Quadratic differential systems possessing invariant ellipses: a complete classification in the space $\mathbb{R}^{12}$

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a work in common with

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The main motivation of our study is to obtain the full geometry of quadratic differential systems possessing an invariant conic and in particular an invariant ellipse.

By the geometry of such systems we mean giving all their phase portraits as well as their bifurcation diagrams and in addition all the information regarding invariant ellipses.

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We achieve this in our Main Theorem which also gives us an algorithm for deciding for any quadratic differential system whether it possesses an invariant ellipse or not. This theorem opens the road for determining the phase portraits of all quadratic systems possessing an invariant ellipse as well as their bifurcation diagram, both in affine invariant form.

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- **(A_1)** *If* $\eta < 0$ *and* $\widetilde{N} \neq 0$, *then the system could possess at most one invariant ellipse. Moreover, the necessary and sufficient conditions for the existence of such an ellipse are given in* DIAGRAM *1, where we can also find the conditions for the ellipse to be real or complex.*

(A_2) *If $\eta < 0$ and $\widetilde{N} = 0$, then the system either has no invariant ellipse or it has an infinite family of invariant ellipses.*

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(A_3) *If $C_2 = 0$, then the system either has no invariant ellipse or it has an infinite family of invariant ellipses. Moreover, the necessary and sufficient conditions for the existence of a family of invariant ellipses are given in* DIAGRAM 2*, where we can also find the conditions for the ellipses to be real or/and complex. In addition, this system possesses a real invariant line and the positions of the invariant ellipses with respect to this line are presented in* FIGURE 2.

(B) *A non-degenerate quadratic system possesses an algebraic limit cycle, which is an ellipse, if and only if* $\widehat{\gamma}_1 = \widehat{\gamma}_2 = 0$, $\eta < 0$, $\mathcal{T}_3\mathcal{F} < 0$, $\widehat{\beta}_1\widehat{\beta}_2 \neq 0$, *and one of the following sets of conditions is satisfied:*

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Moreover, we see in DIAGRAM *1 how these limit cycles are displayed in the 12-parameter space.*

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$$\dot{x} = 1 - cy - x^2 - axy - (b+1)y^2, \quad \dot{y} = x(c + ax + by). \qquad (1)$$

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These systems possess the invariant conic

$$\Phi(x, y) = x^2 + y^2 - 1 = 0.$$

According to [J. Llibre, G. Świrszcz: *Bull. Sci. Math.* **131**(2007)], this conic is a limit cycle if and only if

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$$\mathcal{T}_3\mathcal{F} = a^2c^2(a^2 + b^2 - c^2)\big[a^2 + (b-2)^2\big]^2/8,$$

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Theorem

A quadratic system possesses an algebraic limit cycle of degree 2 if and only if $\eta < 0$, $\mathcal{T}_3\mathcal{F} < 0$, $\widehat{\beta}_1\widehat{\beta}_2 \neq 0$, $\widehat{\gamma}_1 = \widehat{\gamma}_2 = 0$ *and one of the following sets of conditions is satisfied:*

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Proof of the statement **(*B*)** of the Main Theorem

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(about the construction of invariant polynomials)

Proof of the statement $(\boldsymbol{B})$ of the Main Theorem

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Thanks a lot for your attention!

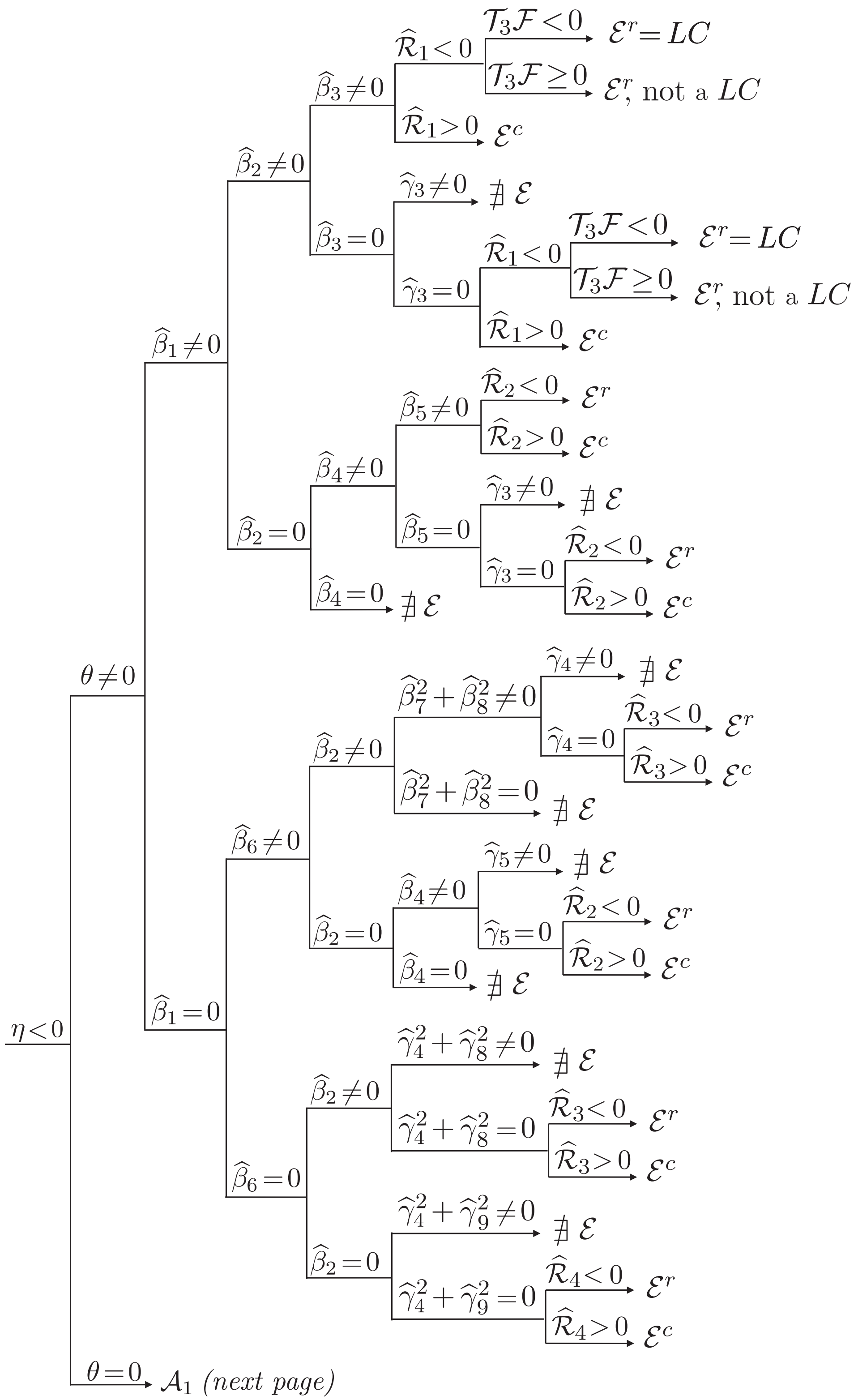


DIAGRAM 1: **The existence of invariant ellipses: QS with one real and two complex infinite singularities**.

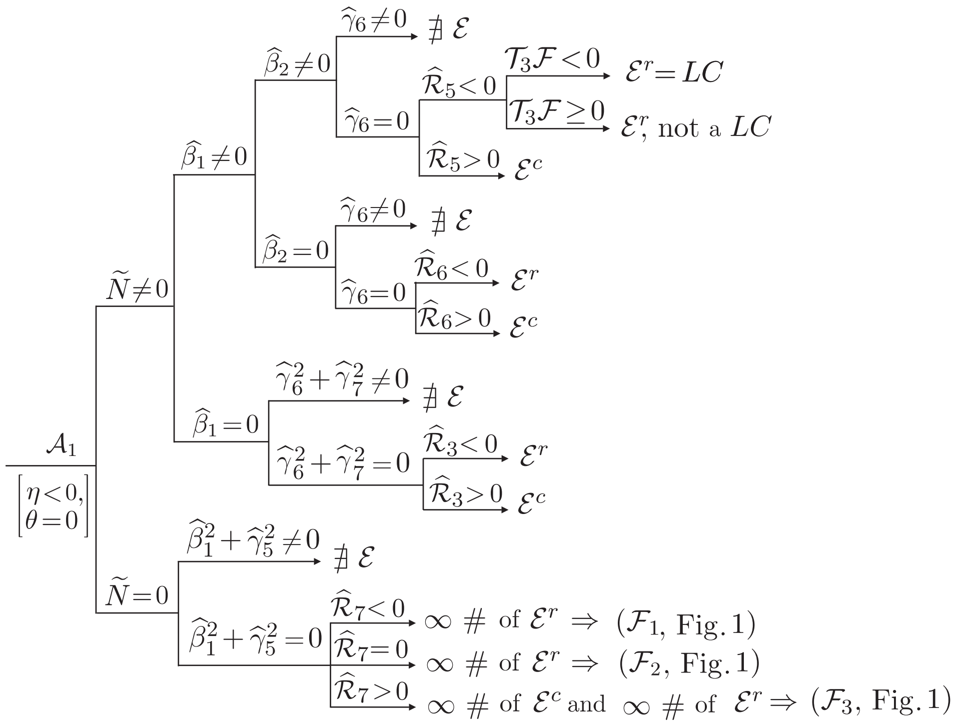
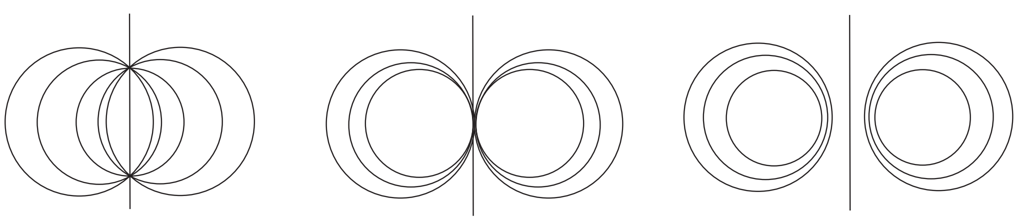


Diagram 1 (*cont.*): **The existence of invariant ellipses: QS with one real and two complex infinite singularities**.

$$\dot{x} = 2xy, \quad \dot{y} = b - x^2 + y^2. \tag{1}$$

$$\Phi(x,y) = b + qx + x^2 + y^2 = 0, \tag{2}$$



$\mathcal{F}_1\,(b<0)$ $\mathcal{F}_2\,(b=0)$ $\mathcal{F}_3\,(b>0)$

Figure 1: **The family** (2) **of invariant ellipses of systems** (1)**.**

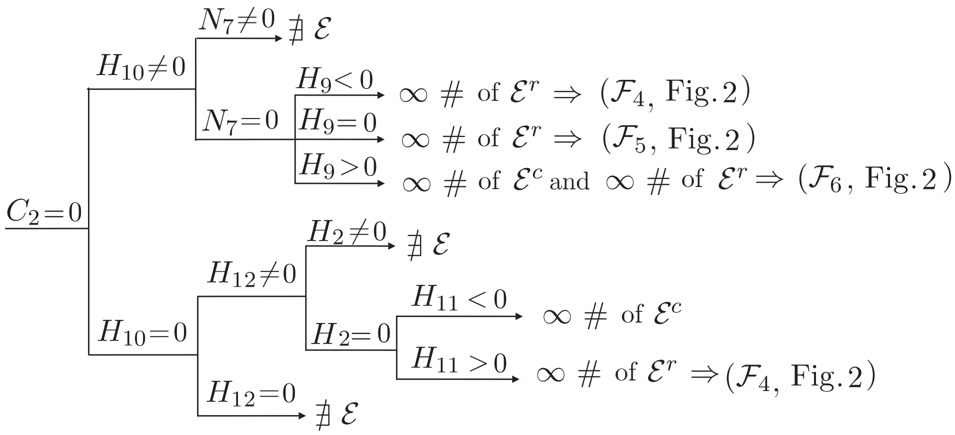


Diagram 2: **The existence of invariant ellipses: QS with infinite line filled up with singularities**.

$$\dot{x} = a + y + x^2, \quad \dot{y} = xy, \tag{3}$$

$$\widetilde{\Phi}(x, y) = a + 2y + x^2 + m^2y^2 = 0. \tag{4}$$

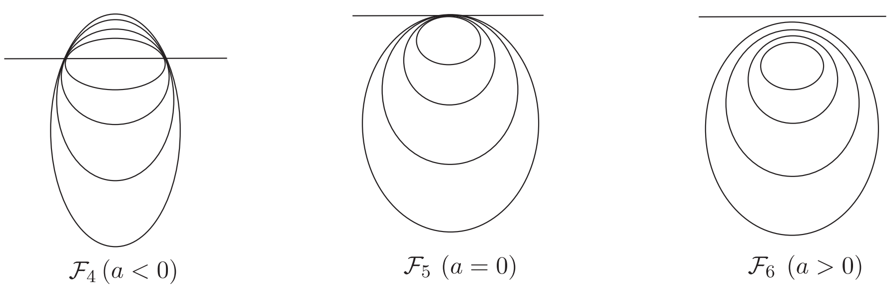


Figure 2: **The family** (4) **of invariant ellipses of systems** (3)**.**