Towards combinatorial monomialization of generalized power series

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Abstract Generalized power series extend the standard ones: Their support ranges over a product of well ordered subsets of non negative real numbers. Those convergent with real coefficients are the blocks to build the category $\mathcal{G}$ of generalized analytic manifolds. Questions linked with the resolution of singularities in the standard real analytic geometry are posed to this new category. The local monomialization for $\mathcal{G}$–functions is already done in [4]. In the present work we are focused on the global result. In order to achieve a global statement, we need to overcome the intrinsic difficulties of $\mathcal{G}$: existence of regular subvarieties which cannot be used as centers of admissible blow–ups and, when it is possible, different non-isomorphic ways of blowing–up come into play. The very first step of our job is to get a *combinatorial monomialization* that only uses monomial transformations.

Introduction A **real generalized formal power series** is by definition a series $s := s(X) := \sum_\alpha s_\alpha X^\alpha$ such that $\mathrm{Supp}\,(s) := \{\alpha \in \mathbb{R}^m_{\geq 0} : s_\alpha \neq 0\} \subset \Lambda_1 \times \Lambda_2 \times \cdots \times \Lambda_m$, $\Lambda_j \subset \mathbb{R}_{\geq 0}$ is well ordered They form a no noetherian local ring $\mathbb{R}[[X^*]]$. If $\mathrm{Supp}(s) \subset \mathbb{R}^k_{\geq 0} \times \mathbb{N}^n$ we write $s \in \mathbb{R}[[X^*, Y]]$, $X = (X_1, X_2, \ldots, X_k)$ are the generalized variables and $Y = (Y_1, Y_2, \ldots, Y_n)$ are the analytic ones; *cf.* [3].

Examples Generalized power series appear in different contexts:

A) In **Algebraic Geometry**: *Puiseux series* as parametrizations of plane analytic irreducible curves. The series $\sum_{k=0}^\infty x^{2-2^{-k}}$ parametrizes the curve $\{y^2 + x^2 y + x^2 = 0\}$ over $\mathbb{Z}_2$.

B) In **Differential Equations**: Solutions of $xy' = \lambda y + \cdots$ are of the form $y = h(x^\lambda)$, where h is a standard power series.

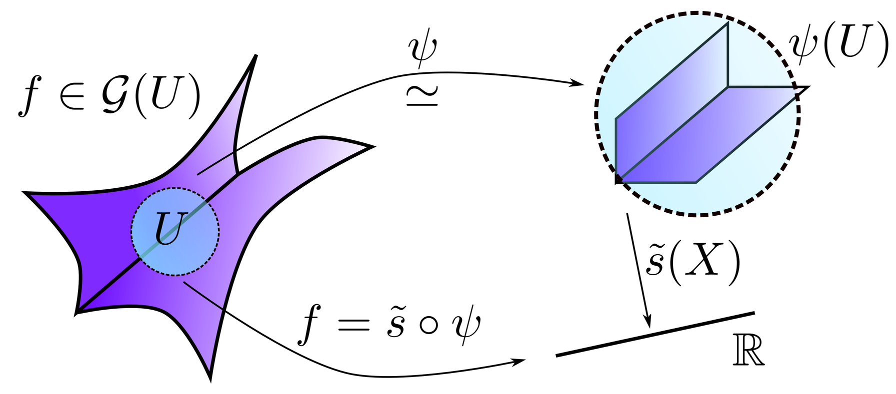
C) In **Functional Equations**: The series $F(x) = \sum_{j=0}^\infty \frac{1}{2^j} x^{2-\frac{1}{2^j}}$ is a solution of $f(x) = x + \frac{1}{2} x f(\sqrt{x})$.

D) An **Ordinary Dirichlet Series**: The function $g(t) = \sum_{j\geq 1} a_j j^{-t}$ after the change $t = \log x$ produces $\sum_{j\geq 1} a_j x^{\log j} \in \mathbb{R}[[x^*]]$. Its support $\{\log j : j \geq 1\}$ is an increasing sequence of real numbers.

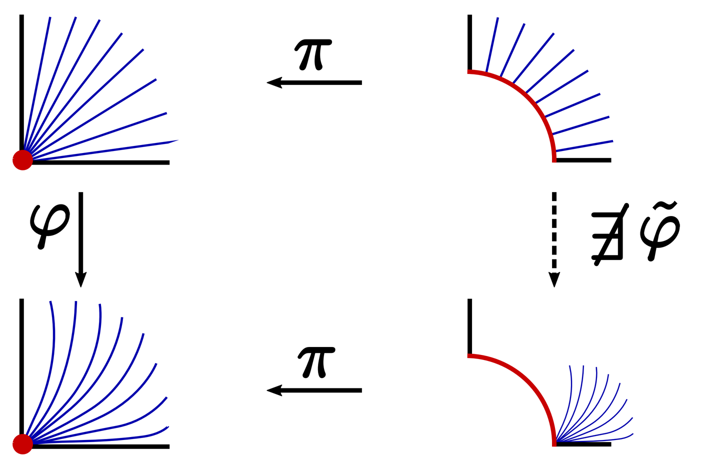
Each power series $s \in \mathbb{R}[[X^*]]$ has a finite representation: $s = \sum_{j=1}^k U_j X^{\alpha_j}$, $U_j \in \mathbb{R}[[X^*]]$ is a unit and $\{\alpha_1, \alpha_2, \ldots, \alpha_k\} =: \mathrm{Supp}_{\min}(s) \subset \mathrm{Supp}\,(s)$ is the minimal support of s. We say that s is of monomial type if $|\mathrm{Supp}_{\min}(s)| = 1$. The Newton polyhedron associated to s, $\mathcal{N}\,(s)$, has a finite number of m-dimensional faces and the vertices set is a subset (possibly proper) of $\mathrm{Supp}_{\min}(s)$.

Generalized Analytic Manifolds

The set of convergent series is a subring $\mathbb{R}\{X^*\} \subset \mathbb{R}[[X^*]]$. The sum of an element $s(X) \in \mathbb{R}\{X^*\}$ determines a continuous function $\tilde{s}\colon U \subset \mathbb{R}^n \times \mathbb{R}^k_{\geq 0} \to \mathbb{R}$ analytic in $\mathrm{Int}(U)$. So we can introduce the category of generalized analytic manifolds, $\mathcal{G}$, as the category whose objects, $\mathcal{G}$–manifolds, are topological spaces with boundary and corners locally $\mathcal{G}$–isomorphic to the local models $\mathbb{G}^m := \left(\left|\mathbb{R}^m_{\geq 0}\right|, \mathcal{G}_M\right)$, where $\mathcal{G}_M$ stands for the sheaf of germs of generalized analytic functions. $\mathcal{G}$ is a category with products and gluing and morphisms of $\mathcal{G}$ are **locally of monomial type** *cf.* [4].

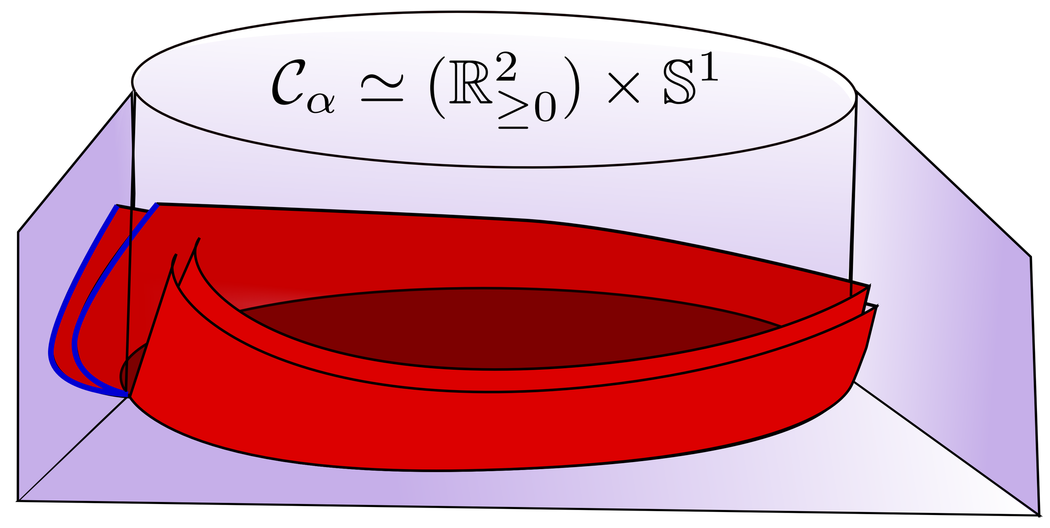


Admissible centers and Blowing–up morphisms



There are non-isomorphic local blow–ups: Let us consider $\lambda > 0$ and let $\varphi\colon \mathbb{R}^2_{\geq 0} \to \mathbb{R}^2_{\geq 0}$ be the $\mathcal{G}$–automorphism given by $(x_1, x_2) \mapsto (x_1^\lambda, x_2)$. Considering the standard blow–up of $(0,0) \in \mathbb{R}^2_{\geq 2}$ it is obvious that there no exist a lifting $\tilde{\varphi}$ of φ doing the diagram on the left commutative if $\lambda \neq 1$.

Due to this two peculiarities on $\mathcal{G}$, if we want to consider a regular subvariety $C \subset M$ as an **admissible center to blow–up** we require a technical definition that passes through standardizations; *i.e.* morphisms $\varphi\colon U \subset M \to A$, $C \subset U$ and A real analytic manifold with boundary and corners such that $\varphi(C)$ has normal crossings in A. Thus, via the usual blow–up of $\varphi(C) \subset A$, we define the blow–up morphism of M centered at C respect to φ as the pull-back of φ and $\pi^A_{\varphi(C)}$. That is the way blow–ups morphisms are settled down in $\mathcal{G}$.



As it was said above, there exist global subvarieties with normal crossings that cannot be considered as blowing–up centers. The figure on the right represents an *exotic cylinder* $\mathcal{C}_\alpha$ with parameter $\alpha \in \mathbb{R}_{>0}$. It is obtained by gluing two copies of $(\mathbb{R}^2_{\geq 0}) \times \mathbb{R}$, say U_1 and U_2 with coordinates (x_1, x_2, y) and (z_1, z_2, w), respectively, under the relation $(x_1, x_2, y) \sim (z_1, z_2, w)$ if and only if $x_1 = z_1$, $x_2 = z_2$ and $y = \frac{1}{w}$ if $y > 0$, or $x_1 = z_1^\alpha$, $x_2 = z_2$ and $y = \frac{1}{w}$ if $y < 0$. Whether $\alpha \neq 1$, $\mathcal{C}_\alpha$ is not standardizable, therefore $\mathbb{S}^1 \subset \mathcal{C}_\alpha$ is not a permissible center of blowing–up in the category $\mathcal{G}$.

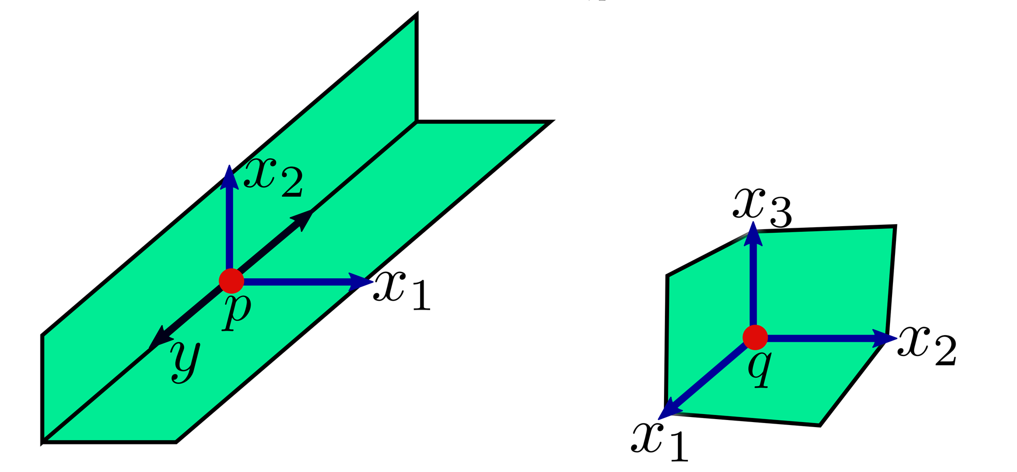
Local monomialization

Let M be a generalized analytic manifold, $p \in |M|$ and $f \in \mathcal{G}_M$. We say that f admits a local monomialization process *if there exists a* ***finite family*** $\{\pi_j\colon W_j \to M,\, L_j\}_{j\in J}$ *where each π_j is the composition of finitely many local blow–ups (with admissible centers) and each L_j is a compact subset of W_j such that $\cup \pi_j(L_j)$ is a neighborhood of p such that $f \circ \pi_j$ is of monomial type over L_j, for any j.*

The main result proved in [4] shows the existence of local monomialization in any dimension. The main steps in the proof given in [4] are: First, to get a monomial representation respect to the generalized variables, then apply the Weierstrass' preparation theorem to the standard variables and, finally, following quite similar arguments carried out in the standard case one completes the proof; *c.f.* [2]. Being just a local result, One does not need to worry about compatibility of the local blowing–ups.

Towards the Global Monomialization

A global monomialization *of $f \in \mathcal{G}_{M,p}$ is a finite composition $\pi\colon \tilde{M} \to M$ of blowing–ups with* global *admissible centers such that $f \circ \pi$ is locally monomial over $\pi^{-1}(p)$.*



$f_p = x_1^{\alpha_1} x_2^{\alpha_2} \cdot U_p$, $U_p(0,0,y) \not\equiv 0$

$f_q = x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} \cdot U_p$, $U_p(0,0,0) \neq 0$

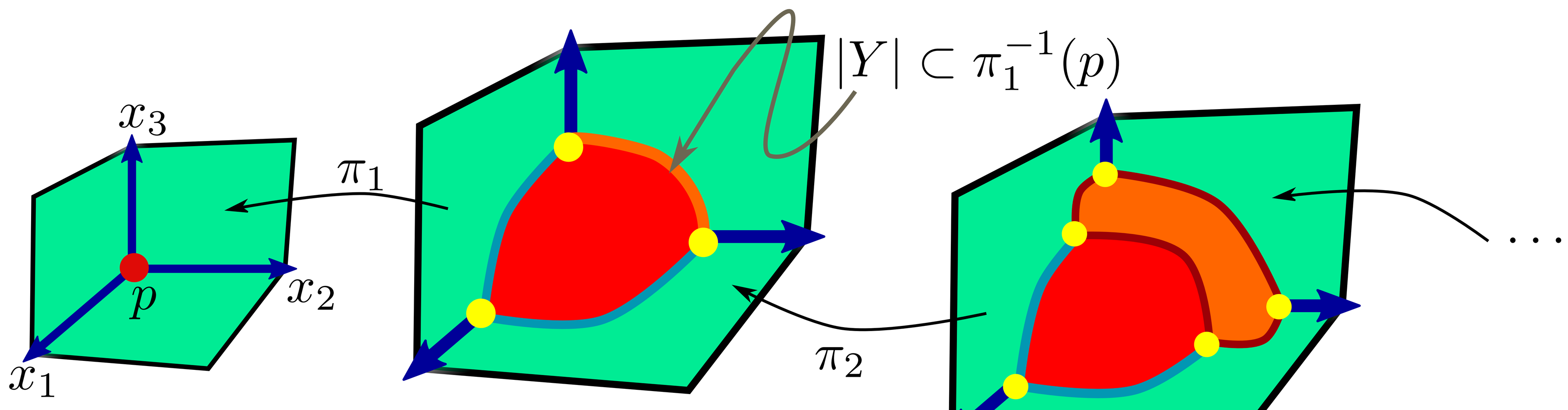
In search of a global result, we split the problem into two main stages: First, we try to get a monomial presentation respect to the generalized variables, the so called combinatorial monomialization. Once we have that situation, we shall try to adapt the strategy employed in the standard case, via the Weierstrass' preparation theorem (respect to the analytic variables) to control the local part and we shall explore in detail for some similar idea to the maximal contact set up in [1] to assure that the global counterpart is coherent.

Let $M \in \mathcal{G}$ be a generalized manifold and $f \in \mathcal{G}(M)$. We say that f is combinatorially monomial on $(M, \partial M)$, where ∂M denotes the boundary of M, if for every point $p \in M$ there exist local coordinates centered at p, (X_p, Y_p) (where X_p are the boundary components of p), such that $f_p \in \mathcal{G}_{M,p}$ is of monomial type respect to the generalized variables p; *i.e.* if $f_p = X_p^\alpha \cdot U$, where $U \in \mathcal{G}_{M,p}$ is such that $U(0, Y_p) \not\equiv 0$ (equivalently, the Newton polyhedron of f respect to the generalized variables has only one vertex).

Note that the boundary of M, $\partial M \subset M$, defines a normal crossings divisor in M and we have a stratification of ∂M naturally. The closed connected subsets of the strata are called combinatorial centers. By construction, each combinatorial center is, indeed, a admissible one. We have the following [5]:

MAIN RESULT (Global Combinatorial Monomialization of Generalized Analytic Functions, dim 3). ***Let $M \in \mathcal{G}$ be a three–dimensional generalized manifold. Let $f \in \mathcal{G}(M)$. Then there is a finite sequence of blowing–up morphisms, $\pi_N\colon M_N \to M_0 = M$, with combinatorial centers, such that the total transform of f is combinatorially monomial on M_N.***

Ideas of the proof. Due to the manner one glues the local charts of a generalized manifold, linear automorphisms of $\mathbb{R}^3$ play an relevant role because they enclose how the gluing is made along combinatorial centers. Using them, we can encode the combinatorial structure of the strata of the divisor of spaces obtained by blowing–up. Furthermore, the Newton polyhedron evolves according to these linear transformations. The proof consist in establishing a good strategy to this combinatorial game in order to win, *i.e.*, the Newton polyhedron has a unique vertex.



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