Complex Cellular Structures

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Goal and motivation

The goal is to parameterize bounded algebraic (or analytic) complex subsets of $\mathbb{C}^n$, i.e.

- find a collection of standard local models $U_\alpha \subset \mathbb{C}^n$ and a class of "good" holomorphic maps $\phi_\alpha : U_\alpha \to \mathbb{C}^n$, such that
- for any F holomorphic on a standard polydisc $B \subset \mathbb{C}^n$ as above there exist finitely many maps $\phi_i : U_i \mapsto \mathbb{C}^n$ such that $\cup \phi_i(U_i) \supset B$ and
  1. $F \circ \phi_i : U_i \mapsto \mathbb{C}$ is "simple"
  2. the maps ϕ_i depend well on parameters, moreover
  3. their number and complexity is roughly the same as the complexity of F whenever defined (algebraic, Pfaffian, Noetherian?)

We were motivated by a field of transcendental number theory born from

Bombieri-Pila theorem

Let $X \subset [0,1]^2$ be an analytic but *not algebraic* irreducible curve. Then the number $N(H;X)$ of rational points of height H on X grows slower than any positive degree of H: $\forall \epsilon > 0\ \exists C(\epsilon)$ s.t. $N(H;X) \leqslant C(\epsilon) H^\epsilon$.

Note that this is a real result. We want to approach it from $\mathbb{C}$.

Existing results

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- Yomdin-Gromov algebraic lemma (see below): U_i are real cubes, ϕ_i are C^r-smooth maps with bounded C^r norm, their number is reasonable. But only real and not even analytic result.

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- Yomdin-Gromov algebraic lemma (see below): U_i are real cubes, ϕ_i are C^r-smooth maps with bounded C^r norm, their number is reasonable. But only real and not even analytic result.

We paid by increasing the family of local models U_α, and get everything we wanted. How? Using a simple lemma on functions of one complex variable (instead of sophisticated algebraic geometry).

The Yomdin-Gromov Algebraic Lemma

Theorem (Yomdin-Gromov Algebraic Lemma)

Let $X \subset [0,1]^\ell$ be a set of dimension μ defined by polynomial equations or inequalities of total degree β. Then for every $r \in \mathbb{N}$ there exists a collection of C^r-smooth maps $\phi_j : [0,1]^\mu \to X$ whose images cover X and $\|\phi_j\|_r \leqslant 1$. Moreover the number of maps is bounded by a constant $C = C(\ell, \mu, \beta, r)$.

Crucial: uniformness in parameters.

But: the maps are only C^r-smooth, and not holomorphic!

- The Y-G theorem is the key step in Yomdin's proof of Shub's entropy conjecture for smooth maps. It also plays a crucial role in Pila-Wilkie's work on the density of rational points in definable sets.
- Y-G is useful because it allows us to do "Taylor approximations" on semialgebraic (or subanalytic) sets.
- Analyzing the dependence of $C(\ell, \mu, \beta, r)$ on β and r is important for both Yomdin's and Pila-Wilkie's directions.

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Yomdin-Gromov complexification: naive approach

Denote $D(r) = \{|z| < r\}$. Let $0 < \delta < 1$.

We define "local models" U_i to be standard polydiscs $D^\mu(1)$.

"Good" maps: C^r-smooth maps should be upgraded to

We say that a holomorphic map $f : D^\mu(1) \mapsto \mathbb{C}^\ell$ is δ-extendable if f can be holomorphically extended to $D^\mu(\delta^{-1})$.

Why? Cauchy formulas give control on all derivatives of f on $D^\mu(1)$.

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Wanted result

Let $X \subset \mathbb{C}^\ell$ be an algebraic set of dimension μ and complexity β. Then there is a finite collection of maps $\phi_j : D(1)^\mu \to X$ whose image cover $X \cap D(1)^n$ such that

- ϕ_j are $1/2$-extendable with $\|\phi_j\|_{D(2)^\mu} \leqslant 2$, and
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Key (counter)example

For $X = \{xy = \epsilon\} \subset \mathbb{C}^2$ one needs $\sim \log\log \epsilon^{-1}$ such maps as $\epsilon \to 0$.

Reminder on hyperbolic domains

A domain $U \subset \mathbb{C}$ whose complement consists of more than one point is called *hyperbolic*.

Theorem (Uniformization theorem)

For every hyperbolic $U \subset \mathbb{C}$ there is a holomorphic universal covering map $\pi : D \to U$ where $D = D(1)$ is the unit disc.

The Poincaré metric $(1-|z|^2)^{-1}|dz|$ on D is invariant under the conformal automorphisms of D and induces a canonical hyperbolic metric on U.

Lemma (Schwartz-Pick)

Let $f : U \to U'$ be a holomorphic map between hyperbolic domains. Then

$$\operatorname{dist}(f(z), f(w); U') \leqslant \operatorname{dist}(z, w; U), \qquad \forall z, w \in U \tag{1}$$

Corollary: $\operatorname{diam}(f(W), U') \leqslant \operatorname{diam}(W, U)$ for any $W \subset U$.

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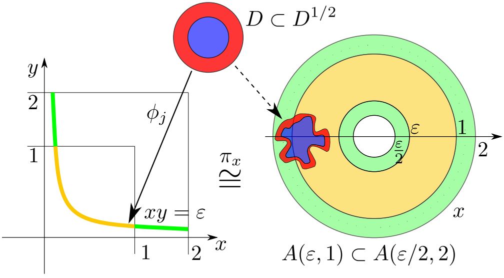
Recall our example $K \subset X$

$$K = \{xy = \varepsilon, |x|, |y| \leqslant 1\}$$
$$X = \{xy = \varepsilon, |x|, |y| < 2\}$$

By projection to x,
$K \simeq A(\varepsilon, 1) \subset X \simeq A(\varepsilon/2, 2)$,
where $A(r_1, r_2) = \{r_1 < |z| < r_2\}$



Computation

If $f : D(2) \to X$ is holomorphic then by Schwarz-Pick

$$\operatorname{diam}(f(D(1)); X) \leqslant \operatorname{diam}(D(1); D(2)) = \log \sqrt{3}. \qquad (2)$$

On the other hand $\operatorname{diam}(K; X) \sim \log \log \varepsilon^{-1}$.

Conclusion: to cover K by $\phi_j(D(1))$ we will need at least $\log \log \varepsilon^{-1}$ maps!

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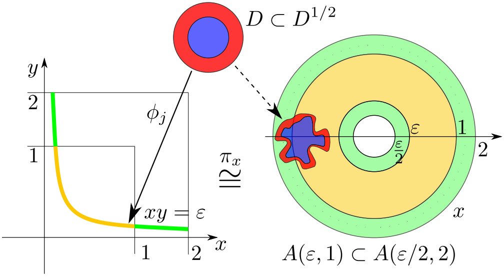

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Real cells

In tame geometry, the notion of a *cylindrical cell* is defined inductively as follows:

- A cell of length one $\mathcal{C} \subset \mathbb{R}$ is a point or an interval.
- A cell $\mathcal{C} \subset \mathbb{R}^{\ell+1}$ length $\ell + 1$ is a set of the form

$$\mathcal{C} = \mathcal{C}_{1..\ell} \odot \mathcal{F} := \{\mathbf{x}_{1..\ell+1} : \mathbf{x}_{1..\ell} \in \mathcal{C}_{1..\ell},\ \mathbf{x}_{\ell+1} \in \mathcal{F}(\mathbf{x}_{1..\ell})\} \tag{3}$$

  where $\mathcal{C}_{1..\ell}$ is a cell of length ℓ and the fiber $\mathcal{F}$ is

$$\mathcal{F}(\mathbf{x}_{1..\ell}) = \{f(\mathbf{x}_{1..\ell})\} \qquad \text{or} \qquad \mathcal{F}(\mathbf{x}_{1..\ell}) = (f_1(\mathbf{x}_{1..\ell}), f_2(\mathbf{x}_{1..\ell}))$$

  where f or $f_1 < f_2$ are continuous functions on $\mathcal{C}_{1..\ell}$ (i.e. $\mathcal{F}$ is a cell of length one depending on $\mathbf{x}_{1..\ell}$).

Note that every cell is homeomorphic to a real cube of dimension $\dim \mathcal{C}$.

Definition

A cell decomposition (C.D.) of a set $X \subset \mathbb{R}^\ell$ is a covering $X = \bigcup_\alpha \mathcal{C}_\alpha$ by (pairwise disjoint) cells.

Theorem (Cellular decomposition)

Every semialgebraic set can be subdivided into cells.

Denote $\pi_{1..k}(\mathbf{x}_{1..\ell}) = \mathbf{x}_{1..k}$.

- C.D. of $X \implies$ C.D. of $\pi_{1..k}(X)$.
- C.D. of $X \implies$ C.D. of $\pi_{1..k}^{-1}(p)$.

A polynomial P is *compatible* with a cell $\mathcal{C}$ if $P|_{\mathcal{C}} \equiv 0$ or $P|_{\mathcal{C}}$ is non-vanishing. Equivalently P has a constant sign on $\mathcal{C}$.

Theorem

$P_1, \dots, P_k$ polynomials $\implies \mathbb{R}^\ell = \bigcup_\alpha \mathcal{C}_\alpha$ with $\mathcal{C}_\alpha, P_j$ pairwise compatible.

Second theorem implies the first.

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Theorem

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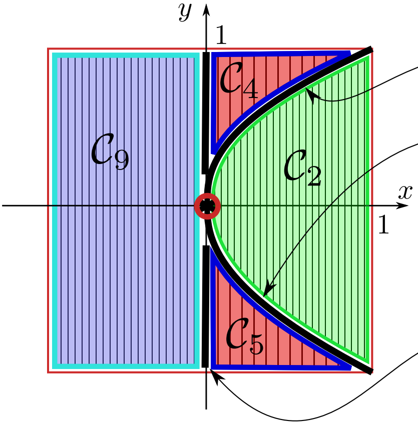
Cell Decompositions

$$\{y^2 = x\} \subset \{|x|, |y| < 1\}$$



$\mathcal{C}_1 = \{(x,y)|0 < x < 1, y = \sqrt{x}\}$

$\mathcal{C}_2 = \{0 < x < 1, -\sqrt{x} < y < \sqrt{x}\}$

$\mathcal{C}_3 = \{(x,y)|0 < x < 1, y = -\sqrt{x}\}$

$\mathcal{C}_4 = \{0 < x < 1, \sqrt{x} < y < 1\}$

$\mathcal{C}_5 = \{0 < x < 1, -1 < y < -\sqrt{x}\}$

$\mathcal{C}_6 = \{(0,0)\}$

$\mathcal{C}_7 = \{x = 0, 0 < y < 1\}$

$\mathcal{C}_8 = \{x = 0, -1 < y < 0\}$

$\mathcal{C}_9 = \{-1 < x < 0, -1 < y < 1\}$

Complex cells

Instead of fibers which are points or intervals, we take $\mathcal{F}$ to be one of

$$* = \{0\} \qquad D(r) = \{|z| < |r|\}$$
$$D_\circ(r) = \{0 < |z| < |r|\} \qquad A(r_1, r_2) = \{|r_1| < |z| < |r_2|\}$$

where r or r_1, r_2 are *holomorphic* bounded functions on $\mathcal{C}_{1..\ell}$ and $r \neq 0$ or $0 < |r_1| < |r_2|$, respectively.

Example

$$D_\circ(1) \odot A(\mathbf{z}_1, 2) = \{\mathbf{z}_{1,2} : 0 < |\mathbf{z}_1| < 1,\ |\mathbf{z}_1| < |\mathbf{z}_2| < 2\}.$$

As a convenience our fibers are always centered at the origin.

Definition

A holomorphic function $F \in \mathcal{O}(\mathcal{C})$ is compatible with $\mathcal{C}$ if F is identically zero or non-vanishing on $\mathcal{C}$.

δ-extensions of complex cells

Holomorphicity means we can talk about analytic continuation. For $0 < \delta < 1$ the δ-extension is defined inductively by

$$\mathcal{C}^\delta := \mathcal{C}^\delta_{1..\ell} \odot \mathcal{F}^\delta, \quad \text{where}$$

$$*^\delta = * \qquad D^\delta(r) = D(\delta^{-1} r)$$

$$D^\delta_\circ(r) = D_\circ(\delta^{-1} r) \qquad A^\delta(r_1, r_2) = A(\delta r_1, \delta^{-1} r_2)$$

assuming that r or r_1, r_2 continue holomorphically to $\mathcal{C}^\delta_{1..\ell}$ and still satisfy $r \neq 0$ or $0 < |r_1| < |r_2|$ there.

For $D_\circ(1) \odot A(\mathbf{z}_1, 2)$ we have $0 < |r_1| < |r_2|$ on $D^\delta_\circ(1)$ for $\delta \geqslant 1/2$. Therefore $(D_\circ(1) \odot A(\mathbf{z}_1, 2))^\delta = D_\circ(\delta^{-1}) \odot A(\delta|\mathbf{z}_1|, 2\delta^{-1})$ is well-defined for $1/2 \leqslant \delta < 1$.

This is the principal new ingredient missing in the real context. The hyperbolic geometry of $\mathcal{C} \subset \mathcal{C}^\delta$ will play a key role in our approach.

Complex cellular decomposition

If $f : \mathcal{C} \to \hat{\mathcal{C}}$ maps $\mathbf{z} \to \mathbf{w}$ we say that f is *prepared* if f is holomorphic and bounded on $\mathcal{C}$ and for $j = 1, \dots, \ell$

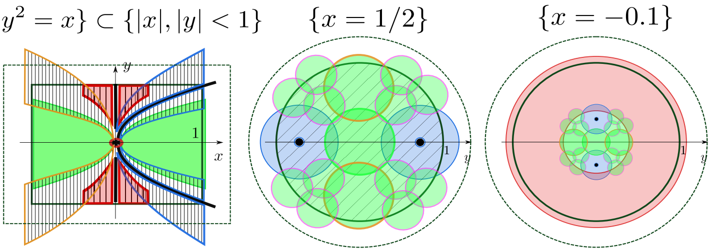
$$\mathbf{w}_j = \mathbf{z}_j^{\mu_j} + \phi_j(\mathbf{z}_{1..j-1}), \qquad \mu_j \in \mathbb{N}_{>0}. \tag{4}$$

- The image of a prepared map is a more accurate analog of a real cell.
- f *admits* δ*-extension* if it continues holomorphically to $f : \mathcal{C}^\delta \to \hat{\mathcal{C}}$.

Theorem (CPT)

Let $\mathcal{C}$ admits δ-extension and $F_1, \dots, F_k \in \mathcal{O}_b(\mathcal{C}^\delta)$. Then there exists a finite collection of prepared cellular maps $f_j : \mathcal{C}_j \to \mathcal{C}^\delta$ which admit δ-extensions such that the $f_j(\mathcal{C}_j^\delta)$ are compatible with each F_i and cover $\mathcal{C}$. If $\mathcal{C}$, F_i are algebraic of complexity β, then the number of maps is $\operatorname{poly}_\ell(\beta, k, \delta)$ and f_j, $\mathcal{C}_j$ are algebraic of complexity $\operatorname{poly}_\ell(\beta, k)$.

For example, the cells for which all F_i vanish give a "uniformization" by cells of the set of common zeros of F_i.



1) $* \odot *$ 2) $* \odot D_\circ(1)$ 3) $D_\circ(1) \odot *$ 4) $D_\circ(1) \odot D(z/2)$

4-5) $D_\circ(1) \odot D_\circ(z/2)$ 7-12) $D_\circ(1) \odot D_\circ(z/4)$ 13) $D_\circ(0.4) \odot A(\frac{5}{4}z, 1)$

Cell decomposition of $\mathcal{C} = D(1) \odot D(1) \subset \mathcal{C}^\delta = D(\delta) \odot D(\delta)$ compatible with $F(x, y) = y^2 - x$ and two cuts by $\{x = \text{const}\}$. E.g.

$$\phi_{13} : D_\circ(0.4) \odot A(\tfrac{5}{4}z, 1) \mapsto \mathcal{C}^\delta, \qquad \phi_{13}(z, w) = (z^2, w).$$

For $x > x_0$ only discs, and for $x < x_0 \ll 1$ one should use annulus (red cell), exactly as for $\{xy = \epsilon\}$. Two points $\{\pm\sqrt{x}\}$ form a *cluster*.

Corollaries: Yomdin-Gromov

If $\mathcal{C}$, $F_1, \ldots, F_k$ are real then CPT gives real C.D. with extras (analytic continuation of maps). This implies effective bounds on

Yomdin-Gromov constant

The constant $C = C(\ell, \mu, \beta, r) = \text{poly}_\ell(\beta) \cdot r^\mu$. Moreover, the maps ϕ_j can be chosen to be semialgebraic of complexity $\text{poly}_\ell(\beta, r)$.
Alternatively, there are $\text{poly}_\ell(\beta)$ Yomdin-Gromov $(A, 2)$-mild maps ϕ_j:

$$\forall \alpha \in \mathbb{N}^\mu \quad \|D^\alpha \phi\| \leqslant \alpha! \left(A|\alpha|^2\right)^{|\alpha|}, \qquad A = \text{poly}_\ell(\beta).$$

Similar bounds for $\mathbb{R}_{an}$-definable families (constants depend on family).
This implies tight bounds on the tail entropy and volume growth for analytic maps, conjectured by Yomdin in 1991.
Applications to counting rational points on algebraic and transcendental varieties.

Corollaries: resolution of singularities

Theorem (Classical Uniformization theorem)

Let $F_1, .., F_k \in \mathcal{O}_b(B)$. Then B can be covered by images of maps $f_j : B_j \to B$ such that $f_j^* F_i$ is a monomial times a unit. Moreover the maps are of a special form.

Complex cells analogue: Monomialization Lemma

Let $F : \mathcal{C}^{\{\rho\}} \to \mathbb{C} \setminus \{0\}$ be holomorphic and bounded. Then on $\mathcal{C}^{\{\rho\}}$ we have $F = \mathbf{z}^\alpha \cdot U(\mathbf{z})$ where $\alpha \in \mathbb{Z}^\ell$, $\log U$ is holomorphic, univalued in $\mathcal{C}^{\{\rho\}}$ and

$$\operatorname{diam}(\log U(\mathcal{C}); \mathbb{R}) < O_f(1) \cdot \rho,$$

with $|\alpha(F)|, O_F(1) = \operatorname{poly}_\ell(\beta)$ in algebraic case.

- The exponent α is defined topologically.
- Nontrivial since $\mathcal{C}$ is not necessarily compact.

Key argument: Domination Lemma

Domination Lemma

Let $f : \mathcal{C}^\delta \to \mathbb{C} \setminus \{0,1\}$. Then either f of f^{-1} is uniformly bounded on $\mathcal{C}$ from above by some constant $C = C(\ell, \delta)$ *independent of* $\mathcal{C}$.

Key Miracle in dim $= 1$:

For $\mathcal{C} = A(\epsilon, 1)$, we have $\operatorname{diam}(\mathcal{C}; \mathcal{C}^\delta) \sim \log\log\epsilon^{-1} \to \infty \quad \text{as} \quad \epsilon \to 0$.
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Corollary of dim $\mathcal{C} = 1$ case: Little and Big Picard Theorems

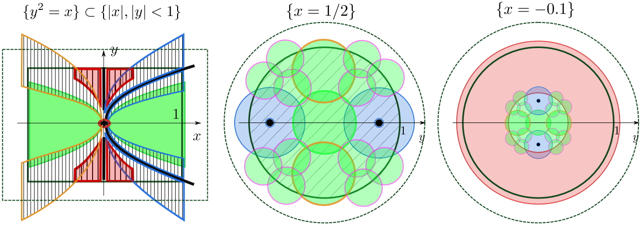
1. Let $f : \mathbb{C} \mapsto \mathbb{C} \setminus \{0,1\}$ be an entire function. Then $f \equiv \text{const}$.
2. Let $f : D_\circ(1) \mapsto \mathbb{C} \setminus \{0,1\}$ be a holomorphic function. Then f has at most a pole at 0.

Proof: 1) Either f or f^{-1} is bounded by C on any $D(r)$, i.e. on $\mathbb{C}$.
2) Either f or f^{-1} is bounded by C on $D_\circ(\frac{1}{2})$, i.e. is holomorphic at 0.

How Domination Lemma works

Inductive step

Let $X_\epsilon = \{x_i(\epsilon)\}_{i=1}^n \subset D(1)$ be holomorphically depending on $\epsilon \in E$. How to cover $D(1) \setminus X_\epsilon$ by cells with extensions?



Relative distance $\frac{x_1 - x_2}{x_1 - 1}$ changes from 0 to ∞ as $\epsilon \in (0, 1)$.

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Fulton-McPherson compactification

Describes confluences scenarios of X_ϵ.

Definition

Cluster is a subset $Y_\epsilon \subset X_\epsilon$ of points which are closer one-to-another than to other points: for any $x_i, x_j \in Y_\epsilon$, $x_k \in X_\epsilon \setminus Y_\epsilon$ we have $|x_i - x_j| \ll |x_i - x_k|$.

To X_ϵ corresponds a tree of clusters. One can read it from $|\alpha_{ijk}(\epsilon)|$, where

$$\alpha_{ijk} : E \to \mathbb{C} \cup \{\infty\}, \qquad \alpha_{ijk}(\epsilon) = \frac{x_i - x_j}{x_i - x_k}$$

How Domination Lemma works (continuation)

Answer: To cover $D(1) \setminus X_\epsilon$ cover smallest clusters by discs, add annulus to go to next cluster, cover this bigger cluster by discs, etc.

Can be done analytically in ϵ as long as the tree of clusters doesn't change.

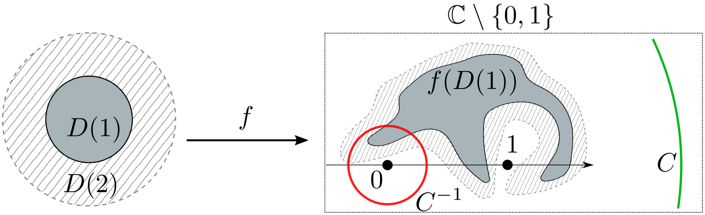
Domination Lemma to help!

Let $f : \mathcal{C}^\delta \mapsto E$ be a cell compatible with all α_{ijk}. Then

$$\alpha_{ijk} : \mathcal{C}^\delta \mapsto \mathbb{C} \setminus \{0, 1\},$$

so is either not too big or not too small *uniformly on* $\mathcal{C}$. Thus the cluster trees for all $X_\epsilon, \epsilon \in f(\mathcal{C})$ are the same!

Domination Lemma Proof: classical result for D



Schwarz-Pick: $\operatorname{diam}\left(f\left(D(1)\right), \mathbb{C}\setminus\{0,1\}\right) \leq \operatorname{diam}\left(D(1), D(2)\right) = \log\sqrt{3}$

Domination Lemma for D

Let $R : D(2) \to \mathbb{C}\setminus\{0,1\}$. Then R is uniformly bounded on $D(1)$, either above or below, by some absolute constant C.

Proof: Take $C > 0$ s.t. $\operatorname{dist}\left(\{|z| = C\}, \{|z| = C^{-1}\}; \mathbb{C}\setminus\{0,1\}\right) > \log\sqrt{3}$.

Domination Lemma for maps to $D(1)$ and $\mathcal{C} = A$

Domination Lemma: maps to $D(1)$

Let $R : \mathcal{C}^\delta \to D$. Then $\operatorname{diam}(R(\mathcal{C}), D) \leqslant \Delta = \Delta(\ell, \delta)$ *independent of* $\mathcal{C}$.

- Let $A = A(r_1, r_2)$ and $f : A^\delta \to D(1)$. We equip the domain and range with their hyperbolic metrics.
- The diameter of $A \subset A^\delta$ is unbounded as $r_1/r_2 \to 0$ (this was the whole point of allowing annuli). However the diameter of $S_1 = \{|z| = r_1\}$ and $S_2 = \{|z| = r_2\}$ in A^δ *is* bounded by some $\rho = \rho(\delta)$:

$$\operatorname{diam}(S_1, A^\delta) \leqslant \operatorname{diam}(S_1, A(\delta r_1, \delta^{-1} r_1)) = \frac{\pi^2}{2\,|\log \delta|} = \rho.$$

- By the open mapping theorem $\partial F(\mathcal{C}) \subset F(\partial \mathcal{C}) = F(S_1) \cup F(S_2)$, and the latter two have diameter bounded by ρ in D.
- Elementary geometry: if the boundary of a bounded *planar* domain has bounded diameter, then the diameter of the domain is similarly bounded.

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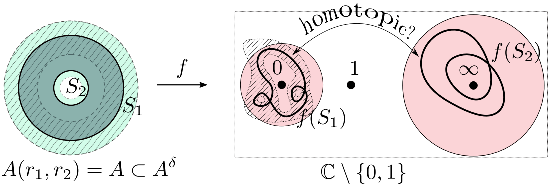
- Let $A = A(r_1, r_2)$ and $f : A^\delta \to \mathbb{C} \setminus \{0,1\}$. We equip the domain and range with their hyperbolic metrics.
- Suppose f takes very small and very large values. By the maximum principle it must take them on S_1 and S_2.
- The hyperbolic diameter of $f(S_1)$ and $f(S_2)$ in $\mathbb{C} \setminus \{0,1\}$ is bounded by ρ. In particular if $f(S_1)$ is very close to 0 at one point then it is uniformly close to 0, and similarly for $f(S_2)$ and ∞.
- S_1, S_2 are homotopic in A, so $f(S_1), f(S_2)$ are homotopic in $\mathbb{C} \setminus \{0,1\}$. But one lives near 0 and the other near ∞, so they are in fact contractible.



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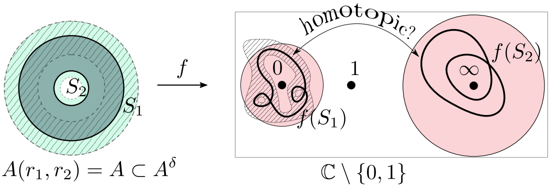

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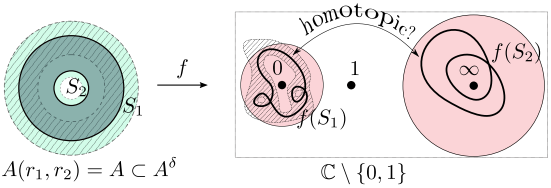

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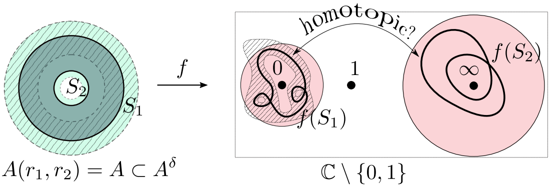

Proof of Domination Lemma in dim = 1 (cont.)

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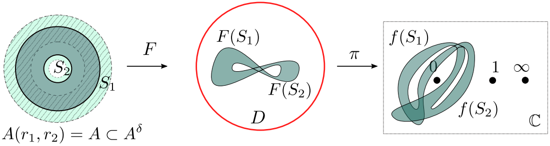

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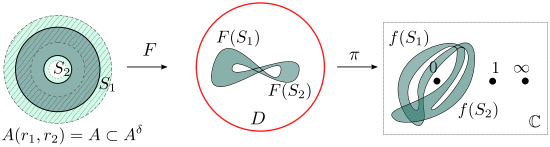

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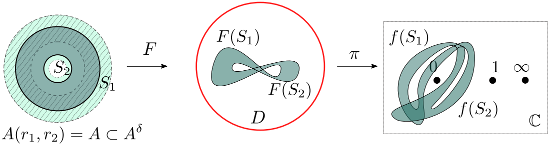

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