Resonance of Isochronous Oscillators

David Rojas

Universitat de Girona, Catalonia, Spain

Advances in Qualitative Theory of Differential Equations 2019

Joint work with Rafael Ortega

Universitat de Girona

This research has been partially supported by the MINECO/FEDER grants MTM2017-82348-C2-1-P and MTM2017-86795-C3-1-P.

Concept of resonance

Resonance: All solutions are unbounded, meaning that each solution $x(t)$ satisfies

$$|x(t_n)| + |\dot{x}(t_n)| \to +\infty$$

for some sequence $\{t_n\}$.

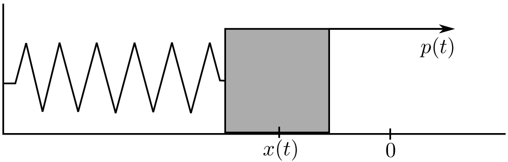
In fact in the results we prove:

$$\lim_{t \to +\infty} (|x(t)| + |\dot{x}(t)|) \to +\infty$$

Resonance for the harmonic oscillator

Consider the harmonic oscillator with period 2π perturbed by a periodic forcing

$$\ddot{x} + n^2 x = p(t), \;\; n = 1, 2, \ldots$$



$$\text{Resonance} \iff \hat{p}_n := \frac{1}{2\pi}\int_0^{2\pi} p(t) e^{-int} dt \neq 0$$

More general isochronous oscillators

Robert Roussarie, Open Problems Session of the II Symposium on Planar Vector Fields (Lleida, 2000)

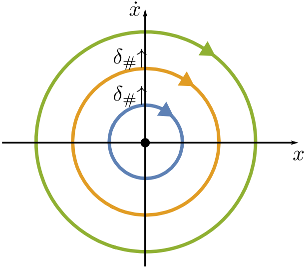
How this phenomena generalize for nonlinear isochronous centers?

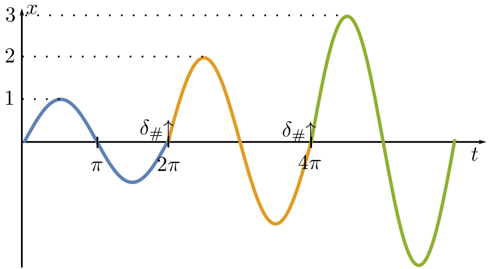
Rafael Ortega

Periodic perturbations of an isochronous center, Qualitative Theory of Dynamical Systems 3 (2002) 83-91.

$$\ddot{x} + V'(x) = \varepsilon\delta_{\#}(t),$$

$\delta_{\#}(t)$ periodic $\delta -$ function.





Regularization: $p(t)$ close to $\delta_{\#}(t) \longrightarrow \ddot{x} + V'(x) = \varepsilon p(t)$,
Hypothesis: V' Lipschitz-continuous.

Resonance for nonlinear oscillators

Consider an oscillator with equation

$$\ddot{x} + V'(x) = 0,\ x \in \mathbb{R}$$

and assume that it has an isochronous center at the origin with period $T = 2\pi$.
We are interested in the class of 2π-periodic functions $p(t)$ such that all the solutions of the non-autonomous equation

$$\ddot{x} + V'(x) = \varepsilon p(t)$$

are unbounded.

Goal

To identify a general class of forcings leading to resonance.
Our main result can be interpreted as a nonlinear version of the condition $\hat{p}_n \neq 0$.

Let:

- $\mathcal{C} = (\mathbb{R}/2\pi\mathbb{Z}) \times [0, \infty)$ with coordinates (θ, r),
- $\varphi(t, r)$ the solution of $\ddot{x} + V'(x) = 0$, $x(0) = r$, $\dot{x}(0) = 0$.
- $\psi(t, r)$ the complex-valued solution of

$$\ddot{y} + V''(\varphi(t, r))y = 0, \ \ y(0) = 1, \ \ \dot{y}(0) = i.$$

- $\Phi_p : \mathcal{C} \to \mathbb{C}$,

$$\Phi_p(\theta, r) := \frac{1}{2\pi} \int_0^{2\pi} p(t - \theta)\psi(t, r) dt.$$

Harmonic oscillator: $\psi(t, r) = \cos(nt) + \frac{i}{n} \sin(nt)$

$$\implies \tfrac{1}{n}|\hat{p}_n| \leq |\Phi_p| \leq |\hat{p}_n|.$$

Sufficient condition for resonance

Let $V \in C^2(\mathbb{R})$ satisfying $V(0) = 0,\ xV'(x) > 0$ if $x \neq 0$, with all solutions of $\ddot{x} + V'(x) = 0$ 2π-periodic.

Theorem

Assume that V'' is bounded over $\mathbb{R}$ and the condition

$$\inf_{\mathcal{C}} |\Phi_p(\theta, r)| > 0$$

holds for some $p \in L^1(\mathbb{T})$. Then the perturbed equation is resonant for small $\varepsilon \neq 0$.

Lets sketch the proof

Second Massera's Theorem: If all solutions of $\dot{z} = F(z, t)$, $z \in \mathbb{R}^2$, $F(\cdot, t)$ 2π-periodic, are globally defined in the future and at least one of them bounded then a 2π-periodic solution exists.

Take $\epsilon_n \downarrow 0$ and suppose x_n 2π-periodic sol. $\ddot{x} + V'(x) = \varepsilon_n p(t)$. Define $y_n = x_n - X_n$, where X_n solution of $\ddot{x} + V'(x) = 0$ with same initial conditions as x_n.

y_n solution of

$$\ddot{y} + y \int_0^1 V''((1-\lambda)x_n(t) + \lambda X_n(t))d\lambda = \epsilon_n p(t), \; y(0) = \dot{y}(0) = 0.$$

This produce: $\|y_n\|_{L^\infty(\mathbb{T})} \leq C|\varepsilon_n| \|p\|_{L^1(\mathbb{T})}$.

Also y_n solution of

$$\ddot{y} + V''(X_n(t))y = \varepsilon_n p(t) - q_n(t)$$

with

$$q_n(t) = y_n(t) \int_0^1 \left[V''((1-\lambda)x_n(t) + \lambda X_n(t)) - V''(X_n(t)) \right] d\lambda.$$

Thus: $\frac{1}{\varepsilon_n} q_n(t) \to 0$ and $\frac{1}{|\varepsilon_n|} \|q_n\|_{L^\infty(\mathbb{R})} \leq C \|p\|_{L^1(\mathbb{T})} \|V''\|_{L^\infty(\mathbb{R})}$.

Solutions $X_n(t)$ write $\varphi(t-\theta_n, r_n)$ and $\psi(t-\theta_n, r_n)$ is a 2π-periodic nontrivial solution of $\ddot{y} + V''(X_n(t))y = 0$.
Particularly $\|\psi\|_{L^\infty(\mathbb{T})} \leq C$.
Fredholm alternative:

$$\varepsilon_n \int_0^{2\pi} p(t)\psi(t-\theta_n, r_n)dt - \int_0^{2\pi} q_n(t)\psi(t-\theta_n, r_n)dt = 0.$$

$$\Phi_p(\theta_n, r_n) = \frac{1}{2\pi} \int_0^{2\pi} \frac{q_n(t)}{\varepsilon_n} \psi(t-\theta_n, r_n)dt \to 0 \text{ as } n \to +\infty.$$

□

There are many of those isochronous potentials?

M. Urabe
Potential forces which yield periodic motions of a fixed period,
J. Math. and Mech. 10 (1961), 569–578.

Consider the initial value problem

$$\frac{dX}{dx} = \frac{2\pi}{T}\frac{1}{1+S(X)},\ X(0)=0,$$

with S an analytic odd function satisfying

$$C_0 := \sup_{X\in\mathbb{R}} |S(X)| < 1,\ C_1 := \sup_{X\in\mathbb{R}} |XS'(X)| < +\infty.$$

The solution $X(x)$ is defined in $\mathbb{R}$ and $V'(x) = X(x)X'(x)$ produce an isochronous center of period T.

Example: $\alpha \arctan X, |\alpha| < \frac{2}{\pi}$. **Example with** $C_1 = +\infty$**:** $\frac{1}{2}\sin X$.

Necessity of the condition: only partially

Previous Theorem is a **sufficient condition for resonance** but it is not too far from being also necessary. A **partial converse** of the main theorem holds: a periodic solution exists when the function Φ_p has a non-degenerate zero.

Proposition

Assume V in the conditions of the Theorem and Φ_p having a non-degenerate zero (θ_*, r_*) with $r_* > 0$. Then the perturbed equation has a 2π-periodic solution for small ε.

- Harmonic oscillator:

$$\text{Resonance} \iff \hat{p}_n := \frac{1}{2\pi}\int_0^{2\pi} p(t)e^{-int}dt \neq 0.$$

- Nonlinear oscillator:

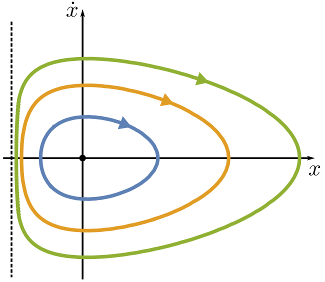
$$\text{Resonance} \text{ ``}\!\iff\!\text{'' } \Phi_p := \frac{1}{2\pi}\int_0^{2\pi} p(t-\theta)\psi(t,r)dt \neq 0.$$

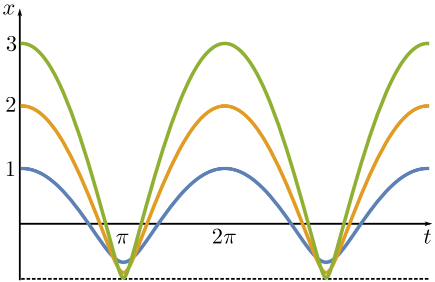
...And isochronous potentials not globally defined?

Until now we have talked about oscillators defined on $\mathbb{R}$ but there are also oscillators producing an **isochronous center** and **having a singularity**. A well-known example is the Pinney equation

$$\ddot{x} + \frac{1}{4}\left(x + 1 - \frac{1}{(x+1)^3}\right) = 0,$$

defined for all $x \in (-1, +\infty)$.





Theorem

Let $p \in L^1(\mathbb{T})$ be a function satisfying the resonance condition. Then all the solutions of equation

$$\ddot{x} + \frac{1}{4}\left(x + 1 - \frac{1}{(x+1)^3}\right) = \varepsilon p(t)$$

are unbounded for sufficiently small $\varepsilon \neq 0$.

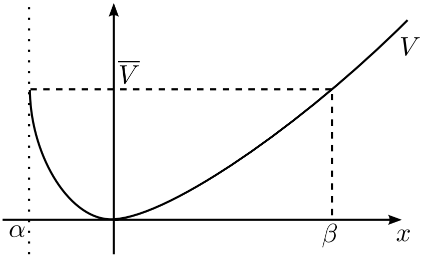
This resonance result deals with the specific Pinney equation but the method of proof can be extended to a larger class of potentials with strong singularity.

If consider $p(t) = a_0 + a_1 \cos t + b_1 \sin t$,

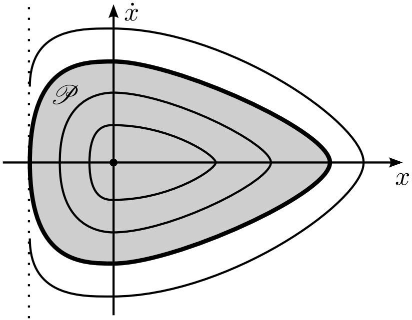
Corollary

Pinney equation is resonant if $a_1^2 + b_1^2 > 9a_0^2$.

Even bounded isochronos centers!

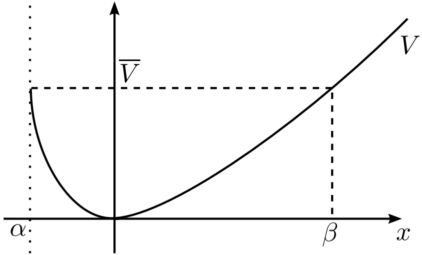


- Massera's Theorem

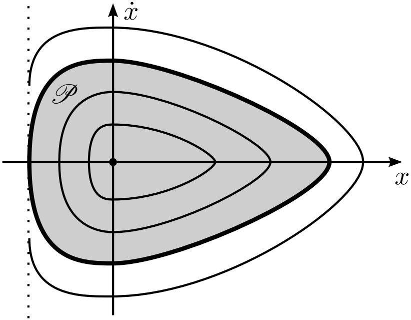


But only on compact subsets...
... for the moment!

Even bounded isochronos centers!



- Massera's Theorem... ✗
- Montgomery's Fixed Point Theorem ✓



But only on compact subsets...
... for the moment!

Some related open problems

(*a*) Either the result concerning the identification of the forcings producing resonance for isochronous oscillators defined in the whole plane and the construction by Ortega require the oscillator to be Lipschitz-continuous. We expect that no specific regularity of the potential is needed or at least weaker properties.

(*b*) We give a sufficient condition of resonance for the Pinney equation perturbed by a linear trigonometric function. It would be interesting to study if the perturbed equation have periodic orbits for $a_1^2 + b_1^2 \leq 9a_0^2$.

(*c*) The results presented deal with nonlinear isochronous oscillators with one degree of freedom. In more degrees of freedom, the notion of isochronicity is strongly related with superintegrability, at least in the Hamiltonian framework. It would be interesting to relate properly superintegrable Hamiltonian systems with isochronicity and to construct resonance of such systems.

Many thanks for your attention

References:

R. Ortega, *Periodic perturbations of an isochronous center*, Qualitative Theory of Dynamical Systems 3 (2002) 83-91.

R. Ortega, D. R., *Periodic oscillators, isochronous centers and resonance*, Nonlinearity 32 (2019) 800-832.

D. R., *Resonance of bounded isochronous oscillators*, Preprint.

M. Urabe, *Potential forces which yield periodic motions of a fixed period*, J. Math. and Mech. 10 (1961) 569–578.