains bounded as $\lambda \to 0$.

- Boundedness of $\angle L_\lambda$ means that under the moderate growth assumption the order of L_λ is continuous at $\lambda = 0$ and not just semicontinous (does not drop down as it could).
- In other words, L_λ considered as a *perturbation* of L_0, is nonsingular.
- Almost literal analog of the Fuchsian condition, except that it occurs parameter-wise and not t-wise.

Example

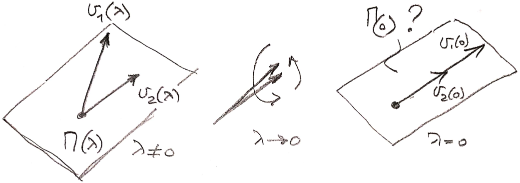
For $L = \lambda^2 \partial^2 + 1$ any solution $f(t, \lambda) = c(\lambda) \sin(\lambda^{-1} t)$ does not grow moderately as $\lambda \to 0$ for any analytic choice $c(\lambda) \not\equiv 0$.

Visualization of Grigoriev theorem

Vocabulary.

- Homogeneous linear ODE of order $n \iff n$-dimensional $\mathbb{C}$-subspace in the space of analytic functions (outside of singularities).
- If $\{L_\lambda\}$ is an OPF, then solutions of this equation depend analytically on λ for $\lambda \neq 0$,
- Under the moderate growth conditions, solutions can be multiplied by a suitable power of λ so that they would stay analytic also for $\lambda = 0$.

Finite-dimensional "model". Consider a tuple of analytic vector-functions $v_i : (\mathbb{C}, 0) \to \mathbb{C}^N$, $\lambda \longmapsto v_i(\lambda)$, $i = 1, \dots, n \leqslant N$, linear independent for all $\lambda \neq 0$. Let Π_λ be the n-subspace in $\mathbb{C}^N$ spanned by $v_i(\lambda)$.



Question. Is there a limit position of Π_λ as $\lambda \to 0$?

Why Grigoriev theorem is obvious in the hindsight?

Grigoriev theorem (geometric reformulation)

If $v_i(\lambda)$ are analytic, then the limit position of $\varPi_\lambda$ as $\lambda \to 0$ exists: one can find find analytic vector functions $w_1 : (\mathbb{C}, 0) \to \mathbb{C}^N$ which span the same $\varPi_\lambda$ for $\lambda \neq 0$ but remain linear independent also at $\lambda = 0$.

1. **Elementary approach.** If $n = 1$, then any analytic vector function $v_1(\lambda) \not\equiv 0$ is of the form $\lambda^s w_1(\lambda)$ with $s \geqslant 0$ and $w_1(0) \neq 0$.

   By induction, one can assume that $v_1(\lambda)$ is the first coordinate constant vector in $\mathbb{C}^N$, and $v_2(\lambda)$ has identically zero first coordinate and not identically zero $\operatorname{mod} v_1$.
2. **Removable singularity theorem.** The Grassmannian manifold $G_{n,N}$ is compact. The tuple $v_1, \dots, v_n$ is a holomorphic map of $(\mathbb{C}, 0)$ to this Grassmannian, hence subject to the *removable singularity theorem*.
3. The infinite-dimension case requires only minor technical changes.

First corollary: boundedness of slope for Fuchsian operators

Let $L = L_0$ be a *globally Fuchsian operator*, and $L_{\alpha\beta\gamma\delta}$, a 4-parametric family of operators obtained from L_0 by Möbius transformations $t \longmapsto z = \left(\frac{\alpha t+\beta}{\gamma t+\delta}\right)$ such that $\det\left(\begin{smallmatrix}\alpha & \beta \\ \gamma & \delta\end{smallmatrix}\right) \neq 0$.

Theorem.

Consider the projective space $\mathbb{P}^3 = \{\Lambda = (\alpha : \beta : \gamma : \delta)\}$ and the slope of the operator $\angle L_\Lambda$ on it defined outside of the algebraic hypersurface $\det \Lambda = 0$. The slope $\angle L_\Lambda$ is *globally bounded* on $\mathbb{P}^3$.

Proof. Consider any analytic family $\lambda : (\mathbb{C}, 0) \to \mathbb{P}^3$, $\lambda \mapsto \Lambda(\lambda)$, $\det \Lambda(0) \not\equiv 0$. For any solution f of $Ly = 0$ its composition $f\left(\frac{\alpha t+\beta}{\gamma t+\delta}\right)$ grows moderately anywhere as $\lambda \to 0$ (moderate growth is stable by composition). Hence $\angle L_\Lambda$ is locally bounded by Grigoriev theorem.

$\mathbb{P}^3$ is compact $\Longrightarrow$ the slope $\angle L_\Lambda$ is globally bounded on $\mathbb{P}^3$. The bound depends on L (not explicitly) and is conformal invariant of L. $\square$

Can one have an explicit bound for this *conformally invariant* slope?

Semialgebraic sets: the tamest algebra

Definition.

Semialgebraic subsets of $\mathbb{R}^N$ are those defined by polynomial equations and inequalities of the form $p(x) = 0$ (resp., $q(x) > 0$) and their finite unions.

Topological complexity obviously depends on the dimension N and the degrees d of the (in)equations. What about the size of *bounded* sets?

Example. $S = \{-c \leqslant x \leqslant c^{-1}\} \subseteq \mathbb{R}^1, \quad c > 0$. Depends on c!

Denote height: $\mathbb{Q} \to \mathbb{N}$ be the height$(m/n) = \max(|m|, |n|)$. For $p \in \mathbb{Q}[x_1, \dots, x_N]$ its height is the maximum of heights of coefficients.

Theorem (constructive quantifier elimination).

If all polynomials p, q are defined over $\mathbb{Q}$ and their heights are $\leqslant M$, then $\operatorname{diam} S < +\infty \implies \operatorname{diam} S \leqslant M^{d^{O(N)}}$. The $O(N)$ is explicit.

Explicit bound for the slope

Consider a differential operator $L \in \mathbb{Q}[t][\partial] = \sum_{k=0}^{n} A_k(t)\partial^{n-k}$, $A_k \in \mathbb{Q}[t]$. Assume that:

1. $\operatorname{height} A_k \leqslant M, \quad \deg A_k \leqslant d, \quad \gcd(A_0, \dots, A_n) = 1;$
2. L is globally Fuchsian on $\mathbb{P}$.

Theorem.

The conformally invariant slope of L does not exceed $M^{\mathrm{Poly}(d,n)}$ with an explicit polynomial in the exponent.

Assume in addition that:

3. all characteristic exponents at all singularities of L are real.

Theorem.

The number of isolated zeros of all solutions $N < +\infty$ is finite and explicitly bounded by a similar expression.

More examples

If L is a parametric family differential operators which depending rationally (and over $\mathbb{Q}$) on N (complex projective) parameters with at most m singularities, which *for some reasons* remains Fuchsian for all values of the parameters, then slope is explicitly uniformly bounded.

Super-example (Infinitesimal Hilbert 16th problem).

Consider a general planar algebraic curve

$$H(x,y)=0, \qquad H=\sum_{0\leqslant i+j\leqslant n}\lambda_{ij}x^iy^j, \qquad \varLambda=\{\lambda_{ij}\}\in\mathbb{P}^{(n+1)(n+2)/2}.$$

Consider a general Abelian integral (period) $I(t,\varLambda,c)=\oint_{\delta(t)}\sum_{i,j\leqslant n}c_{ij}x^iy^j\,\mathrm{d}x$ over a homological cycle $\delta(t)\subset\{H=0\}$, $t=\lambda_{00}$.

Then $I=I(t)$ satisfies a parametric family of Fuchsian equations of order n^2 depending on $\varLambda$ exactly in the way described above.

Zero-one coefficients of input + explicit derivation of Picard–Fuchs equations.