Real analytic vector fields with first integral and separatrices

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General setting: separatrices

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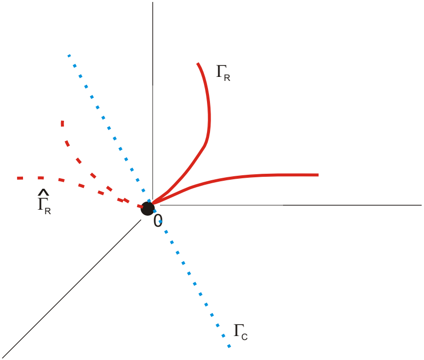
X is an analytic v. f. on a nbhd U of $0 \in \mathbb{R}^n$, $X(0) = 0$.

Separatrix (analytic or formal, real or complex)

A $\left\{ \begin{array}{c} \text{real (or complex) analytic curve: } \Gamma_{\mathbb{R}} (\text{or } \Gamma_{\mathbb{C}}) \\ \text{formal real (or complex) curve: } \widehat{\Gamma}_{\mathbb{R}} (\text{or } \widehat{\Gamma}_{\mathbb{C}}) \end{array} \right\}$ which is invariant for X.



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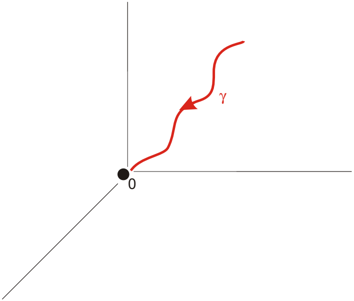
Characteristic orbit

An integral curve (trajectory) $\gamma : [0, \infty[\to U$ of X (its image) such that $\lim_{t\to\infty} \gamma(t) = 0$ and

$$\lim_{t\to\infty} \frac{\gamma(t)}{\|\gamma(t)\|} \text{ exists}$$



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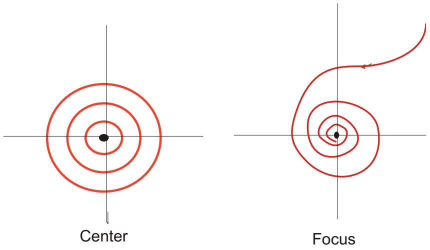
- There are examples with no $\Gamma_{\mathbb{R}}$ or $\widehat{\Gamma}_{\mathbb{R}}$:



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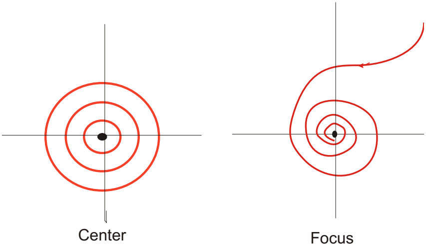

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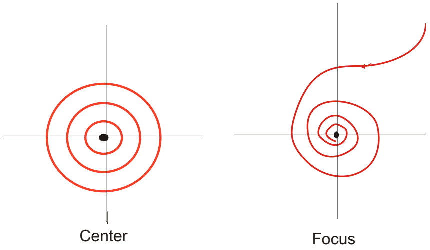

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- There are also examples with characteristic orbits without $\widehat{\Gamma}_{\mathbb{R}}$ (Corral-Sanz, 2011).

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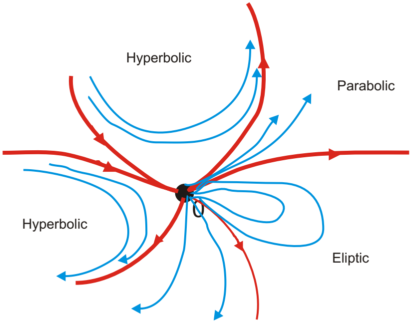
Sketch.-

- If X is of the type *center-focus* then $I_0(X) = 1$.
- If X has characteristic orbits, it follows from *Bendixson's Formula*

$$I_0(X) = 1 + \frac{e - h}{2}.$$



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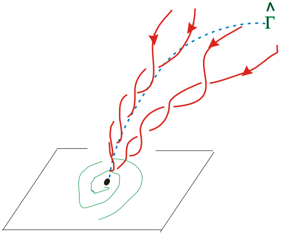

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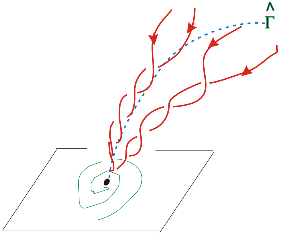

- **Brunella's Theorem, 1998**: If $\mathrm{Sing}(X) = \{0\}$, there always exists a characteristic orbit.

General dimension n

Moussu's Theorem, 1997: If $X = \nabla f$ is the gradient of an analytic function $f : (\mathbb{R}^n, 0) \to \mathbb{R}$ then X has $\Gamma_{\mathbb{R}}$.

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Let X be a real analytic vector field at $(\mathbb{R}^3, 0)$. If X has a non-constant analytic *first integral f* (i.e. $df(X) = 0$) then X has a $\widehat{\Gamma}_{\mathbb{R}}$.

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Related results of vector fields tangent to a foliation (a *flag of foliations*).

- **Lins Neto-Cerveau, 2017** If X is holomorphic at $(\mathbb{C}^3, 0)$ and tangent to a (singular) codimension 1 foliation (i.e. $\omega(X) = 0$ for some integrable 1-form ω) then X has a $\Gamma_{\mathbb{C}}$.

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- **Cano and Roche, 2014**: If X is analytic at $(\mathbb{K}^3, 0)$ and tangent to a foliation then it admits a *reduction of singularities* by blow-ups.

About convergence of the formal separatrix

Include Risler's example in a family of planar vector fields

$$X = (y^2 + x^4 + \mathbf{z^2})\frac{\partial}{\partial x} + A(x,y)\frac{\partial}{\partial y}.$$

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The result is only valid in dimension three:

- If $n = 2m$ is even, consider a polycenter $X = X_1 + \cdots + X_m$, where $X_j = -y_j\partial_{x_j} + x_j\partial_{y_j}$.

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- If $n = 2m + 1 \geq 5$ is odd, consider $X = X_1 + \cdots + X_{m-1} + Y$, where X_j is a center and Y is Gomez-Mont and Luengo's example.

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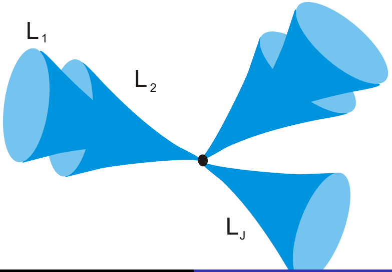

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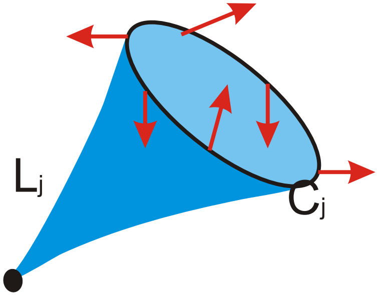

Ideas of the proof. Generalizing Bendixson's formula

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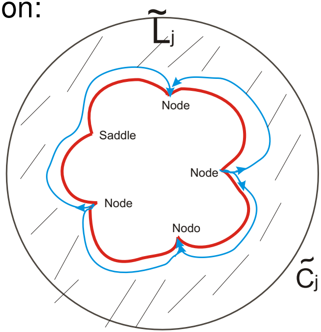
The proof uses the same argument of existence of sectors and Bendixson's formula inside L_j, using that $(\overline{L_j}, 0) \simeq (\mathbb{R}^2, 0)$.
By a process of reduction of singularities of $\overline{L_j}$, together with a reduction of singularities of the foliation generated by $X|_{L_j}$, we have the situation:



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End of the proof.- We finish by showing that $I_{C_{j_0}}(X) = 0$ for such j_0 (by "pushing" the vector field $X|_{L_{j_0}}$ near C_{j_0} to a nearby simply connected fiber).

Proof of the proposition

After an embedded resolution of singularities $\pi : M \to \mathbb{R}^3$ of the function f (with real oriented blow-ups), we have the situation