Correspondence between Godbillon-Vey sequence and Françoise algorithm

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Abstract

We study local integrable foliations and their deformations. We introduce the so-called Françoise algorithm for computing the first nonzero Melnikov function for the displacement map, and describe what we call the Godbillon-Vey sequence for a deformation. We establish the correspondence between these objects and give some results on the integrability of the deformation in terms of the length of the sequences (see [4]).

Preliminary Notions

Let $\gamma_0 \subset \mathbb{C}^2$ be a regular curve, Σ a transversal to γ_0, F a holomorphic function defined on a tubular neigborhood $U \subset \mathbb{C}^2$ of γ_0, formed by regular curves $\gamma(t) \subset F^{-1}(t)$, $t \in F(\Sigma)$, with $\gamma_0 = \gamma(t_0)$.

Consider the integrable foliation $dF = 0$ and its holomorphic deformation

$$dF + \epsilon\omega = 0 \tag{1}$$

in U. We are interested in the displacement function Δ of (1) along $\gamma(t)$. It can be developed as

$$\Delta(t) = \sum_{i\geq 1} \epsilon^i M_i(t). \tag{2}$$

The functions $M_i(t)$ are called *Melnikov functions*. If $\Delta \equiv 0$, this means that (1) has a first integral in a neighborhood of $\gamma(t)$. If not, then there exists a *first nonzero* Melnikov function M_μ.

Françoise algorithm

Françoise algorithm allows to compute the first nonzero Melnikov function M_μ. Namely,

Theorem 1. *[2] Let (2) be the displacement map of (1). Assume that $M_i \equiv 0$, for $i = 1, ..., k$. Then, $M_{k+1}(t) = (-1)^{k+1} \int_{\gamma(t)} g_k\omega$, where $g_0 = 1$, and g_i, r_i verify*

$$g_{i-1}\omega = g_i dF + dr_i, \quad i = 1, ..., k. \tag{3}$$

Definition (Françoise pairs and Françoise sequence). We call any pair (g_i, r_i), verifying (3) an i-th Françoise pair associated to the deformation (1) and call the sequence (g_i, r_i), $i = 0, 1, ...$ a Françoise sequence. We say that the length of a Françoise sequence is ℓ, if ℓ is the smallest index such that $g_{\ell+1} = 0$. If there does not exist such an index, we say that the sequence is of infinite length.

Classical Godbillon-Vey sequence

The classical Godbillon-Vey sequence is associated to a foliation defined by a single one-form

$$\omega = 0. \tag{4}$$

It is a sequence of one-forms $\omega_0 = \omega, \omega_i, i = 1, ...$ such that the formal one-form

$$\Omega = d\epsilon + \omega_0 + \sum_{i=1} \frac{\epsilon^i}{i!}\omega_i \tag{5}$$

in $\mathbb{C}^2 \times \mathbb{C}$ verifies the *formal integrability condition*

$$\Omega \wedge \tilde{d}\Omega = 0. \tag{6}$$

Here $\tilde{d} = d_\epsilon + d$ denotes the total differential with respect to x, y, ϵ. Condition (6) is equivalent to

$$\begin{aligned} d\omega_0 &= \omega_0 \wedge \omega_1, \\ d\omega_1 &= \omega_0 \wedge \omega_2, \\ \cdots & \quad \cdots \\ d\omega_n &= \omega_0 \wedge \omega_{n+1} + \textstyle\sum_{k=1}^n \binom{n}{k} \omega_k \wedge \omega_{n-k+1}. \end{aligned}$$

We say that the *Godbillon-Vey sequence is of length n* if the forms ω_k vanish for $k \geq n$.

Definition. Let K be a differential field, G a function and K_G the extension of K by G. We say that the extension K_G is: *Darboux*, *Liouville* or *Riccati*, respectively, if it belongs to a finite sequence of field extensions starting from the field K: the extensions in each step are either algebraic or given, respectively, by solutions of the equations $dG = \eta_0$, $dG = G\eta_1 + \eta_0$ or $dG = G^2\eta_2 + G\eta_1 + \eta_0$, with η_i one-forms with coefficients in the corresponding field extensions.

In that case, we call the function G Darboux, Liouville or Riccati with respect to K.

In [1], Casale relates the length n of the Godbillon-Vey sequence to the type of first integral of the foliation given by (4):

Theorem 2.

(i) There exists a Godbillon-Vey sequence of length 1 if and only if (4) has a Darboux first integral.

(ii) There exists a Godbillon-Vey sequence of length 2 if and only if (4) has a Liouvillian first integral.

(iii) There exists a Godbillon-Vey sequence of length 3 if and only if (4) has a Riccati first integral.

Godbillon-Vey sequence for a deformation

We give a well-adapted version of Godbillon-Vey sequences to study deformations of integrable foliations as in (1). The Godbillon-Vey sequence gives a condition for verifying if the integrability at the level $\epsilon = 0$ extends to $\epsilon \neq 0$.

We define the form

$$\Omega = Rd\epsilon + (dF + \epsilon\omega)G, \tag{7}$$

with

$$G = \sum_{i=0} \epsilon^i G_i, \quad R = \sum_{i=0} \epsilon^i R_{i+1} \tag{8}$$

unknown functions and $G_0 \equiv 1$.

We give a relative version of the definition of different types of first integral for the deformation (1).

Definition. We denote by $K_{F,\omega}$ the field associated to the deformation (1). That is, the smallest differential field in a tubular neighborhood U of a cycle γ_0 containing the functions given by coefficients of dF and ω.

Let $F_\epsilon = \sum_{i=0}^{\ell} \epsilon^i F_i$, $\ell < \infty$, be a first integral of (1). We say that it is Darboux, Liouville or Riccati, respectively, if all F_i are in the corresponding extension of the field $K_{F,\omega}$.

Theorem 3. *[3] There exists a solution (G, R) of the equation*

$$\Omega \wedge \tilde{d}\Omega = 0 \tag{9}$$

if and only if the deformation preserves formal integrability along γ, i.e., $\Delta \equiv 0$.

Proof. It follows by Frobenius Theorem. □

We also consider the Godbillon-Vey equation up to order k with Ω, G, R given by (7) and (8):

$$\Omega \wedge \tilde{d}\Omega = 0 \mod \epsilon^{k+1}. \tag{10}$$

Definition (Godbillon-Vey pairs and Godbillon-Vey sequence for a deformation). We call any pair (G_i, R_i) verifying (10), an i-th *Godbillon-Vey pair associated to the deformation* (1). The sequence (G_i, R_i), $i = 0, 1, \ldots$ is the *Godbillon-Vey sequence associated to the deformation*. We say that the length of a Godbillon-Vey sequence associated to the deformation is ℓ, if ℓ is the smallest index such that $G_{\ell+1} = 0$. If there does not exist such an index, we say that the sequence is of infinite length.

Main theorems

Here we state our two main results. The first establishes the relationship between the Françoise pairs and the Godbillon-Vey pairs associated to the deformation. In particular it shows that the minimal length of Françoise sequences and Godbillon-Vey sequences coincide:

Theorem 4.

(i) The Melnikov functions M_i, $i = 1, \ldots, k$, are identically equal to zero if and only if one can solve the equation

$$\Omega \wedge \tilde{d}\Omega = 0 \mod \epsilon^{k+1}, \tag{11}$$

(ii) For each choice of the Françoise sequence (g_i, r_i), $i = 1, \ldots, k$, the Godbillon-Vey sequence (G_i, R_i), $i = 1, \ldots, k$, can be chosen verifying the equations

$$G_i = (-1)^i g_i, \quad R_i = (-1)^{i+1} i r_i. \tag{12}$$

(iii) If Ω verifies (11) then

a) there exists a function $N = 1 + \sum_{i=1}^k \epsilon^i n_i$ such that

$$\Omega = N\tilde{d}F_\epsilon \mod \epsilon^{k+1}.$$

Then the function F_ϵ is of the form

$$F_\epsilon = F + \sum_{i=1}^{k} (-1)^{i+1} \epsilon^i r_i. \tag{13}$$

and

$$\tilde{d}F_\epsilon = \tilde{R}d\epsilon + \tilde{G}\,(dF + \epsilon\omega)\,. \tag{14}$$

b) Let $\tilde{G}$ and $\tilde{R}$ be given in (14) and (G_i, R_i), $i = 1, \ldots, k$, be its coefficients as in (8). Then the functions (g_i, r_i), $i = 1, \ldots, k$, given by (12) are Françoise pairs.

Our second result gives the type of local first integral F_ϵ of the deformation (1) if the length of its Françoise sequence is finite. The first result is that the first integral is in a finite sequence of extensions of Darboux type. The second shows that it is in a single extension of Liouvillian type.

Theorem 5. *Let $\eta_\epsilon = dF + \epsilon\omega$ as in (1) be such that there exists a Françoise sequence of finite length ℓ.*

(i) Then (1) admits a univalued first integral which is Darboux with respect to the field $K_{F,\omega}$ of the deformation (1).

(ii) Then there exists a meromorphic form $\tilde{\theta}_\epsilon$ verifying the Godbillon-Vey sequence of length 2:

$$\begin{aligned} d\eta_\epsilon &= \eta_\epsilon \wedge \tilde{\theta}_\epsilon \\ d\tilde{\theta}_\epsilon &= 0, \end{aligned}$$

such that there exists a (possibly multivalued) first integral $\tilde{F}_\epsilon$ of (1) verifying

$$d\tilde{F}_\epsilon = f\eta_\epsilon,$$

where

$$df = f\tilde{\theta}_\epsilon$$

is a (possibly multivalued) function in a tubular neighborhood U of the cycle γ_0.

In particular, the function f belongs to a Liouville extension of $K_{F,\omega}$ and $\tilde{F}_\epsilon$ belongs to a Darboux extension of this Liouville extension.

Remark. Note that we are restricting our study to a tubular neighborhood U of a cycle γ_0. A first integral F_ϵ which is Darboux in U can be more complicated (Liouville, Riccati,...) when studied globally.

Remark. In Theorem 5 (ii) we prove in particular that if the deformation (1) has a finite Françoise sequence, then it has a Liouvillian first integral. The converse is an interesting question.

Remark. In Theorem 5 we suppose that (1) has a Françoise sequence of finite order. What happens in the case of $\ell = \infty$? In particular, is it possible to give a condition assuring that a deformation (1) has a Liouville or a Riccati first integral in these terms?

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