Integrability and Linearizability of a Family of Three-dimensional Quadratic Systems

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1. Outline

We present necessary and sufficient conditions for both integrability and linearizability of a three dimensional vector field having a $(1:-2:1)$–resonance at the origin. We choose for our investigation a system with quadratic nonlinearities and in general non of the axes planes is invariant. So we deal with a nine parametric family of quadratic systems. Some techniques like Darboux method are used to prove the sufficiency of the obtained conditions. For a particular three parametric subfamily we provide conditions to guarantee the non existence of a polynomial first integral.

Motivation

- Integrability is interesting and it's fun!!!
- There are many works for Lotka–Volterra: Bobienski and Żołądek , Cairó and Llibre, Aziz and Christopher, Basov and Romanovski, Moulin-Ollagnier, Christodoulides and Damianou, Gao and Liu, Gonzalez-Gascon and Peralta Salas, Aziz and Christopher and Llibre and Pantazi,...
- There are some works for non Lotka–Volterra: Dukarić and Oliveira and Romanovski, Hu and Han and Romanovski, Aziz...
- There are also some works related with polynomial first integrals: Labrunie, Moulin-Ollangneir, Llibre and Valls, Leach and Miritzis, Cairó...

Notation

$$\begin{aligned} \dot{x} &= x + axy + bxz + cyz,\\ \dot{y} &= -2y + dxy + exz + fyz,\\ \dot{z} &= z + gxy + hxz + kyz. \end{aligned} \tag{1}$$

Our main objective is to find conditions on the parameters such that system (1) possesses the two independent first integrals

$$\phi_1 = x^2y + O(x,y,z) \quad \text{and} \quad \phi_2 = yz^2 + O(x,y,z). \tag{2}$$

We found the necessary conditions for both integrability and linearizability at the origin via factorized Gröbner basis using Reduce and minAssGTZ algorithm in Singular. To prove the sufficiency we exhibit first integrals of the form (2).

2. The main Theorem

Consider system (1). The origin is integrable if and only if one of the following conditions are satisfied:

1) $2b^2 - 4bf - ce + 2f^2 = k = h = g = d = a = 0$
2) $g = f = e = d = c = b = 0$
3) $h = g = f = d = c = b = a + k = 0$
4) $k = g = f = e = d + h = b = a = 0$
5) $h = g = f = e = d = c = 0$
6) $g = f = e = d = c = a - k = 0$
7) $k = g = c = a = 0$
8) $bd + 2bh + fh = af - 2bk - fk = ad + 2ah - dk = g = e = c = 0$
9) $2d^2 - 4dh - eg + 2h^2 = k = f = c = b = a = 0$
10) $h = f = e = d = b = 0$
11) $k = h = e = d = c = b + f = a = 0$

Moreover, the system is linearizable if and only if conditions above holds except condition (8).

3. Some known results

Theorem-1[Aziz–Christopher] Assume that system (1) is a Lotka–Volterra system. Consider that it is integrable and there exists a function $\xi = x^\alpha y^\beta z^\gamma(1 + O(x,y,z))$ such that $X(\xi) = k\xi$ for some constant $k = \alpha\lambda + \beta\mu + \gamma\nu$, then the system is linearizable.

Theorem-2[Aziz] Consider the system

$$\dot{X} = \lambda X, \qquad \dot{y} = y(-\mu + A(X,Z)) + B(X,Z), \qquad \dot{Z} = \nu Z,$$

where $\lambda, \mu, \nu \in \mathbb{Z}^+$. The system is linearizable if there exist functions α and γ such that $\dot{\alpha} + \gamma B = -\mu\alpha$ and $\dot{\gamma} + \gamma A = 0$, where $A(X,Z)$ and $B(X,Z)$ are functions of X and Z.

Theorem-3[Goriely/Llibre, Yu and Zhang] Consider the differential system

$$\frac{d\mathbf{x}}{dt} = \mathbf{P}(\mathbf{x}), \quad \mathbf{x} = (\mathbf{x_1}, \ldots, \mathbf{x_n}) \in \mathbb{R}^{\mathbf{n}},$$

with $\mathbf{P}(x) = \big(P_1(x), \ldots, P_n(x)\big)$ and $P_i \in \mathbb{R}[x_1, \ldots, x_n]$ for $i = 1, \ldots, n$. Assume that it admits a Jacobi multiplier and $n-2$ first integrals functionally independent. Then the system admits an additional first integral functionally independent with the previous $n-2$ first integrals.

4. A taste of the proof

Case 2

We have a Lotka–Volterra system

$$\dot{x} = x(1 + ay), \quad \dot{y} = -2y, \quad \dot{z} = z(1 + hx + ky).$$

and has $x = 0$, $y = 0$ and $\exp(y)$ invariant.
For $a \neq 0$ has the Darboux first integral $H = x^2y(\exp(y))^a$ and the Darboux Jacobi multiplier $M_1 = (xyz)^{-1}$ and so admits an additional first integral.

The transformation $(X,\ Y) = (x\exp(\frac{a}{2}y),\ y)$ linearizes the first two equations as

$$\dot{X} = X, \qquad \dot{Y} = -2Y.$$

The change of coordinates $Z = z\exp(-\phi)$ with ϕ an analytic function of the variables X and Y will give $\dot{Z} = Z$ if and only if

$$\dot{\phi}(X,Y) = X\frac{\partial\phi}{\partial X} - 2Y\frac{\partial\phi}{\partial Y} = hx(X,Y) + ky(Y).$$

We have

$$x = X\exp\left(-\frac{a}{2}Y\right) = \sum_{n\geq 1}\frac{1}{(n-1)!}\left(-\frac{a}{2}\right)^{n-1}XY^{n-1},$$

and note that x contains no terms of the form $(X^2Y)^n$.

For $a = 0$ we have the two independent Darboux first integrals

$$H_1 = x^2y, \qquad H_2 = z^2y\exp\left(-2hx + ky\right).$$

The change of coordinates $(X,\ Y,\ Z) = \left(x,\ y,\ z\exp\left(-hx + \frac{k}{2}y\right)\right)$ linearizes the system.

Case 10

$$\dot{x} = x + axy + cyz, \qquad \dot{y} = -2y, \qquad \dot{z} = z + gxy + kyz,$$

and has invariant $f_1 = y = 0$ and $f_2 = cz^2 + (a-k)xz - gx^2 = 0$ and $E = \exp(y)$.

For $a \neq -k$ the system has the first integral $H_1 = f_2^2 f_1^2 E^{a+k}$ and the Jacobi multiplier $M = E^{(a+k)/2}$. Hence, admits an additional first integral independent of H_1.

For $a = -k$ the system is divergence free and has the first integral $H = f_1 f_2$.

For both cases the system will linearize under the transformation

$$(X,Y,Z) = (\ell_1 E^{\frac{\alpha_1}{2}}, y, \ell_2 E^{\frac{\alpha_2}{2}}),$$

where $\ell_1 = r_1 x + cz$ and $\ell_2 = r_2 x + cz$ such that

$$\begin{aligned} r_1 &= \frac{1}{2}\Big(a - k + \sqrt{a^2 - 2ak + 4cg + k^2}\Big),\\ r_2 &= \frac{1}{2}\Big(a - k - \sqrt{a^2 - 2ak + 4cg + k^2}\Big),\\ \alpha_1 &= \frac{1}{2}\Big(a + k + \sqrt{a^2 - 2ak + 4cg + k^2}\Big),\\ \alpha_2 &= \frac{1}{2}\Big(a + k - \sqrt{a^2 - 2ak + 4cg + k^2}\Big). \end{aligned}$$

Polynomial first integrals

We consider the subfamily of system (1) (a Lotka-Volterra system)

$$\begin{aligned} \dot{x} &= x(1 + ay + bz),\\ \dot{y} &= -2y,\\ \dot{z} &= z(1 + hx + ay). \end{aligned} \tag{3}$$

Theorem If (i) $h = 0$ and $a \neq 0$ or (ii) $ha \neq 0$ then the three parametric family (3) has no polynomial first integrals.

Corollary The family (3) for the cases 2),5) and 6) of the main Theorem has no polynomial first integrals. The same holds for case 8) when $h = 0$ and $a \neq 0$.

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