# Qualitative studies of some biochemical models

Valery Romanovski

University of Maribor
and
CAMTP – Center for Applied Mathematics and Theoretical Physics

AQTDE, Castro Urdiales
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## Outlines:

- Introduction
- Some basics from the elimination theory
- Applications of the elimination theory to detecting Hopf bifurcations
- Invariant surfaces
- Limit cycles in a three dimensional model

References:


- Y. Li, V.G. Romanovski, Hopf bifurcations in a Predator-Prey Model with an Omnivore, preprint, 2019.
- Y. Xia, M. Grašič, W. Huang and V. G. Romanovski, Limit Cycles in a Model of Olfactory Sensory Neurons, *International Journal of Bifurcation and Chaos,* Vol. 29, No. 3 (2019) 1950038.
- V. Antonov, W. Fernandes, V. G. Romanovski and N. L. Shcheglova, First integrals of the May-Leonard asymmetric system, *Mathematics*, vol. 7, no. 3 (2019) 1-15.

# Predator-prey model

$$\frac{dx}{dt} = x(\alpha - \beta y), \frac{dy}{dt} = -y(\gamma - \delta x) \tag{1}$$

- $y$ is the number of some predator;
- $x$ is the number of its prey;
- $\frac{dx}{dt} = \dot{x}$ and $\frac{dy}{dt} = \dot{y}$ represent the growth of the two populations against time $t$.

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- $x$ is the number of its prey;
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- System (1) is called Lotka-Volterra system

# May-Leonard model

May and Leonard (SIAM J. Appl. Math., 1975):

$$\begin{aligned} \dot{x} =& x(1 - x - \alpha y - \beta z), \\ \dot{y} =& y(1 - \beta x - y - \alpha z), \\ \dot{z} =& z(1 - \alpha x - \beta y - z). \end{aligned} \tag{2}$$

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**Some studies on classical May-Leonard system:**

- May and Leonard (1975), dynamic aspects;
- Schuster, Sigmund and Wolf (1979), dynamic aspects;
- Leach and Miritzis (2006), first integrals;
- Blé, Castellanos, Llibre and Quilantán (2013), integrability.

# May-Leonard asymmetric model

$$\begin{aligned} \dot{x} =& x(1 - x - \alpha_1 y - \beta_1 z), \\ \dot{y} =& y(1 - \beta_2 x - y - \alpha_2 z), \\ \dot{z} =& z(1 - \alpha_3 x - \beta_3 y - z). \end{aligned} \tag{3}$$

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**Some studies on May-Leonard asymmetric system:**

- Chi, Hsu and Wu (1998), dynamic aspects;
- van der Hoff, Greeff and Fay (2009), dynamic aspects;
- Antonov, Dolićanin, R. and Tóth (2016), periodic solutions, first integrals.

Chi, Hsu and Wu (SIAM J. Appl. Math. 1998) studied (3) under assumptions

$$0 < \alpha_i < 1 < \beta_i \ (1 \le i \le 3). \tag{4}$$

$$A_i = 1 - \alpha_i, \quad B_i = \beta_i - 1, \quad (1 \le i \le 3).$$

**Chi, Hsu and Wu showed:**

under (4) system (3) has a unique interior equilibrium $P$, which is locally asymptotically stable if $A_1A_2A_3 > B_1B_2B_3$, and if $A_1A_2A_3 < B_1B_2B_3$, then $P$ is a saddle point with a one-dimensional stable manifold. They also have shown that if $A_1A_2A_3 \neq B_1B_2B_3$, then the system does not have periodic solutions, and if

$$A_1A_2A_3 = B_1B_2B_3, \tag{5}$$

then there is a family of periodic solutions.

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$$\dot{x} = x(1 - bx - y - z),\ \dot{y} = y(-c + x),\ \dot{z} = z(-e + fx + gy - \beta z). \quad (6)$$

$x$ – the density of prey, $y$ – the density of its predator, $z$ – of the scavenger population. $b$ is the carrying capacity of the prey, $\beta$ is of the scavenger, $c$ is the death rate of the predator in the absence of prey, $e$ is the death rate of the scavenger in the absence of its food ($y$ and $x$), $f$ is the efficiency that $z$ preys upon $x$, $g$ is the degree of efficiency that the scavenger benefits from carcasses of predator $y$.

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In all models:

- Right hand sides are polynomial or rational functions
- Depend on many parameters

# Elimination of variables

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The variety of the ideal $I=\langle f_1,\dots,f_m\rangle\subset k[x_1,\dots,x_n]$ in $k^n$, denoted $\mathbf{V}(I)$, is the zero set of all polynomials of $I$,

$$\mathbf{V}(I)=\{A=(a_1,\dots,a_n)\in k^n | f(A)=0 \quad \text{for all } f\in I\},$$

where $k$ is a field, e.g. $=\mathbb{Q},\mathbb{R}\ \mathbb{C}$.
We want to eliminate $x_1,\dots,x_\ell$ $(\ell<n)$ from $f_1(x_1,\dots,x_n)=\cdots=f_m(x_1,\dots,x_n)=0$.
For an ideal $I$ in $k[x_1,\dots,x_n]$ we denote by $\mathbf{V}(I)$ its variety. Let us fix $\ell\in\{0,1,\dots,n-1\}$. The *$\ell$-th elimination ideal* of $I$ is the ideal $I_\ell=I\cap k[x_{\ell+1},\dots,x_n]$. Any point $(a_{\ell+1},\dots,a_n)\in\mathbf{V}(I_\ell)$ is called a *partial solution* of the system $\{f=0 : f\in I\}$.

The projection of a variety in $k^n$ onto $k^{n-\ell}$ is not necessarily a variety.

**Theorem (Closure Theorem)**

*Let $V = \mathbf{V}(f_1, \dots, f_s)$ be an affine variety in $\mathbb{C}^n$ and let $I_\ell$ be the $\ell$-th elimination ideal for the ideal $I = \langle f_1, \dots, f_s \rangle$. Then $\mathbf{V}(I_\ell)$ is the smallest affine variety containing $\pi_\ell(V) \subset \mathbb{C}^{n-\ell}$ (that is, $\mathbf{V}(I_\ell)$ is the Zariski closure of $\pi_\ell(V)$).*

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Elimination "by hand":

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The reduced GB of $I = \langle xy - 1, xz - 1\rangle$ with lex $x > y > z$ is $\{xz - 1, y - z\}$. $\Longrightarrow I_1 = \langle y - z\rangle$. $\Longrightarrow \mathbf{V}(I_1)$ is the line $y = z$. Partial solutions are $\{(a, a) : a \in \mathbb{C}\}$. $(a, a)$ for which $a \neq 0$ can be extended to $(1/a, a, a)$, except of $(0, 0)$.

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**Theorem (Extension Theorem)**

*Let $I = \langle f_1, \ldots, f_s\rangle$ be a nonzero ideal in the ring $\mathbb{C}[x_1, \ldots, x_n]$ and let $I_1$ be the first elimination ideal for $I$. Write the generators of $I$ in the form $f_j = g_j(x_2, \ldots, x_n)x_1^{N_j} + \tilde{g}_i$, where $N_j \in \{\mathbb{N} \cup 0\}$, $g_j \in \mathbb{C}[x_2, \ldots, x_n]$ are nonzero polynomials, and $\tilde{g}_j$ are the sums of terms of $f_j$ of degree less than $N_j$ in $x_1$. Consider a partial solution $(a_2, \ldots, a_n) \in \mathbf{V}(I_1)$. If $(a_2, \ldots, a_n) \notin \mathbf{V}(g_1, \ldots, g_s)$, then there exists $a_1$ such that $(a_1, a_2, \ldots, a_n) \in \mathbf{V}(I)$.*

# Conditions for existence of Hopf bifurcations

$$\dot{x} = x(1 - bx - y - z), \;\; \dot{y} = y(-c + x), \;\; \dot{z} = z(-e + fx + gy - \beta z). \quad (6)$$

System (6) has 6 equilibrium points, but all coordinates are positive only at $A(x_0, y_0, z_0)$,

$$x_0 = c, \;\; y_0 = -\frac{b\beta c - \beta + cf - e}{\beta + g}, \;\; z_0 = \frac{c(f - bg) - e + g}{bet + g}. \quad (7)$$

The Jacobian at $A$ is

$$J = \begin{pmatrix} -bc & -c & -c \\ \frac{-bc\beta+\beta+e-cf}{\beta+g} & 0 & 0 \\ \frac{f(-e+g+c(f-bg))}{\beta+g} & \frac{g(-e+g+c(f-bg))}{\beta+g} & \frac{\beta(e-cf+bcg-g)}{\beta+g} \end{pmatrix}. \quad (8)$$

The eigenvalues of $J$ are complicated. The characteristic polynomial of $J$:

$$p(u) = \frac{1}{\beta+g}((-\beta-g)u^3+(\beta(e-cf-g)+bc(\beta(-1+g)-g))u^2+ (c(e(-1+f)+f(c-cf-g+bcg)+\beta(-1+b(c+e-cf-g)+b^2cg)))u - c(\beta(bc-1)-e+cf)(e-cf-g+bcg). \quad (9)$$

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Let $u_1 = -b_0$ be a real root of $p(u)$.
Thus, $p(u)$ can be written in the form

$$\tilde{p}(u) = -(u+b_0)(u^2+w^2) \quad (10)$$

if two eigenvalues of $J$ are pure imaginary ($u_{1,2} = \pm iw$). Equating the coefficients of $u$ on both sides of $p(u) = \tilde{p}(u)$:

$$
\begin{aligned}
&bc(\beta(-g)+\beta+g)+b_0(\beta+g)+\beta(cf-e+g)=0,\\
&\beta\left(bc^3(bg-f)+c^2(be-2bg+f)+b_0w^2+c(g-e)\right)+\\
&c^3f(bg-f)+c^2(-beg+2ef-fg)+b_0gw^2+ce(g-e)=0,\\
&\beta\left(bc^2(bg-f+1)+c(be-bg-1)+w^2\right)+\\
&c^2f(bg-f+1)+gw^2+c(e(f-1)-fg)=0.
\end{aligned}
\tag{11}
$$

$p(u)$ can be represented as $\tilde{p}(u)=-(u+b_0)(u^2+w^2)$ only for those values of parameters of (6) for which system (11) has a solution. To find such values of parameters we eliminate from (9) $b_0, w$.

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• $w$ should be different from zero.

We add to (11) the equation $1-vw=0$, where $v$ is a new variable, and then eliminate $b_0, w, v$.

We compute in $\mathbb{Q}[v, w, b_0, b, f, g, \beta, e, c]$ a Gröbner basis $\tilde{G}$ (consists of 30 polynomials) of the ideal with respect to the lexicographic term order with $v \succ w \succ b_0 \succ b \succ f \succ g \succ \beta \succ e \succ c$ and find that the third elimination ideal is $\langle F \rangle$ generated by

$$F = b^3\beta c^2(\beta(-1+g)-g)g+(e-cf-g)(\beta f(e-cf)+(\beta+e-(\beta+c)f)g)
b(cg(e(1-f+g)+f(c(-1+f-g)+g))+\beta^2(c^2f^2+(e-g)^2+c(1-2ef+2fg
\beta c(cf(-1+f+g-2fg)+g(1+f+2g-2fg)+e(1-g+f(-1+2g))))-
(b^2c(cfg^2+\beta^2(c+e-cf-g-2eg+2cfg+2g^2)+\beta g(e-g+c(1+g-fg)))). \quad (12$$

Denote by $D$ the discriminant $p(u)$.

**Theorem**

*If all the coefficients of* (6) *and the coordinates of* $A$ *are positive, then* $J$ *has a pair of pure imaginary eigenvalues if and only if* $F = 0$ *and* $D < 0$.

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Proof. By the Closure Theorem for "almost all" values of parameters $b, f, g, \beta, e, c$ satisfying the condition $F(b, f, g, \beta, e, c) = 0$ our system has a solution. However it can happen that for some values of parameters it does not hold.

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Proof. By the Closure Theorem for "almost all" values of parameters $b, f, g, \beta, e, c$ satisfying the condition $F(b, f, g, \beta, e, c) = 0$ our system has a solution. However it can happen that for some values of parameters it does not hold. We show that under the conditions of the theorem every solution of $F(b, f, g, \beta, e, c) = 0$ can be extended to a complete solution.

The Gröbner basis $\tilde{G}$ contains the polynomials

$\tilde{g}_1 = (g+\beta)b_0 - bg\beta c + bgc + b\beta c + f\beta c + g\beta - \beta e$

$\tilde{g}_2 = (g+\beta)w^2 + b^2 g\beta c^2 + bfgc^2 - bf\beta c^2 - bg\beta c + b\beta ec + b\beta c^2 - f^2c^2 - fgc + fec + fc^2 - \beta c - ec,$

$\tilde{g}_3 = c(-\beta c + b\beta c^2 + \beta e - ce + e^2 - \beta cf + c^2 f - cef - \beta g + b\beta cg - eg + bceg)v + \beta cw + b\beta cw - \beta ew + \beta cfw + \beta gw + cgw + bcgw - b\beta cgw$

$\tilde{g}_4 = (c(\beta+e)(-e+cf)^2(\beta+g)^2)v + h_4(\beta, c, e, f, g, b, w),$

where $h_4$ has a long expression.

**Theorem (Extension Theorem)**

*Let $I = \langle f_1, \ldots, f_s \rangle$ be a nonzero ideal in the ring $\mathbb{C}[x_1, \ldots, x_n]$ and let $I_1$ be the first elimination ideal for $I$. Write the generators of $I$ in the form $f_j = g_j(x_2, \ldots, x_n)x_1^{N_j} + \tilde{g}_i$, where $N_j \in \{\mathbb{N} \cup 0\}$, $g_j \in \mathbb{C}[x_2, \ldots, x_n]$ are nonzero polynomials, and $\tilde{g}_j$ are the sums of terms of $f_j$ of degree less than $N_j$ in $x_1$. Consider a partial solution $(a_2, \ldots, a_n) \in \mathbf{V}(I_1)$. If $(a_2, \ldots, a_n) \notin \mathbf{V}(g_1, \ldots, g_s)$, then there exists $a_1$ such that $(a_1, a_2, \ldots, a_n) \in \mathbf{V}(I)$.*

- The coefficient of $b_0$ in $\tilde{g}_1$ does not vanish for the positive values of parameters, by the Extension Theorem (ET) every positive solution $(\hat{b}, \hat{f}, \hat{g}, \hat{\beta}, \hat{e}, \hat{c})$ of $F = 0$ can be extended to $(\hat{b}_0, \hat{b}, \hat{f}, \hat{g}, \hat{\beta}, \hat{e}, \hat{c})$ in the variety of $J_2$. From the form of $\tilde{g}_1$ $\Longrightarrow$ $\hat{b}_0$ is real.

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- ($\tilde{g}_2$ and the ET) $\Longrightarrow$ the partial solution $(\hat{b}_0, \hat{b}, \hat{f}, \hat{g}, \hat{\beta}, \hat{e}, \hat{c})$ can be extended to a point $(\hat{w}, \hat{b}_0, \hat{b}, \hat{f}, \hat{g}, \hat{\beta}, \hat{e}, \hat{c})$ in the variety of $J_1$.

$\tilde{g}_3 = c(-\beta c + b\beta c^2 + \beta e - ce + e^2 - \beta cf + c^2 f - cef - \beta g + b\beta cg - eg + bceg)v + \beta cw + b\beta cw - \beta ew + \beta cfw + \beta gw + cgw + bcgw - b\beta cgw$
$\tilde{g}_4 = (c(\beta + e)(-e + cf)^2(\beta + g)^2)v + h_4(\beta, c, e, f, g, b, w)$.
($\tilde{g}_3$ , $\tilde{g}_4$ and the ET) $\Longrightarrow$ the partial solution $(\hat{w}, \hat{b}_0, \hat{b}, \hat{f}, \hat{g}, \hat{\beta}, \hat{e}, \hat{c})$ can be extended to a complete solution unless $e - cf = bc - 1 = 0$. However in such case $A$ has coordinates $(c, 0, 0)$, which contradicts our assumption that all coordinates of $A$ are positive.

Thus, if the parameters of (6) satisfy $F = 0$, then

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- A cubic polynomial with real coefficients has a pair of complex conjugate roots if and only if its discriminant is negative. Since $D < 0$ if the roots are $\alpha \pm i\gamma$, $p(u) = (u + b_0)(u^2 - 2\alpha u + \alpha^2 + \gamma^2)$ we conclude $p(u)$ has two complex roots if $w = \gamma$ is real, in which case the roots are $u_{1,2} = \pm iw$. $\square$

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*Remark.* Elimination ideals for studying such problem were used recently in N. Kruff, S. Walcher. Coordinate-independent criteria for Hopf bifurcations. Discrete & Continuous Dynamical Systems, doi: 10.3934/dcdss.2020075

The condition $F = 0$, $D < 0$ is rather general. We can use `Reduce`. of MATHEMATICA for some simplification.
*Example.* In (6) let us set $e = 5, g = 3, \beta = 2$ and $c = 4$. Then

```
Reduce[F == 0 && D < 0 && b >0 && f > 0 && y0>0 && z0 >0,
```

yields

$$\frac{1}{2} < f < \frac{1}{4}\left(\sqrt{46} - 2\right)$$

and $b$ is a root of the cubic equation, with respect to $\alpha$, $21 - 50f + 8f^2 + 16f^3 + (180 - 68f - 24f^2)\alpha + (-168 - 88f)\alpha^2 + 48\alpha^3 = 0$. If these conditions are fulfilled then the corresponding system (6) has a center manifold passing through the point $A$ and the Jacobian at $A$ has a pair of pure imaginary eigenvalues.

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\begin{aligned}
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\dot{x}_2 &= w_1 x_1 + h.o.t. \\
\dot{x}_3 &= -w_2 x_4 + h.o.t. \\
\dot{x}_2 &= w_2 x_3 + h.o.t. \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
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$$\tilde{p}(u) = a(p_k u^k + \cdots + p_1 u + p_0)(u^2 + w_1^2)(u_2 + w_2^2)$$

Eliminate $p_k, \ldots, p_1, p_0, w_1, w_2$.

To understand the dynamics of a model described by systems of ODEs it is important to know:

- Singular points
- First integrals
- Invariant surfaces

# Invariant surfaces in polynomial systems

$$\dot{x} = P(x,y,z), \quad \dot{y} = Q(x,y,z), \quad \dot{z} = R(x,y,z), \tag{13}$$

the maximal degree of polynomials $P, Q, R$ is $m$.

**Definition**

A surface $H = 0$ ($H$ is a polynomial) is an invariant surface of (13) iff

$$\mathcal{X}(H) := \frac{\partial H}{\partial x}P + \frac{\partial H}{\partial y}Q + \frac{\partial H}{\partial z}R = K\,H \tag{14}$$

$K$ – a polynomial of degree at most $m-1$.
$H$ – a Darboux polynomial of (13)
$K$ – a cofactor.

# Invariant planes in May-Leonard system

- Problem: find all invariant planes of May-Leonard system

  $\dot{x} = x(1-x-\alpha_1 y-\beta_1 z),\ \dot{y} = y(1-\beta_2 x-y-\alpha_2 z),\ \dot{z} = z(1-\alpha_3 x-\beta_3 y-z).$

$$H(x, y, z) = h_{000} + h_{100}x + h_{010}y + h_{001}z.$$

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**Theorem**

*System (3) has an invariant plane passing through the origin and different from the planes $x=0$, $y=0$, and $z=0$ if one of the following conditions holds:*

*1)* $\alpha_2 = \beta_1,\ \beta_2 \neq 1,$

*2)* $\alpha_1 = \beta_3,\ \alpha_3 \neq 1,$

*3)* $\alpha_3 = \beta_2,\ \beta_3 \neq 1,$

*4)* $\beta_3 = \frac{2-\alpha_1-\alpha_2+\alpha_1\alpha_2-\alpha_3+\alpha_1\alpha_3+\alpha_2\alpha_3-\alpha_1\alpha_2\alpha_3-\beta_1-\beta_2+\beta_1\beta_2}{(\beta_1-1)(\beta_2-1)},$

*5)* $\beta_1 = \alpha_3 = 1,\ (-1+\alpha_1)(-1+\beta_3) \neq 0,$

*6)* $\beta_2 = 1, \alpha_1\ (-1+\alpha_2)(-1+\beta_1) \neq 0.$

Proof. We look for an invariant plane in the form

$$H(x,y,z) = h_{100}x + h_{010}y + h_{001}z. \tag{15}$$

with the corresponding cofactor

$$K(x,y,z) = c_0 + c_1x + c_2y + c_3z. \tag{16}$$

Substituting $H(x,y,z)$ and $K(x,y,z)$ into

$$\mathcal{X}(H) = KH$$

and comparing the coefficients of similar terms:

$$g_1 = g_2 = \cdots = g_9 = 0 \tag{17}$$

where

$$\begin{aligned}
g_1 =& h_{001} - c_0 h_{001}, \\
g_2 =& -h_{001} - c_3 h_{001}, \quad g_3 = h_{010} - c_0 h_{010}, \\
g_4 =& -h_{010} - c_2 h_{010}, \quad g_5 = -\beta_3 h_{001} - c_2 h_{001} - \alpha_2 h_{010} - c_3 h_{010}, \\
g_6 =& h_{100} - c_0 h_{100}, \quad g_7 = -h_{100} - c_1 h_{100}, \\
g_8 =& -\beta_2 h_{010} - c_1 h_{010} - \alpha_1 h_{100} - c_2 h_{100},
\end{aligned} \tag{18}$$

We are looking for planes passing through the origin $\Rightarrow h_0 = 0$. Denote by $J = \langle g_1, g_2, \ldots, g_9 \rangle$ the ideal generated by polynomials of system (18). To obtain the conditions for existence of invariant planes we have to eliminate from (18) the variables $h_i$ and $c_i$, that is, to compute the 7-th elimination ideal of $J$ in the ring

$$\mathbb{Q}[h, c, \alpha, \beta] := \mathbb{Q}[h_{100}, h_{010}, h_{001}, c_0, c_1, c_2, c_3, \alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3].$$

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We impose the condition that polynomial (15) is not a constant and it is different from $H_1, H_2,\ H_3$.
$H(x, y, z) = h_{100}x + h_{010}y + h_{001}z$
defines a plane different from $x = 0$, $y = 0$, $z = 0$ if at least two from the coefficients $h_{100}, h_{010}, h_{001}$ are different from zero.

In the polynomial form:

- $1 - wh_{100}h_{010} = 0,$
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with $w$ being a new variable.

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Denote

- this elimination ideal by $J_7^{(1)}$;
- its variety by $V_1 = \mathbf{V}(J_7^{(1)})$.

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The output gives the 6 conditions of the theorem.

Invariant surfaces of degree 2: $H(x, y, z) = 1+h_{100}x+h_{010}y+h_{001}z+h_{200}x^2+h_{110}xy+h_{101}xz+h_{020}y^2+h_{011}yz+h_{002}z^2.$

Invariant surfaces of degree 2: $H(x,y,z) = 1+h_{100}x+h_{010}y+h_{001}z+h_{200}x^2+h_{110}xy+h_{101}xz+h_{020}y^2+h_{011}yz+h_{002}z^2$.
The computational procedure yields 88 conditions on the parameters $\alpha_i, \beta_i$ of the May-Leonard asymmetric system for existence of an invariant surface of degree two not passing through the origin.

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We say, that two conditions for existence of invariant surfaces are *conjugate* if one can be obtained from another by means of one of transformations:

$$\alpha_1 \to \alpha_3,\ \beta_1 \to \beta_3,\ \alpha_2 \to \alpha_1,\ \beta_2 \to \beta_1,\ \alpha_3 \to \alpha_2,\ \beta_3 \to \beta_2,$$
$$\alpha_1 \to \alpha_2,\ \beta_1 \to \beta_2,\ \alpha_2 \to \alpha_3,\ \beta_2 \to \beta_3,\ \alpha_3 \to \alpha_1,\ \beta_3 \to \beta_1,$$
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**Theorem**

*System (3) has an irreducible invariant surface not passing through the origin if one of the following conditions or conjugated to it holds:*

1. $\alpha_2 = \beta_1 = \beta_2 - 1/2 = \alpha_1 - 3 = 0$
2. $\alpha_2 = \beta_1 = \beta_2 - 3 = \alpha_1 - 3 = 0$
3. $\beta_3 = \beta_1 = \alpha_3 + \beta_2 - 1 = \alpha_2 + 1 = \alpha_1 - \alpha_3 - 1 = 0$
4. $\beta_3 = \beta_1 = \alpha_3 + 1 = \beta_2 - 3 = \alpha_2 + 1 = \alpha_1 - 1/2 = 0$
5. $\beta_3 = \beta_1 = \alpha_3 - 3 = \beta_2 - 3 = \alpha_2 - 3/2 = \alpha_1 + 1 = 0$
6. $\beta_1 = \beta_3 - 3 = \alpha_3 - 3 = \beta_2 - 1 = \alpha_2 - 1/2 = \alpha_1 - 1 = 0$
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10. $\beta_1 = \beta_3 - 1/2 = \alpha_3 - 2 = \beta_2 - 3 = \alpha_2 - 3/2 = \alpha_1 - 1/2 = 0$
11. $\beta_1 = \alpha_3 = \beta_2 - \beta_3 - 1 = \alpha_2 + \beta_3 - 2 = \alpha_1 + \beta_3 - 1 = 0$
12. $\beta_1 = \beta_3 - 3 = \alpha_3 + \beta_2 - 4 = \alpha_2 + 1 = \alpha_1 - \alpha_3 + 2 = 0$
13. $\beta_3 - 1/2 = \alpha_3 - 1/2 = \alpha_2 - 3 = \beta_1 - 3 = \alpha_1 + \beta_2 - 2 = 0$
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For finding the surfaces of the second degree the computations over the field $\mathbb{Z}_p$ were used.
$H(x,y,z) =$
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- Modular computations:

Choose a prime number $p$ and do all calculations modulo $p$, that is, in $\mathbb{Z}_p = \mathbb{Z}/p$.

Reconstruct (lift) $r/s \in \mathbb{Q}$ given its image $t \in \mathbb{Z}_p$.
Algorithm by P. Wang ($\lfloor \cdot \rfloor$ stands for the floor function):
Step 1. $u = (u_1, u_2, u_3) := (1, 0, m),\ v = (v_1, v_2, v_3) := (1, 0, c)$
Step 2. While $\sqrt{m/2} \leq v_3$ do
$\{q := \lfloor u_3/v_3 \rfloor,\ r := u - qv, u := v,\ v := r\}$
Step 3. If $|v_2| \geq \sqrt{m/2}$ then error()
Step 4. Return $v_3, v_2$
Given an integer $c$ and a prime number $p$ the algorithm produces integers $v_3$ and $v_2$ such that $v_3/v_2 \equiv c \pmod{p}$, that is, $v_3 = v_2 c + pt$ with some $t$. If such a number $v_3/v_2$ does need not exist. If this is the case, then the algorithm returns "error()".

# Example

$$
\begin{aligned}
f_1 =& 8x^2y^2 + 5xy^3 + 3x^3z + x^2yz,\\
f_2 =& x^5 + 2y^3z^2 + 13y^2z^3 + 5yz^4,\\
f_3 =& 8x^3 + 12y^3 + xz^2 + 3,\\
f_4 =& 7x^2y^4 + 18xy^3z^2 + y^3z^3.
\end{aligned}
\tag{19}
$$

Under the lexicographic ordering with $x > y > z$ a Groebner basis for $I$ is

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G = \{x, y^3 + \frac{1}{4}, z^2.\} \tag{20}
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Rational reconstruction yields (20).

Calculations for the case

$$H(x,y,z) = h_{100}x + h_{010}y + h_{001}z + h_{200}x^2 + h_{110}xy + h_{101}xz + h_{020}y^2 + h_{011}y$$

turned out computationally unfeasible even over $\mathbb{Z}_p$.

# Darboux first integral

Let $n$ be an arbitrary natural number, $H_i$ be algebraic invariant surfaces of

$$\dot{x} = P(x,y,z), \quad \dot{y} = Q(x,y,z), \quad \dot{z} = R(x,y,z), \tag{22}$$

with the corresponding cofactors $K_i$ ($i = 1, 2, \ldots, n$).
A *Darboux first integral* of system (22) is a function of the form

$$\Psi(x,y,z) = \prod_{i=1}^{n} H_i(x,y,z)^{\lambda_i},$$

where

$$\sum_{i=1}^{n} \lambda_i K_i = 0 \tag{23}$$

and $\lambda_i$ are some constants.

# Darboux first integral

Let $n$ be an arbitrary natural number, $H_i$ be algebraic invariant surfaces of

$$\dot{x} = P(x,y,z), \quad \dot{y} = Q(x,y,z), \quad \dot{z} = R(x,y,z), \tag{22}$$

with the corresponding cofactors $K_i$ $(i = 1, 2, \dots, n)$.
A *Darboux first integral* of system (22) is a function of the form

$$\Psi(x,y,z) = \prod_{i=1}^{n} H_i(x,y,z)^{\lambda_i},$$

where

$$\sum_{i=1}^{n} \lambda_i K_i = 0 \tag{23}$$

and $\lambda_i$ are some constants.
Using the obtained invariant surface a number of Darboux first integrals of the May-Leonard system was constructed.

## Periodic solutions in the May-Leonard system

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- but they mentioned that the family arises as the result of Hopf bifurcations.
- In fact there is another mechanism for existence of the family.

Under condition 4) of Theorem 2 we have:

$$\beta_3 = \frac{2-\alpha_1-\alpha_2+\alpha_1\alpha_2-\alpha_3+\alpha_1\alpha_3+\alpha_2\alpha_3-\alpha_1\alpha_2\alpha_3-\beta_1-\beta_2+\beta_1\beta_2}{(\beta_1-1)(\beta_2-1)},$$

$$H_4 = -x+\alpha_3 x+\beta_2 x-\alpha_3\beta_2 x+y-\alpha_1 y-\alpha_3 y+\alpha_1\alpha_3 y+z-\beta_1 z-\beta_2 z+\beta_1\beta_2 z \tag{24}$$

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The Darboux first integral

$$\Psi = x^{\alpha_1} y^{\alpha_2} z^{\alpha_3} H_4^{\alpha_4} \tag{25}$$

$\alpha_1(-1+\beta_1)$ $\alpha_1(-1+\beta_1)(-1+\beta_2)$ $\alpha_1(1-\alpha_2+\alpha_2\alpha_3-\alpha_3\beta_1-\beta_2+\beta_1\beta_2)$

For simplicity we take the parameters
$\beta_1 = 1/4,\ \beta_2 = 11/10,\ \alpha_1 = 5/4,\ \alpha_2 = 4/5,\ \alpha_3 = 3/2, \beta_3 = 2/3.$
In this case system (3)

$$\dot{x} = x(-x-\frac{5y}{4}-\frac{z}{4}+1),\ \dot{y} = y(-\frac{11x}{10}-y-\frac{4z}{5}+1),\ \dot{z} = z(\frac{3x}{2}+\frac{2y}{3}+z-1). \tag{26}$$

and the singular point $P$ has the coordinates

$$x_0 = 1/3,\ y_0 = 1/2,\ z_0 = 1/6.$$

**Proposition**

System (26) has a family of periodic solutions in a neighborhood of the singular point $P(1/3, 1/2, 1/6)$.

# Proof:

Moving the origin to the singular point by the substitution

$$u = x - x_0, v = y - y_0, \; w = z - z_0$$

and then performing the linear change of coordinates

$$\begin{aligned} u =& 2X + 370Y/249, \\ v =& 3X - Y - 15\sqrt{10}Z/83, \\ w =& X + 1/249(-235Y + 77\sqrt{10}Z) \end{aligned}$$

we obtain from (26)

$$\dot{X} = -X - 6X^2 + \frac{10450Y^2}{268671} + \frac{38048\sqrt{10}YZ}{806013} - \frac{10450Z^2}{268671},$$

$$\dot{Y} = \frac{Z}{3\sqrt{10}} - 6XY + \sqrt{\frac{2}{5}}XZ - \frac{2090Y^2}{39923} + \frac{16979\sqrt{\frac{2}{5}}YZ}{39923} + \frac{2090Z^2}{39923},$$

$$\dot{Z} = -\frac{Y}{3\sqrt{10}} - \sqrt{\frac{2}{5}}XY - 6XZ + \frac{19187\sqrt{10}Y^2}{119769} + \frac{7730YZ}{119769} - \frac{19187\sqrt{10}Z^2}{119769}.$$

- By the Center Manifold Theorem $\exists$ an analytic center manifold $X = h(Y, Z)$ passing through $X = Y = Z = 0$.
- Expanding the first integral (25) into power series

$$\Psi(X, Y, Z) = Y^2 + Z^2 + h.o.t.$$

- $\Rightarrow$ in a neighborhood of the origin there exists a family of periodic orbits formed by the intersection of the graphs of $X = h(Y, Z)$ and $\Psi = c$ $(0 < c < c_0)$.

# Lyapunov functions on the center manifold

$$\dot{\mathbf{x}} = A\mathbf{x} + F(\mathbf{x}) = G(\mathbf{x}), \tag{27}$$

$\mathbf{x} = (x, y, z)$, the matrix $A$ has the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ and $\lambda_1 < 0$, $\lambda_2 = i\omega$, $\lambda_3 = -i\omega$, $F$ is a vector-function, which is analytic in a neighborhood of the origin and such that its series expansion starts from quadratic or higher terms, and $G(x) = (G_1(x), G_2(x), G_3(x))^T$.

By the Center Manifold Theorem the system has a center manifold defined by a function $x = f(y, z)$. After a linear transformation and rescaling of time system:

$$\begin{aligned} \dot{u} &= \quad -v + P(u, v, w) = \widetilde{P}(u, v, w) \\ \dot{v} &= \qquad u + Q(u, v, w) = \widetilde{Q}(u, v, s) \\ \dot{w} &= -\lambda w + R(u, v, w) = \widetilde{R}(u, v, w). \end{aligned} \tag{28}$$

## Theorem

Suppose that for (27) there exists a function

$$\Psi(\mathbf{x}) = \sum_{k+l+m=2}^{\infty} a_{klm} x^k y^l z^m \tag{29}$$

$$\mathcal{X}(\Psi) := \frac{\partial \Psi(x)}{\partial x} G_1(\mathbf{x}) + \frac{\partial \Psi(\mathbf{x})}{\partial y} G_2(\mathbf{x}) + \frac{\partial \Psi(\mathbf{x})}{\partial z} G_3 = g_1(y^2+z^2)^2 + g_2(y^2+z^2)^3 + \dots . \tag{30}$$

Let

$$x = f(y, z, \alpha^*) \tag{31}$$

be the center manifold of system (27) corresponding to the value $\alpha^*$ of parameters of the system and

$$q(\mathbf{x}, \alpha^*) = \sum_{k+l+m=2} a_{klm} x^k y^l z^m \tag{32}$$

Let $q_1(y,z,\alpha^*)$ be $q(\mathbf{x},\alpha^*)$ evaluated on (31). Assume that $q_1(y,z,\alpha^*)$ is positively defined quadratic form and

$$g_1(\alpha^*) = g_2(\alpha^*) = \cdots = g_k(\alpha^*) = 0, \qquad g_{k+1}(\alpha^*) \neq 0. \tag{33}$$

Then,
1) if $g_{k+1}(\alpha^*) < 0$, the corresponding system (27) has a stable focus at the origin on the center manifold, and if $g_{k+1}(\alpha^*) > 0$ then the focus is unstable.
2) if it is possible to choose perturbations of the parameters $\alpha$ in system (27) such that

$$|g_1(\alpha_k)| \ll |g_2(\alpha_{k-1})| \ll \ldots |g_k(\alpha_1)| \ll |g_{k+1}(\alpha^*)|, \tag{34}$$

$\alpha_{j+1}$ is arbitrary close to $\alpha_j$ and the signs of $g_s(\alpha_m)$ in (34) alternate, then system (27) corresponding to the parameter $\alpha_k$ has at least k limit cycles on the center manifold.

Proof. 1) Since $q_1$ is positively defined the function $\Psi$ restricted to the center manifold is positively defined in a small neighborhood of the origin. The derivative of $\Psi$ with respect to the vector field on the center manifold has the same sign as $g_{k+1}(\alpha^*)$. Thus, by the Lyapunov theorem the origin is a stable focus on the center manifold if $g_{k+1}(\alpha^*) < 0$ and unstable focus if $g_{k+1}(\alpha^*) > 0$.

2) Assume for determinacy that $g_{k+1}(\alpha^*) < 0$. Under the condition of the theorem the equality $\Psi(\mathbf{x}, \alpha^*) = c$ $(c \in (0, c_1])$ defines in a small neighborhood of the origin near the center manifold (31) a family of cylinders which are transversal to the center manifold. Let $C_1$ be the curve formed by the intersection of the cylinder $\Psi(\mathbf{x}, \alpha^*) = c_1$ and the center manifold $M(\alpha^*)$ of system (27) defined by (31). If $c_1$ is sufficiently small then $C_1$ is an oval on $M(\alpha^*)$ and the vector field is directed inside $C_1$, since

$$\mathcal{X}(\Psi(\mathbf{x}, \alpha^*)) = g_{k+1}(\alpha^*)(y^2 + z^2)^{k+2} + h.o.t$$

and $g_{k+1}(\alpha^*) < 0$.

By the assumption of the theorem there is $\alpha_1$ arbitrary close to $\alpha^*$ and such that $g_k(\alpha_1) > 0$. Then for some $c_2 < c_1$ the intersection of the cylinder $\Phi(\mathbf{x}, \alpha_1) = c_2$ ($c_2 \in (0, c_1]$) defines a curve $C_2$ on the center manifold $x = f(y, z, \alpha_1)$ such that the vector field of system (27) is directed outside of $C_2$ (since $g_k(\alpha_1) > 0$). Since the perturbation is arbitrary small the curve $C_1$ is transformed to a curve $C_1^{(1)}$ such that the vector field on $C_1^{(1)}$ still is directed inside the curve. Then by the Poincaré-Bendixon theorem there is a limit cycle on the center manifold $x = f(y, z, \alpha_1)$ in the ring bounded by $C_2$ and $C_1^{(1)}$. Continuing the procedure on the center manifold corresponding to a parameter $\alpha_k$ we obtain $k$ curves $C_1^{(k)}$, $C_2^{(k-1)}, \dots, C_k$, such that the the vector field on $C_1^{(k)}$ is directed inside the curve, the vector field on $C_2^{(k-1)}$ is directed outside of the curve, the vector field on $C_3^{(k-2)}$ is directed inside the curve and so on. Then, in each ring bounded by the curves $C_i^{(j)}$ system (27) corresponding to the parameter $\alpha_k$ has at least one limit cycle on the center manifold $x = f(y, z, \alpha_k)$. $\square$

We now investigate Hopf and degenerate Hopf bifurcations near the singular point $A$ of (6). We limit consideration to the case when one of the eigenvalues is equal to $-1$. For the characteristic polynomial $p(u)$ we have that $p(-1) = 0$ if

$$g = \frac{\beta(1+c)(-1+bc)(-1-e+cf) + c(e-cf)(1+e-(1+c)f)}{(-1+bc)(1+\beta(1+c)(-1+bc) - ce + c(1+c)f)} \qquad (35)$$

and the two other eigenvalues are

$$\lambda_{2,3} = \mu \pm \sqrt{\nu}, \qquad (36)$$

where

$$\mu = -\frac{c(bc-1)(b\beta(c+1) + cf - e + f - 1)}{2(\beta(c+1)(bc-1) + c(cf - e + f - 1))} \qquad (37)$$

$$\nu^2 = \frac{\nu_1}{\nu_2}, \qquad (38)$$

$$\nu_1 = c(bc-1)(\beta^2(c+1)^2(bc-1)(b(b+4)c-4) + 2\beta(c+1)(bc-1)(-c((b+4)e+$$
$$(b+4)c(c+1)f-2e) + c(-(c+1)f+e+1)(c(-f((b+4)c+b+3)+be+b+4e)+3e+$$

From (37) we see that $\lambda_{2,3}$ can be pure imaginary if $\mu = 0$, that is, if

$$f = \frac{-b\beta(c+1) + e + 1}{c+1}. \tag{39}$$

## Theorem

*Assume that for system* (6) *conditions* (35) *and* (39) *are fulfilled. Then the system has a center manifold $W$ passing through the equilibrium point $A$, and $A$ is a center or a focus for the flow of* (6) *restricted to $W$, if and only if*

$$\beta > 1 \wedge b > 0 \wedge \Bigg(\Bigg(b < \frac{e+1}{\beta c + \beta} \wedge c > 0 \wedge ((e > 0 \wedge e + 1 \leq \beta) \vee (\beta <$$

$$ce + c \leq \beta c + \beta))) \vee \left(c > \frac{\beta}{-\beta + e + 1} \wedge e + 1 > \beta \wedge b < \frac{1}{c}\right)\Bigg). \tag{40}$$

## Proof.

When conditions (35) and (39) hold, the eigenvalues of the Jacobian at $A$ are $-1$ and $\pm\sqrt{\nu}$.
The Jacobian has a pair of purely imaginary eigenvalues if $\nu < 0$.
To find such conditions we solve the the semialgebraic system

$$x_0 > 0 \wedge y_0 > 0 \wedge z_0 > 0 \wedge \beta > 0 \wedge g > 0 \wedge e > 0 \wedge c > 0 \wedge f > 0 \wedge b > 0 \wedge \nu < 0$$

where $\nu$ is defined by (38) and $x_0, y_0, z_0$ are the coordinates of the point $A$ defined by (7), with respect to the variables $\beta, e, c$ and $b$. Solving the system with `Reduce` of MATHEMATICA, we obtain the condition given in the statement of the theorem. Thus, under the condition the system has a center manifold passing through $A$ and $A$ is either a center or a focus on the center manifold. □

We move the origin to $A$ this point by performing the substitution $X = x - x_0, \ Y = y - y_0, \ Z = z - z_0$. Then, writing in the transformed system $x, y, z$ instead of $X, Y, Z$, we obtain

$$\begin{aligned}
\dot{x} &= -(c + x)(bx + y + z) = X(x, y, z),\\
\dot{y} &= -\frac{x(b\beta c - bc - \beta y - \beta + 1)}{\beta} = Y(x, y, z),\\
\dot{z} &= ((bc - \beta z - 1)((\beta - 1)x(bc - 1)(b\beta(c + 1) - e - 1) + \beta(y(b(\beta - 1)c(c +\\
&\quad + e + 1) + (\beta - 1)(c + 1)z(bc - 1))))/((\beta - 1)\beta(c + 1)(bc - 1)) = Z(x,
\end{aligned} \tag{41}$$

We look for

$$\Phi(\mathbf{x}) = \sum_{j+l+m=2}^{3} a_{jlm} x^j y^l z^m \tag{42}$$

such that

$$\mathcal{X}(\Phi) := \frac{\partial \Phi(x)}{\partial x} X(x, y, z) + \frac{\partial \Phi(\mathbf{x})}{\partial y} Y(x, y, z) + \frac{\partial \Phi(\mathbf{x})}{\partial z} Z(x, y, z) = g_1 (y^2 + z^2)^2 + O(||\mathbf{x}||^5). \tag{43}$$

The quadratic part of (42) is

$$\Phi_2 = \frac{1}{2} a_{101} \left( \gamma_1 y^2 + \gamma_2 x^2 + \gamma_3 xy + \gamma_4 yz + \gamma_5 z^2 + \gamma_6 xz \right), \qquad (44)$$

where

$$\gamma_1 = \frac{(\beta^2(c+1)(c(bc-1)-1) - \beta(c(2c(bc+b-1)-3)+e) + c(c+1)(bc}{(\beta-1)^2(c+1)(bc-1)^2}$$

$$\gamma_2 = \frac{\beta(c+1)((b-1)c-1) + c(c-e)}{\beta c(c+1)}$$

$$\gamma_3 = \frac{2(b(\beta-1)c(c+1)+e+1)}{(\beta-1)(c+1)(bc-1)}$$

$$\gamma_4 = \frac{2cyz}{bc-1}, \ \gamma_5 = \frac{c}{bc-1}, \ \gamma_6 = 2.$$

$a_{101}$ in (44) can be chosen any,
$g_1 = \frac{h_1(x,y,z) a_{101}}{h_2(x,y,z)}$, where $h_1$ and $h_2$ are long polynomials.

We now look for a series expansion of the center manifold of system (41)

$$x = \sum_{i+j=1}^{\infty} \alpha_{ij} y^i z^j = H(y,z) \tag{45}$$

$$\dot{x} - \dot{y}\frac{\partial H}{\partial y} - \dot{z}\frac{\partial H}{\partial z} = 0,$$

where the left-hand side is evaluated for $x$ as given by (45).
Computing the first two terms of the series expansion (45) we obtain

$$x = \frac{(e+1)y}{b(\beta-1)(c+1)} - \frac{z}{b} + h.o.t. \tag{46}$$

We substitute this expression into (44) obtaining

$$Q(y,z) = \frac{a_{101}\left(b\beta c^3 + b\beta c^2 - bc^3 + bc^2 e - \beta c^2 - 2\beta c - \beta + c^2 - ce\right)}{2b^2(\beta-1)^2\beta c(c+1)^3(bc-1)^2} q(x,y) \tag{47}$$

where
$q = ((1+e)^2 - bc(1+e)^2 + b^2\beta c(1+c)(\beta - c + \beta c + e))y^2 + 2(-1+\beta)(1+c)(-1+bc)(1+e)yz + (-1+\beta)^2(1+c)^2(1-bc)z^2.$

## Theorem

*If for some* $a_{101} \neq 0$ *and some chosen values* $\beta^*, b^*, c^*, e^*$ *of parameters* $\beta, b, c, e$ *of system* (41) *at least one of partial derivative of* $\mu$ *(defined by* (37)*) is not equal to zero, then:*
*(a) if the quadratic form* $Q(y, z)$ *is positive definite and* $g_1 < 0$*, then the corresponding system* (41) *admits a supercritical Hopf bifurcation,*
*(b) if* $Q(y, z)$ *is positive definite and* $g_1 > 0$ *then the system admits a subcritical Hopf bifurcation,*
*(c) if* $Q(y, z)$ *is negative definite and* $g_1 > 0$ *then the system admits a supercritical Hopf bifurcation,*
*(d) if* $Q(y, z)$ *is negative definite and* $g_1 < 0$ *then the system admits a subcritical Hopf bifurcation.*

To study the degenerate Hopf bifurcations of system (41) we need to compute the second focus quantity $g_2$.
We have to computed $g_2$ only for some particular values of the parameters. In order to perform symbolic computations we need to find rational values of parameters for which $g_1$ vanishes. After some computational experiments we found that if

$$\beta = c = 2, \quad e = 3 \tag{48}$$

the polynomial $h_1$ factors as

$$h_1 = (-4 + 21b)(15 + 26b + 56b^2). \tag{49}$$

**Theorem**

*There are systems (6) with two limit cycles in a neighborhood of the singular point at the origin.*

Proof. When $\beta = c = 2$, (47) takes the form

$$Q = \frac{a_{101}(2b(e+4) - e - 7)s(y,z)}{108(1-2b)^2 b^2}. \tag{50}$$

$$s(y,z) = \left(y^2 \left(12b^2(e+4) - 2b(e+1)^2 + (e+1)^2\right) \right. \\ \left. +6(2b-1)(e+1)yz + 9(1-2b)z^2\right).$$

Computing the leading principal minors of the quadratic form in the numerator of (50) we obtain

$\Delta_1 = -(8b - e + 2be = 7)(2b - 48b^2 - 2e + 4be - 12b^2 e - e^2 + 2be^2 - 1)a_{101}$

and $\Delta_2 = -108b^2(-1+2b)(4+e)(-7+8b-e+2be)^2 a_{101}^2$. By Sylvester's criterion the quadratic form (50) is positive definite if $\Delta_1 > 0$ and $\Delta_2 > 0$. Solving with Reduce of MATHEMATICA the semi-algebraic system $\Delta_1 > 0, \Delta_2 > 0, b > 0, e > 0$ with respect to $b, e$ and $a_{101}$ we find that the solution is $0 < b < \frac{1}{2} \wedge e > 0 \wedge a_{101} < 0$. Thus, setting $a_{101} = -1$ we have that the quadric form (50) is positive definite for any $e > 0$ and $0 < b < \frac{1}{2}$.

When condition (48) is satisfied and $b = \frac{4}{21}$, from (49) we have that $g_1 = 0$ and the computations yield

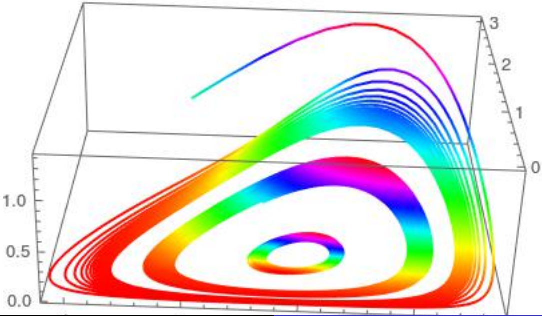
$$g_2 = -\frac{93395925504}{205676731273}.$$

Thus the singular point at the origin is a stable focus on the center manifold.
For $\beta = c = 2$ and $b = \frac{4}{21}$ we have

$$g_1 = \frac{37044(e-3)(e+4)\left(4277e^2 + 14776e + 20156\right)}{13(52e+271)\left(8281e^4 + 44772e^3 + 272728e^2 + 503784e + 1080004\right)}.$$

Then for $e > 3$ but sufficiently close to 3, $|g_1| \ll |g_2|$ and $g_1$ is negative, so a stable limit cycle bifurcates from the origin.
Under the perturbation the matrix of the linear approximation still has a pair of pure imaginary eigenvalues.

Computing $\frac{\partial \mu}{\partial f}$ (where $\mu$ is defined by (37)) we see that it is non-zero if $\beta = c = 2$ and $b = \frac{4}{21}$. Therefore, an unstable limit cycle bifurcates from the origin as the result of a Hopf bifurcation. Since we can choose the perturbation to be arbitrary small, the limit cycle $L$ is preserved, so the perturbed system has two limit cycles.



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Thank you for your attention!