Limit cycle bifurcations near an elementary center and a homoclinic loop

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mainly based on works:

- ✦ Yun Tian and Pei Yu, Bifurcation of small limit cycles in cubic integrable systems using higher-order analysis, *J. Differential Equations*, 264(2018) 5950–5976.
- ✦ Yun Tian and Maoan Han, Hopf and homoclinic bifurcations for near-Hamiltonian systems, *J. Differential Equations*, 262(2017) 3214–3234.
- ✦ Yun Tian and Pei Yu, Bifurcation of ten small-amplitude limit cycles by perturbing a quadratic Hamiltonian system with cubic polynomials, *J. Differential Equations*, 260(2016) 971-990.

Near-integrable Polynomial Systems

Consider near-integrable polynomial systems in the form of

$$\begin{aligned} \frac{dx}{dt} &= M^{-1}(x,y,\mu)H_y(x,y,\mu) + \varepsilon\, p(x,y,\varepsilon,\delta), \\ \frac{dy}{dt} &= -M^{-1}(x,y,\mu)H_x(x,y,\mu) + \varepsilon\, q(x,y,\varepsilon,\delta), \end{aligned} \tag{1}$$

where

- $0 < \varepsilon \ll 1$, μ and δ are vector parameters;
- $H(x,y,\mu)$ is an analytic function in x, y and μ;
- $p(x,y,\varepsilon,\delta)$ and $q(x,y,\varepsilon,\delta)$ are polynomials in x and y, and analytic in δ and ε;
- $M(x,y,\mu)$ is an integrating factor of the system $(1)|_{\varepsilon=0}$.

We suppose that the level curves $H(x,y)=h$ have a family of periodic orbits $L_h \subset \{(x,y)|\ H(x,y)=h, h\in(h_c,h_s)\}$ with two boundaries:

- an elementary center C as the inner boundary,
- a homoclinic loop L_{h_s} as the outer boundary which passes through a hyperbolic saddle S.

Limit cycles in system (1) could be produced

- around the elementary center C,
- by periodic orbits L_h with $h\in(h_c,h_s)$,
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Hopf Bifurcation

To find the maximum number of small limit cycles, we

➢ Compute and solve focus values.
- use computer algebra systems (Maple, Mathematica)
- much easier to solve focus values for the case of near-integrable systems

Normal Form

We obtain the normal form of (1) as follows:

$$\begin{aligned} \frac{dr}{dt} &= r\left[v_0(\varepsilon) + v_1(\varepsilon)r^2 + v_2(\varepsilon)r^4 + \cdots + v_i(\varepsilon)r^{2i} + \cdots\right], \\ \frac{d\theta}{dt} &= \tau_0(\varepsilon) + \tau_1(\varepsilon)\,r^2 + \tau_2(\varepsilon)\,r^4 + \cdots + \tau_k(\varepsilon)\,r^{2k} + \cdots, \end{aligned} \tag{2}$$

where

$$v_i(\varepsilon) = \sum_{k=1}^{\infty} V_{ik}\varepsilon^k, \quad i = 0, 1, 2, \ldots,$$

in which V_{ik} denotes the ith ε^k-order focus value.

Rearrange Terms

Note that $v_i(\varepsilon) = O(\varepsilon)$ since system $(1)|_{\varepsilon=0}$ is an integrable system.

Further, because system (1) is analytic in ε, we can rearrange the terms in (2), and obtain

$$\frac{dr}{dt} = V_1(r)\,\varepsilon + V_2(r)\,\varepsilon^2 + \cdots + V_k(r)\,\varepsilon^k + \cdots\,, \tag{3}$$

where

$$V_k(r) = \sum_{i=0}^{\infty} V_{ik}\, r^{2i+1}, \quad k = 1, 2, \ldots. \tag{4}$$

Eliminate Time t

Similarly, for the normal form of system (1) we have the θ differential equation, given by

$$\frac{d\theta}{dt} = T_0(r) + O(\varepsilon),$$

with $T_0(0) \neq 0$, and thus

$$\frac{dr}{d\theta} = \frac{V_1(r)\,\varepsilon + V_2(r)\,\varepsilon^2 + \cdots + V_k(r)\,\varepsilon^k + \cdots}{T_0(r) + O(\varepsilon)}. \tag{5}$$

Solution of System (5)

Assume the solution $r(\theta, \rho, \varepsilon)$ of (5), satisfying the initial condition $r(0, \rho, \varepsilon) = \rho$, is given in the form of

$$r(\theta, \rho, \varepsilon) = r_0(\theta, \rho) + r_1(\theta, \rho)\varepsilon + r_2(\theta, \rho)\varepsilon^2 + \cdots + r_k(\theta, \rho)\varepsilon^k + \cdots ,$$

with $0 < \rho \ll 1$. Then, $r_0(0, \rho) = \rho$ and $r_k(0, \rho) = 0$ for $k \geq 1$.

If there exists a positive integer K such that $V_k(r) \equiv 0$, $1 \leq k < K$, and $V_K(r) \not\equiv 0$, then we get from (5)

$$\frac{dr}{d\theta} = \frac{V_K(r)\,\varepsilon^K + \cdots}{T_0(r) + O(\varepsilon)},$$

and then

$$r_0(\theta, \rho) = \rho, \quad r_k(\theta, \rho) = 0, \quad 1 \leq k < K, \quad \text{and} \quad r_K(\theta, \rho) = \frac{V_K(\rho)}{T_0(\rho)}\theta.$$

Two Displacement Functions

Thus, the displacement function $d(\rho)$ of system (5) can be written as

$$d(\rho) = r(2\pi, \rho, \varepsilon) - \rho = 2\pi \frac{V_K(\rho)}{T_0(\rho)} \varepsilon^K + O(\varepsilon^{K+1}). \tag{6}$$

Another displacement function is

$$d(h) = M_1(h)\varepsilon + M_2(h)\varepsilon^2 + \cdots + M_k(h)\varepsilon^k + \cdots .$$

- For any integer $K \geq 1$, equation (6) holds if and only if $M_k(h) \equiv 0$, $1 \leq k < K$ and $M_K(h) \not\equiv 0$.
- Moreover, $V_K(\rho)$ for $0 < \rho \ll 1$ and $M_K(h)$ for $0 < h - h_1 \ll 1$ have the same maximum number of isolated zeros.

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Theorem 1.1

Consider system (3) *and assume* $V_k(r) \equiv 0,\ 1 \le k < K$. *Suppose that for an integer* $m \ge 1$, *each* V_{iK}, $0 \le i < m$ *is linear in* δ, *and further the following two conditions hold:*

(i) $\text{rank}\left[\frac{\partial(V_{0K},\cdots,V_{m-1,K})}{\partial(\delta_1,\cdots,\delta_m)}\right] = m$,

(ii) $V_K(r) \equiv 0$, if $V_{iK} = 0, i = 0, 1, \cdots, m-1$.

Then, for any given $N > 0$, *there exist* $\varepsilon_0 > 0$ *and a neighborhood* V *of the center such that system* (1) *has at most* $m-1$ *limit cycles in* V *for* $0 < |\varepsilon| < \varepsilon_0$, $|\delta| \le N$. *Moreover,* $m-1$ *limit cycles can appear in an arbitrary neighborhood of the origin for some values of* (ε, δ).

Application 1

We will apply our method to study the bifurcation of small-amplitude limit cycles in the system

$$\begin{aligned} \frac{dx}{dt} &= a + \frac{5}{2}x + xy + x^3 + \sum_{k=1}^{n} \varepsilon^k p_k(x,y), \\ \frac{dy}{dt} &= -2ax + 2y - 3x^2 + 4y^2 - ax^3 + 6x^2y + \sum_{k=1}^{n} \varepsilon^k q_k(x,y), \end{aligned} \tag{7}$$

where

$$p_k(x,y) = a_{00k} + \sum_{i+j=1}^{3} a_{ijk}\, x^i y^j, \quad q_k(x,y) = b_{00k} + \sum_{i+j=1}^{3} b_{ijk}\, x^i y^j, \tag{8}$$

in which a_{ijk} and b_{ijk} are ε^kth-order coefficients (parameters).

The unperturbed system $(7)|_{\varepsilon=0}$ has a rational Darboux integral,

$$H_0 = \frac{f_1^5}{f_2^4} = \frac{(x^4 + 4x^2 + 4y)^5}{(x^5 + 5x^3 + 5xy + 5x/2 + a)^4},$$

with the integrating factor $M = 20 f_1^4 f_2^{-5}$. For $a < -2^{5/4}$, system $(7)|_{\varepsilon=0}$ has a center at $\mathrm{E}_0 = (-\frac{a}{2}, -\frac{a^2+2}{4})$.

- ➢ find 11 small limit cycles around E_0 by using 12 Melnikov integrals to study the second-order Melnikov function. [Żołądek, 1995]
- ➢ Two of these 12 Melnikov integrals can be expressed as a linear combination of the other ten integrals. [Tian & Yu, 2016]
- ➢ Questions: whether 11 small limit cycles exist in system (7) or not? the maximum number?

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First nonzero $V_k(r) \not\equiv 0$, $k = 1, 2, \dots, 7$

Theorem 1.2 (Tian & Yu, 2018)

(I) *When $V_1(r) \not\equiv 0$ or $V_1(r) \equiv 0$ and $V_2(r) \not\equiv 0$, there exist at most* 9 *small limit cycles around* E_0 *in system* (7) *for all related parameters and ε sufficiently small, and* 9 *small limit cycles exist for some parameter values.*

(II) *System* (7) *can indeed have* 11 *small limit cycles around* E_0 *under perturbations satisfying $V_i(r) \equiv 0$, $1 \le i < 7$, $V_7(r) \not\equiv 0$.*

k	1	2	3	4	5	6	7
$N(k)$	9	9	10	9	9	10	11

Table: Maximum number $N(k)$ of small limit cycles around E_0 in system (7) when $V_i(r) \equiv 0$, $1 \le i < k$, $V_k(r) \not\equiv 0$

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Application 2

Consider the following cubic polynomial system

$$\begin{aligned} \dot{x} &= 10x(8axy - 3x^2 - 9x - 12y^2 - 6), \\ \dot{y} &= 24a - 16ax + 90y + 15xy - 16axy^2 + 60y^3, \end{aligned} \tag{9}$$

which is determined by a Darboux first integral,

$$H_0 = \frac{(xy^2 + x + 1)^5}{x^3(xy^5 + \frac{5}{2}xy^3 + \frac{5}{2}y^3 + \frac{15}{8}xy + \frac{15}{4} + a)^2},$$

where a is a parameter.
When $32a^2 \neq 75$, system (9) has an elementary center (x_e, y_e), given by

$$x_e = \frac{6(8a^2 + 25)}{32a^2 - 75}, \quad y_e = \frac{70\,a}{32a^2 - 75}.$$

- By system (9) with $a = 2$, verify that there exists a class of cubic systems with 11 limit cycles bifurcating from a critical point. [Christopher, 2006]

- With a properly chosen value of a, system (9) can have 12 small limit cycles bifurcating from (x_e, y_e) by proper cubic perturbation. [Yu & Tian, 2014]

Theorem 1.3

For any positive integer K, when $V_k(r) \equiv 0$, $1 \leq k < K$ and $V_K(r) \not\equiv 0$, there exist at most 12 small limit cycles bifurcating from (x_e, y_e) in system (9) under cubic perturbations.

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Next, we consider homoclinic bifurcation in near-Hamiltonian systems of the form

$$\dot{x} = H_y + \varepsilon f(x, y, a),\ \dot{y} = -H_x + \varepsilon g(x, y, a), \tag{10}$$

where $\deg(H(x, y)) = n$ and $\deg(f(x, y)) = \deg(g(x, y)) = m$, a is a vector parameter, and $\varepsilon \in \mathbb{R}$ is small.

Then Melnikov function is given by

$$M(h, a) = \oint_{H=h} g(x, y, a)dx - f(x, y, a)dy,$$

and has a series expansion

$$M(h, a) = \sum_{j \geq 0} b_j(a)(h - h_c)^{j+1}, \quad 0 \leq h - h_c \ll 1. \tag{11}$$

at the end point h_c for Hopf bifurcation.

Expansion of $M(h,a)$ at $h = h_s$

For homoclinic bifurcation, from [Dulac (1923); Roussaire (1986)] $M(h,a)$ has the following expansion

$$\begin{aligned} M(h,a) &= \sum_{j\geq 0}[c_{2j}(a) + c_{2j+1}(a)(h-h_s)\ln|h-h_s|](h-h_s)^j \\ &= c_0(a) + c_1(a)(h-h_s)\ln|h-h_s| + c_2(a)(h-h_s) \\ &\quad + c_3(a)(h-h_s)^2\ln|h-h_s| + \cdots, \quad 0 < h_s - h \ll 1. \end{aligned} \tag{12}$$

- only the first four coefficients c_j, $j = 0,1,2,3$ were obtained in [Han-Ye (1998), Han-Yang-Tarta-Gao (2008)].
- **Objective**: establish a new method to compute other coefficients.

Obviously,

$$c_0(a) = M(h_s, a) = \oint_{L_{h_s}} g dx - f dy.$$

Han-Ye obtained formulas of c_1 and c_2 as follows

$$\begin{gathered} c_1 = -\frac{1}{|\lambda|}\bar{c}_1, \quad c_2 = \bar{c}_2 + \beta\bar{c}_1, \\ \bar{c}_1 = (f_x + g_y)(S, a), \quad \bar{c}_2 = \oint_{L_{h_s}} (f_x + g_y - \bar{c}_1) dt, \end{gathered} \tag{13}$$

where β is a constant, and $\pm\lambda$ are the eigenvalues of the matrix

$$\mathrm{Hess}(S) = \begin{pmatrix} H_{xy}(S) & H_{yy}(S) \\ -H_{xx}(S) & -H_{xy}(S) \end{pmatrix}.$$

The formula of c_3 was given by Han et al. in the form

$$
\begin{aligned}
c_3(a) = \frac{-1}{2|\lambda|\lambda}\{&(a_{12} - 3a_{30} - b_{21} + 3b_{03}) \\
&-\frac{1}{\lambda}[(2b_{02} + a_{11})(3h_{03} - h_{21}) \\
&+(2a_{20} + b_{11})(3h_{30} - h_{12})]\} + bc_1(a)
\end{aligned}
\tag{14}
$$

for some constant b, when H, f and g can be written as

$$
\begin{aligned}
&H(x, y) = h_s + \frac{\lambda}{2}(y^2 - x^2) + \sum_{i+j\geq 3} h_{ij}x^iy^j, \\
&f(x, y, a) = \sum_{i+j\geq 0} a_{ij}x^iy^j, \quad g(x, y, a) = \sum_{i+j\geq 0} b_{ij}x^iy^j.
\end{aligned}
\tag{15}
$$

Lemma 2.1

For the Melnikov function $M(h) = \oint_{L_h} g\mathrm{d}x - f\mathrm{d}y$ *we have*

$$\frac{\partial M}{\partial h} = \oint_{L_h} (f_x + g_y)\mathrm{d}t.$$

By Lemma 2.1, (11) and (12) we have

$$b_0(a) = T_0\bar{b}_0(a), \quad \bar{b}_0(a) = (f_x + g_y)(C, a), \tag{16}$$

and

$$\lim_{h \to h_s^-} \oint_{L_h} (f_x + g_y)\mathrm{d}t = \oint_{L_{h_s}} (f_x + g_y)\mathrm{d}t \in \mathbb{R} \ \Leftrightarrow \ \bar{c}_1(a) = 0,$$

where $T_0 > 0$ is a constant, and $\bar{c}_1 = (f_x + g_y)(S, a)$.

Theorem 2.2 (Tian & Han, 2017)

Suppose there exist analytic functions $P_1(x, y, a)$ *and* $Q_1(x, y, a)$ *such that for* $\bar{b}_0 = \bar{c}_1 = 0$*, the following equality holds*

$$f_x + g_y = H_x(x, y)P_1(x, y, a) + H_y(x, y)Q_1(x, y, a), \tag{17}$$

for $(x, y) \in \bigcup_{h_c \le h \le h_s} L_h$*. Then when* $\bar{b}_0 = \bar{c}_1 = 0$*, we have for* b_1 *in* (11) *and* c_3 *and* c_4 *in* (12)

$$b_1 = T_1 \bar{b}_1(a), \quad c_3(a) = -\frac{1}{2|\lambda|}\bar{c}_3(a), \quad c_4(a) = \frac{1}{2}\bar{c}_4 + \beta_1 \bar{c}_3(a), \tag{18}$$

where T_1 *and* β_1 *are constants with* $T_1 > 0$*, and*

$$\begin{aligned} &\bar{b}_1(a) = (P_{1x} + Q_{1y})(C, a), \quad \bar{c}_3(a) = (P_{1x} + Q_{1y})(S, a), \\ &\bar{c}_4(a) = \oint_{L_{h_s}} (P_{1x} + Q_{1y} - \bar{c}_3(a))\mathrm{d}t. \end{aligned} \tag{19}$$

Theorem 2.2 (continued)

Further, if we let $H(x,y)$ *satisfy* (15), *and*

$$P_1(x,y,a) = \sum_{i+j\geq 0} \tilde{a}_{ij}x^i y^j, \quad Q_1(x,y,a) = \sum_{i+j\geq 0} \tilde{b}_{ij}x^i y^j$$

for (x,y) *near* S, *then* $c_5(a) = \frac{-1}{6|\lambda|}\bar{c}_5 + b\bar{c}_3$ *for some constant* b *under* $\bar{b}_0 = \bar{c}_1 = 0$, *where*

$$\begin{aligned}\bar{c}_5(a) = \ & \frac{1}{\lambda}\{(-3\tilde{a}_{30} - \tilde{b}_{21} + \tilde{a}_{12} + 3\tilde{b}_{03}) - \frac{1}{\lambda}[(2\tilde{b}_{02} + \tilde{a}_{11}) \\ & \cdot(3h_{03} - h_{21}) + (2\tilde{a}_{20} + \tilde{b}_{11})(3h_{30} - h_{12})]\}.\end{aligned} \tag{20}$$

Outline of Proof. Let $\bar{b}_0 = \bar{c}_1 = 0$. We have $b_0 = c_1 = 0$ from (13) and (16). Then by (11) and (12), it is easy to get

$$\frac{\partial M}{\partial h} = 2b_1(h-h_c) + 3b_2(h-h_c)^2 + 4b_3(h-h_c)^3 + \cdots, \tag{21}$$

for $0 \le h - h_c \ll 1$, and

$$\begin{aligned}\frac{\partial M}{\partial h} &= c_2 + 2c_3(h-h_s)\ln|h-h_s| + (c_3+2c_4)(h-h_s) \\ &\quad +3c_5(h-h_s)^2\ln|h-h_s| + (c_5+3c_6)(h-h_s)^2 \\ &\quad +4c_7(h-h_s)^3\ln|h-h_s| + \cdots\end{aligned} \tag{22}$$

for $0 < h_s - h \ll 1$.

On the other hand, by Lemma 2.1 and (17) we have

$$\begin{aligned}\frac{\partial M}{\partial h} &= \oint_{L_h} (H_x P_1 + H_y Q_1)\mathrm{d}t \\ &= \oint_{L_h} Q_1\mathrm{d}x - P_1\mathrm{d}y \equiv M_1(h,a).\end{aligned} \tag{23}$$

Theorem 2.3 (Tian & Han, 2017)

Suppose there exist analytic functions P_1, Q_1, P_2 and Q_2 such that (17) and the following equation

$$P_{1x} + Q_{1y} = H_x(x,y)P_2(x,y,a) + H_y(x,y)Q_2(x,y,a), \quad (x,y) \in U,$$

are satisfied for $\bar{b}_0 = \bar{b}_1 = 0$ and $\bar{c}_1 = \bar{c}_3 = 0$. Then

$$b_2 = T_2\bar{b}_2, \; c_5 = -\frac{1}{6|\lambda|}\bar{c}_5, \; c_6 = \frac{1}{6}\bar{c}_6 + \beta_2\bar{c}_5, \tag{24}$$

where

$$\begin{aligned} &\bar{b}_2 = (P_{2x} + Q_{2y})(C,a), \; \bar{c}_5 = (P_{2x} + Q_{2y})(S,a), \\ &\bar{c}_6 = \oint_{L_{h_s}} (P_{2x} + Q_{2y} - \bar{c}_5)\mathrm{dt}. \end{aligned} \tag{25}$$

Theorem 2.3 (continued)

Further, we have

$$\begin{aligned} c_7(a) &= \frac{-1}{24|\lambda|}\bar{c}_7(a) + O(\bar{c}_5), \\ \bar{c}_7(a) &= \frac{1}{\lambda}\{(-3\bar{a}_{30} - \bar{b}_{21} + \bar{a}_{12} + 3\bar{b}_{03}) - \frac{1}{\lambda}[(2\bar{b}_{02} + \bar{a}_{11}) \\ &\quad \cdot(3h_{03} - h_{21}) + (2\bar{a}_{20} + \bar{b}_{11})(3h_{30} - h_{12})]\}, \end{aligned} \tag{26}$$

if $P_2(x,y)$ *and* $Q_2(x,y)$ *are given by*

$$P_2(x,y,a) = \sum_{i+j\geq 0} \bar{a}_{ij}x^iy^j, \quad Q_2(x,y,a) = \sum_{i+j\geq 0} \bar{b}_{ij}x^iy^j$$

for (x,y) *near* S.

If the Hamiltonian is given by

$$H(x,y)=\int_0^x q(x)\mathrm{d}x+\int_0^y p(y)\mathrm{d}y$$

in system (10), it is easy to find functions

$$P_1(x,y)=\frac{F(x,0)}{q(x)},\quad Q_1(x,y)=\frac{F(x,y)-F(x,0)}{p(y)}, \tag{27}$$

satisfying

$$f_x+g_y=H_x(x,y)P_1(x,y,a)+H_y(x,y)Q_1(x,y,a),$$

for $(x,y)\in U$ when $\bar{b}_0=\bar{c}_1=0$, where

$$F(x,y)=f_x(x,y)+g_y(x,y).$$

Example

Consider the following perturbed polynomial Liénard system

$$\dot{x} = y + \varepsilon f(x, y), \quad \dot{y} = x - x^3 + \varepsilon g(x, y), \tag{28}$$

with the Hamiltonian $H(x, y) = \frac{1}{2}y^2 - \frac{1}{2}x^2 + \frac{1}{4}x^4$, where $0 < \varepsilon \ll 1$, $-f(x, y) = f(-x, -y)$ and $-g(x, y) = g(-x, -y)$.

Then system $(28)|_{\varepsilon=0}$ has

- two centers $C_1 = (1, 0)$ and $C_2 = (-1, 0)$,
- one saddle point $S = (0, 0)$,
- an eight-loop $H(x, y) = 0$,
- periodic orbits $L_1(h)$ and $L_2(h)$ given by $H(x, y) = h$, $h \in (-1/4, 0)$,
- periodic orbits $L(h)$ defined by $H(x, y) = h$, $h \in (0, \infty)$.

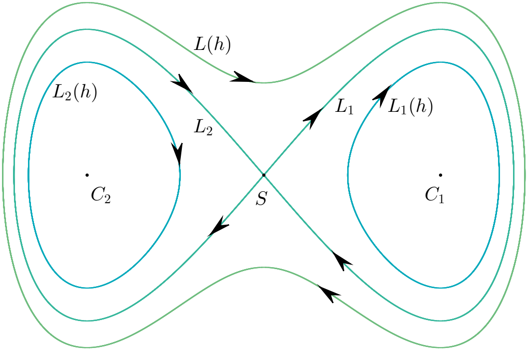

Figure: Phase portrait of a double homoclinic loop.

Theorem 2.4 (Tian & Han, 2017)

Suppose $n = \max(\deg(f(x,y)), \deg(g(x,y)))$. *For* $n = 3, 5, 7, 9$, *system* (28) *can have* $[\frac{7n-6}{3}]$ *limit cycles under proper perturbations with distribution,*

$$\begin{cases} (2,2,1) \text{ for } n = 3, \\ (4,4,1) \text{ for } n = 5, \\ (6,6,2) \text{ for } n = 7, \\ (8,8,3) \text{ for } n = 9, \end{cases}$$

where $[\cdot]$ *denotes the integer part function.*

Thank you for your attention!