Classifications of parabolic Dulac germs

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Dulac or almost regular germs

**Definition [Ilyashenko].**

*Parabolic almost regular germ (Dulac germ):*

- $f \in C^\infty(0, d)$
- extends to a holomorphic germ $f$ to a *standard quadratic domain* $Q$:
  
  $$Q = \Phi\left( \mathbb{C}_+ \setminus \overline{K(0, R)} \right), \quad \Phi(\eta) = \eta + C(\eta + 1)^{\frac{1}{2}}, \quad C, \quad R > 0,$$

  in the *logarithmic chart* $\xi = -\log z$. 
Standard quadratic domain

\[ r_k := r(\varphi_k) \sim e^{-C \sqrt{\frac{|k| \pi}{2}}}, \ k \to \pm \infty, \]

\[ \varphi_k \in \left((k - 1)\pi, (k + 1)\pi\right) \]
\( f \) admits the *Dulac* asymptotic expansion:

\[
f(z) \sim_{z \to 0} 1 \cdot z + \sum_{k=1}^{\infty} z^{\alpha_i} P_i(-\log z),
\]

i.e.

\[
f(z) - z - \sum_{i=1}^{n} z^{\alpha_i} P_i(-\log z) = O(z^{\alpha_n}), \quad n \in \mathbb{N},
\]

\( \alpha_i > 1 \), strictly increasing to \(+\infty\),
\( \alpha_i \) finitely generated \(^1\),
\( P_i \) polynomials.

\( \mathbb{R}_+ \) invariant under \( f \) (i.e. coefficients of \( \hat{f} \) real!)

\(^1\)There exist \( \beta_k, \ k = 1 \ldots n \), such that: \( \alpha_i \in \mathbb{N}\beta_1 + \ldots + \mathbb{N}\beta_n \).
Motivation and history

- *first return maps* for polycycles with hyperbolic saddle singular points – *n* saddle vertices with hyperbolicity ratios $\beta_i > 0$ (Dulac)

- locally at the saddle

\[
\begin{align*}
\dot{x} &= x + \text{h.o.t.} \\
\dot{y} &= -\beta_i y + \text{h.o.t.}
\end{align*}
\]
Motivation and history

- **Dulac’s problem**: accumulation of limit cycles on a hyperbolic polycycle possible?
- Limit cycles = fixed points of the first return map
- Dulac: accumulation $\Rightarrow$ trivial power-log asymptotic expansion of the first return map $\Rightarrow$ trivial germ on $\mathbb{R}_+$ (Dulac’s mistake)
- The problem: Dulac asymptotic expansion does not uniquely determine $f$ on $\mathbb{R}_+$ (add any exponentially small term w.r.t. $x$!), e.g.

$$f(x) \sim x + x^2 - \log x, \quad f(x) + e^{-1/x} \sim x + x^2 - \log x, \ x \to 0$$

- Ilyashenko’s solution: first return maps extendable to a SQD
- SQD **sufficiently large complex domain**: by a variant of maximum modulus principle (*Phragmen-Lindelöf*), Dulac’s expansion uniquely determines the germ on a SQD!
Questions

* goal: theory like the standard theory of Birkhoff, Ecalle, Voronin, Kimura, Leau etc. for parabolic analytic germs Diff(\(\mathbb{C}, 0\))

- **formal classification** of parabolic Dulac germs – by a sequence (!!! not necessarily convergent) of formal power-logarithmic changes of variables

\[
\hat{g} = \hat{\varphi}^{-1} \circ \hat{f} \circ \hat{\varphi},
\]

\(\hat{f}, \hat{g}\) Dulac expansions,
\(\hat{\varphi}(z) = z + h.o.t.\) diffeo- with power-log asymptotic expansion

- **analytic classification** of parabolic Dulac germs

\[
g = \varphi^{-1} \circ f \circ \varphi,
\]

\(f, g\) Dulac germs on \(Q\), \(\varphi(z) = z + o(z)\) analytic on \(Q\)

- \(\varphi\) admits \(\hat{\varphi}\) as its asymptotic expansion?
simpler question: is a Dulac germ **analytically embeddable** in a flow of an analytic vector field $\xi(z) \frac{d}{dz}$ defined on a standard quadratic domain? (= describe *trivial* analytic class)

$$g = \varphi^{-1} \circ f_0 \circ \varphi,$$

$f, f_0$ Dulac germs,

$f_0$ time-one map of an analytic vector field,

$\varphi$ analytic.

**Example**

$$f(z) = z + z^2 + z^3 + \ldots = \frac{z}{1-z} \text{ time-one map of } z^2 \frac{d}{dy}.$$
Historical results - germs of parabolic analytic diffeomorphisms

(Fatou ∼ end of 19th century; Birkhoff ∼ 1950; Ecalle, Voronin ∼ 1980, ...)

\[ f \in \text{Diff}(\mathbb{C}, 0), \quad f(z) = z + a_1 z^{k+1} + a_2 z^{k+2} + \ldots, \quad k \in \mathbb{N} \]

- Formal embedding
  = formal reduction to a time-one map of a vector field:

\[ f_0(z) = \exp\left( \frac{z^{k+1}}{1 + \rho z^k} \frac{d}{dx} \right) \cdot \text{id} = z + z^{k+1} + (\rho + \frac{k + 1}{2}) z^{2k+1} + \ldots \]

Step-by-step elimination of monomials from \( f \):

\[ \varphi_\ell(z) = \begin{cases} a z, & a \neq 1, \\ z + c z^\ell, & \ell \in \mathbb{N} \end{cases} \quad \leftrightarrow \quad \hat{\varphi}(z) = a z + \sum_{k=2}^{\infty} c_k z^k \in \mathbb{C}[[z]] \]

(formal changes of variables)

\[ \Rightarrow (k, \rho), \quad k \in \mathbb{N}, \quad \rho \in \mathbb{C} \ldots \text{formal invariants for } f. \]
Historical results - germs of *analytic diffeomorphisms*

- Is $f$ analytically embeddable, or just formally?
  $\iff$ Does $\hat{\varphi}$ converge to an analytic function at 0?

**Leau-Fatou flower theorem (1987):**
- $2k$ analytic conjugacies $\varphi_i$ of $f$ to $f_0$, all expanding in $\hat{\varphi}$
- defined on $2k$ *petals* invariant under local discrete dynamics
- $k$ attracting directions: $(-a_1)^{-\frac{1}{k}}$; $k$ repelling directions: $a_1^{-\frac{1}{k}}$

$k = 3 \rightarrow 6$ petals, $f(z) = z + z^4 + \ldots$

→ in general, analytic embedding in a flow only on open sectors
→ the analytic class of $f$ in direct relation with this question
FORMAL CLASSIFICATION OF DULAC GERMS
Formal embedding into flows for Dulac germs (non-analytic at 0)

- elimination **term-by-term** by an *adapted* 'sequence' of non-analytic elementary changes of variables:

\[ \varphi(z) = az; \quad \varphi_{\alpha,m}(z) = z + cz^\alpha \ell^m, \quad m \in \mathbb{Z}, \quad \alpha > 0, \quad (\alpha, m) \succ (1, 0). \]

**Example (MRRZ, 2016)**

0. \[ f(z) = z - z^2 \ell^{-1} + z^2 + z^3, \]
1. \[ \varphi_1(z) = z + c_1 z \ell, \quad c_1 \in \mathbb{C}, \]
   \[ f_1(z) = \varphi_1^{-1} \circ f \circ \varphi_1(z) = z - z^2 \ell^{-1} + a_1 z^2 \ell + h.o.t, \]
2. \[ \varphi_2(z) = z + c_2 z \ell^2, \quad c_2 \in \mathbb{R}, \]
   \[ f_2(z) = \varphi_2^{-1} \circ f \circ \varphi_2(z) = z + z^2 \ell^{-1} + a_2 z^2 \ell^2 + h.o.t, \]
3. \[ \varphi_3(z) = z + c_3 z \ell^3, \quad c_3 \in \mathbb{R}, \]
   \[ f_3(z) = \varphi_3^{-1} \circ f \circ \varphi_3(z) = z + z^2 \ell^{-1} + a_2 z^2 \ell^3 + h.o.t, \]
   :
The visualisation of the reduction procedure

drawn the control of the support!
The description of the formal change of variables

- more than just a *formal series composition* of changes of variables: a *transfinite composition*, \( \rightarrow \) produces a *transseries* \( \hat{\varphi} \):
  - in the process, prove that *every change has its successor change*
  - prove the *formal convergence* of composition of changes of variables: by *transfinite induction*\(^1\) in the *formal topology*\(^2\)

\(^1\) a generalization of the mathematical induction from \( \mathbb{N} \) to ordinal numbers: existence of a *successor element* and a *limit element*,
\(^2\) i.e. in each step of composition the support remains well-ordered; the coefficient of each monomial in the support stabilizes in the course of composition.
A broader class closed to embeddings: the class of power-log transseries $\hat{\mathcal{L}}$

...contains both the Dulac germ expansions $f \mapsto \hat{f}$ and the formal changes of variables

\[ \hat{\mathcal{L}} \ldots \hat{f}(z) = \sum_{\alpha \in S} \sum_{k=N_\alpha}^{\infty} a_{\alpha,k} z^{\alpha} \ell^k, \quad a_{\alpha,k} \in \mathbb{R}, \quad N_\alpha \in \mathbb{Z}, \]

$S \subseteq (0, \infty)$ well-ordered (here: finitely gen.)

Similarly we define $\hat{\mathcal{L}}_2$, $\hat{\mathcal{L}}_3$, etc. and

\[ \hat{\mathcal{L}} := \bigcup_{k \in \mathbb{N}} \hat{\mathcal{L}}_k. \]

(iterated logarithms admitted!)
Theorem (Formal embedding theorem for Dulac germs, MRRZ 2016)

\( \hat{f}(z) = z - az^\alpha \ell^m + h.o.t. \) parabolic Dulac, \( a > 0, \alpha > 1, m \in \mathbb{N}_- \).

\[ \Rightarrow \text{formally in } \hat{\mathcal{L}} \text{ conjugated to:} \]

\[
f_0(z) = \exp \left( \frac{-z^\alpha \ell^m}{1 - \frac{\alpha}{2} z^{\alpha-1} \ell^k + (\frac{k}{2} - \rho) z^{\alpha-1} \ell^{k+1}} \frac{d}{dz} \right). \text{id} =
\]

\[= z - z^{\alpha} \ell^m + \rho z^{2\alpha-1} \ell^{2m+1} + h.o.t. \]

\[ \star (\alpha, m, \rho) \in \mathbb{R} \ldots \text{formal invariants for Dulac germ} \]

\[ \star f_0(z) \text{ a time-one map of an analytic vector field on SQD } (\mathbb{Q}_+) \]
Example continued

Example (continued)

\[ f_0(z) = \exp \left( - \frac{z^2 \ell^{-1}}{1 - z \ell^{-1} + (b - \frac{1}{2})z} \right) \cdot \text{id} = \]

\[ = z - z^2 \ell^{-1} + b z^3 \ell^{-1} + h.o.t., \]

\[ f_0 = \hat{\varphi}^{-1} \circ \hat{f} \circ \hat{\varphi}, \quad \hat{\varphi} \in \hat{\mathcal{L}} - \text{a transfinite change of variables} \]
ANALYTIC CLASSIFICATION OF DULAC GERMS
Choice of analytic conjugacy - analytic on standard quadratic domain

Definition [MRR, in progress] $f$ and $g$ Dulac on SQD $Q$ are \textit{analytically conjugated} if there exists

$\varphi(z) = z + o(z)$ analytic on $Q$

$g = \varphi^{-1} \circ f \circ \varphi$ on $Q$.

$\implies \varphi$ admits asymptotic expansion in $\hat{\mathcal{L}}$

$\implies f$ and $g$ formally conjugated in $\hat{\mathcal{L}} \implies$ expansion in $\hat{\mathcal{L}} \subset \hat{\hat{\mathcal{L}}}$.

Another possible classification: $\varphi \in \mathbb{R}\{z\}$ (non-ramified)
'Equivalent' problems:

1. (formal) conjugation of $f$ to $f_0$ (time-one map of an analytic vector field)

2. (formal) Fatou coordinate for $f$

$$\Psi(f(z)) - \Psi(z) = 1 \quad \text{(Abel equation)}$$
$$\hat{\Psi}(\hat{f}(z)) - \hat{\Psi}(z) = 1 \quad \text{(formal Abel equation)}$$

$$\Psi = \Psi_0 \circ \varphi, \ \hat{\Psi} = \Psi_0 \circ \hat{\varphi}$$
Historical results - construction of the Ecalle-Voronin moduli of analytic classification for Diff(\mathbb{C}, 0)

★ simplest formal class \((k = 1, \rho = 0)\);
\[ f_0(z) = \text{Exp}(z^2 \frac{d}{dz}) = \frac{z}{1-z} \]

★ \( f \in \text{Diff}(\mathbb{C}, 0), \ f(z) = z + z^2 + z^3 + o(z^3) \)

\[ \Psi(f(z)) - \Psi(z) = 1 \quad \text{(Abel equation)} \]

**Fatou, 1919:**

▶ unique (up to additive constant) formal solution
\[ \hat{\Psi}(z) \in -1/z + z\mathbb{C}[[z]], \]

▶ unique (up to additive constant) analytic solutions \( \Psi_{\pm}(z) \) on petals \( V_{\pm} \)

▶ \( \Psi_{\pm} \) admit \( \hat{\Psi}(z) \) as asymptotic expansion

→ Fatou coordinates, sectorial trivialisations
Ecalle-Voronoï moduli of analytic classification for $\text{Diff}(\mathbb{C}, 0)$

**Ecalle, Voronoï:** spaces of attr./repelling orbits (spheres!) ”glued” at closed orbits (poles!) by 2 germs of diffeomorphisms:

$$\varphi_0(t) := e^{-2\pi i \Psi_- \circ (\Psi^+)^{-1}(-\frac{\log t}{2\pi i})}, \; t \approx 0,$$

$$\varphi_{\infty}(t) := e^{-2\pi i \Psi_+ \circ (\Psi^-)^{-1}(-\frac{\log t}{2\pi i})}, \; t \approx \infty$$
Ecalle-Voronin moduli of analytic classification for $\text{Diff}(\mathbb{C}, 0)$

Identifications:

$$\left( \varphi_0(t), \varphi_\infty(t) \right) \equiv \left( a \varphi_0(bt), \frac{1}{b} \varphi_\infty\left(\frac{t}{a}\right) \right), \quad a, b \in \mathbb{C}^*$$

(choice of constant in $\Psi_{\pm}$, i.e. coordinates on spheres)

**Theorem Ecalle-Voronin:** After identifications, $(\varphi_0, \varphi_\infty)$ are analytic invariants.

**Realisation theorem:** Each pair $(\varphi_0, \varphi_\infty)$ tangent to identity can be realized as E-V modulus of a germ from the model formal class.

Trivial modulus $(\text{id}, \text{id}) \leftrightarrow$ analytically embeddable germs
Invariant domains (petals) for the local dynamics of a parabolic Dulac germ

**L-F-like theorem, Dulac germs [MRR, in progress].**

\[ f(z) = z + az^\alpha \ell^m + \ldots \] Dulac germ on a SQD \( Q \), \( a \in \mathbb{R} \), \( \alpha > 1 \), \( m \in \mathbb{N}_- \).

\[ \Rightarrow \text{countably many overlapping attracting/repelling petals} \]

\[ V_i^\pm, \ i \in \mathbb{Z}, \ \text{of opening} \ \frac{2\pi}{\alpha-1} \]

\[ \Rightarrow \text{centered at complex directions} \]

\[ (-\text{sgn}(a))^{\frac{1}{\alpha-1}} \text{(attracting)}, \ (\text{sgn}(a))^{\frac{1}{\alpha-1}} \text{(repelling)} \]

(invariant lines for \( f \) tangential to these directions at 0)

**Sketch of the proof.** In the chart \( w = -\frac{1}{a(\alpha-1)}z^{-\alpha+1}\ell^{-m} \) \( f \) almost translation by 1, easier construction of invariant domain.
Dynamics of a Dulac germ (logarithmic chart)

\[ f(z) = z + az^\alpha \ell^m + \ldots, \quad a < 0 \]
(Formal) Fatou coordinate of a Dulac germ

**Theorem [MRRZ2 (2019), MRRp (in progress)]**

$f$ Dulac on SQD $\mathcal{Q}$, $\hat{f}$ its Dulac expansion.

- unique (up to an additive constant) formal Fatou coordinate $\hat{\Psi}$ for $\hat{f}$ in class $\hat{\mathcal{L}}$ (in $\hat{\mathcal{L}}_2$)
- unique (up to additive constants) analytic Fatou coordinates $\Psi_{\pm j}$, $j \in \mathbb{Z}$, on attracting/repelling petals $V_{\pm j}$
- $\Psi_{\pm j}$ admit $\hat{\Psi}$ as transserial asymptotic expansion with respect to integral sums on limit ordinal steps as $z \to 0$ on $V_{\pm j}$

Caution! Transserial asymptotic expansion is not well-defined (unique), if we do not prescribe a canonical summation method on limit ordinal steps (dictated here by Abel equation)!
Non-uniqueness of asymptotic expansion of a germ in $\hat{L}$

→ ambiguity: choice of the sum in $\ell$ at limit ordinal steps

Example

$f(z) = z + z^2 \frac{\ell}{1-\ell} + z^5$

Some possible asymptotic expansions:

\[ \hat{f}_1(z) = z + z^2(\ell + \ell^2 + \ell^3 + \ldots) + z^5 \]
\[ \hat{f}_2(z) = z + z^2(\ell + \ell^2 + \ell^3 + \ldots) - z^3 + z^5, \text{ etc.} \]

\[ \hat{f}_1: \text{ canonical (convergent sum) at the first limit ordinal step:} \]

\[ \ell + \ell^2 + \ell^3 + \ldots \mapsto \frac{\ell}{1-\ell} \]

\[ \hat{f}_2: \ell + \ell^2 + \ell^3 + \ldots \mapsto \frac{\ell}{1-\ell} + e^{-\frac{3}{\ell}} \quad (z = e^{-1/\ell}) \]

Moreover: (?) canonical choice if series in $\ell$ was **divergent** (the case in the Fatou coordinate)
Sketch of the proof / method of summation

\[ f(z) \sim \hat{f}(z) = z + z^{\alpha_1} P_1(-\log z) + z^{\alpha_2} P_2(-\log z) + \ldots \]

- solve (formal) Abel equation by \textit{blocks}

\[ \hat{\Psi}(z + z^{\alpha_1} P_1(\ell^{-1}) + \ldots) - \hat{\Psi}(z) = 1 \]

- \( \hat{\Psi}(z) := \sum z^{\beta_i} \hat{T}_i(\ell) \)

- In each step, \( \hat{T}_i \) obtained solving one differential equation:

\[
\frac{d}{dz} \left( z^{\beta_i} \hat{T}_i(\ell) \right) := z^{\beta_i-1} R(\ell),
\]

\( \ast \) \( \hat{T}_i(\ell) = z^{-\beta_i} \int z^{\beta_i-1} R(\ell) dz, \)

\( \beta_i \) a finite combination of \( \alpha_i \); \( R \) a rational function in \( \ell \).

- \( \ast \) solvable analytically (\( T_i \) analytic on \( Q \)) as well as formally (\( \hat{T}_i \in \mathbb{C}[[z]] \)) by partial integration

\( \hat{T}_i \mapsto T_i \) (integral sum)
\[ \hat{\Psi} := \Psi_\infty + \hat{R}, \text{ where } \Psi_\infty \text{ contains only finitely many infinite blocks} \]

\[ \text{analytic Fatou coordinate on petals: iterative summation of the Abel equation along the orbit of } f/f^{-1}, \text{ after subtracting sufficiently many blocks:} \]

\[ R(f(z)) - R(z) = \delta(z), \]

\[ \delta(z) \text{ of arbitrarily small order.} \]

\[ \Rightarrow R^j_\pm(z) := -\sum_{k=0}^{\infty} \delta(f^{\circ(\pm)k}(z)), \ j \in \mathbb{Z}. \]

Converges locally uniformly on petals \( V^j_\pm. \)

\[ Q.E.D. \]
Example of blocks computation in the Fatou coordinate of a Dulac germ

Example

\[ f(z) = z + z^2 \ell^{-1} + z^3 \Rightarrow \Psi(z + z^2 \ell^{-1} + z^3) - \Psi(z) = 1. \ (*\) \]

Computation of the first block of \( \Psi \) by formal T. expansion of \((*)\):

\[ \Psi'_0(z) z^2 \ell^{-1} = 1 \Rightarrow \Psi_0(z) = \int z^{-2} \ell \, dz \]

➤ Integration by parts: \( \hat{\Psi}_0(z) = z^{-1} \sum_{n \in \mathbb{N}} n! \ell^n \)  
  (divergent series in \( \ell \) in the first block!)

➤ Analytic integration on SQD: \( \Psi_0(z) = \int_*^z y^{-2} \ell(y) \, dy \)

? appropriate sum of divergent series above ? integral sum

\[ \sum_n n! \ell^n \mapsto \int_*^z y^{-2} \ell(y) \, dy \div z^{-1}. \]
Ecalle-Voronom moduli for Dulac germs

- infinitely many attracting/repelling petals indexed by \( \mathbb{Z} \)
- neighboring spheres glued at closed orbits by a germ of a diffeomorphism
- infinite necklace of spheres (spaces of orbits on petals), not closed
Ecalle-Voronin moduli for Dulac germs

**Theorem E-V for Dulac maps** (MRRp)

\( f \) and \( g \) Dulac in the same \( \hat{\mathcal{L}} \)-formal class \((\alpha, m, \rho)\).

- analytic invariants given by a sequence of diffeomorphisms of 0 and \( \infty \) tangent to the identity, up to identifications \((*)\)

\[
\varphi_0^i(t) := e^{-2\pi i \Psi_+^{-1} \circ (\Psi_-)^{-1}(-\frac{\log t}{2\pi i})}, \quad t \approx 0
\]

\[
\varphi_\infty^i(t) := e^{-2\pi i \Psi_-^{-1} \circ (\Psi_+)^{-1}(-\frac{\log t}{2\pi i})}, \quad t \approx \infty, \; i \in \mathbb{Z}
\]

- radii of definition (at least)

\[
|t| < R_i \sim K_1 e^{-Ke^{C\sqrt{i}}}, \; i \to \infty \quad \text{(SQD)}
\]

- identifications \((*)\)

\[
(\varphi_0^i, \varphi_\infty^i; \; R_i)_{i \in \mathbb{Z}} \equiv (\psi_0^i, \psi_\infty^i; \; \tilde{R}_i)_{i \in \mathbb{Z}}
\]

if \( R_i, \; \tilde{R}_i \) bounded as above (possibly different constants) and there exist sequences \((a_i)_{i \in \mathbb{Z}}, \; (b_i)_{i \in \mathbb{Z}}\) in \( \mathbb{C}^* \) such that

\[
\varphi_0^i(t) = a_{i-1} \cdot \psi_0^i\left(\frac{t}{b_i}\right), \quad \varphi_\infty^i(t) = b_i \cdot \psi_\infty^i\left(\frac{t}{a_i}\right), \; i \in \mathbb{Z}.
\]
necklace symmetric w.r.t. $\mathbb{R}_+$-axis

Proof: Schwarz’s reflection lemma,
$f(\mathbb{R}_+) \subseteq \mathbb{R}_+ \Rightarrow f(\overline{z}) = f(z)$.

$\star$ $f$ embeddable analytically on SQD in a vector field $\iff$ modulus trivial, $(\ldots, \text{id}, \text{id}, \ldots)$
Perspectives and comments

- realization of moduli in wider *generalized Dulac* class
- what can be realized really by Dulac corner maps of one saddle or by first return maps of more saddle polycycles (expected: *periodicity* of modules after finitely many)
References


MRRp Mardesic, P., Resman, M., Rolin, J.P., Analytic moduli for parabolic Dulac germs & Realization of moduli for parabolic Dulac germs (in progress)
Thank you for the attention!