Dulac map and time in families of hyperbolic saddles

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joint work with Jordi Villadelprat (URV)

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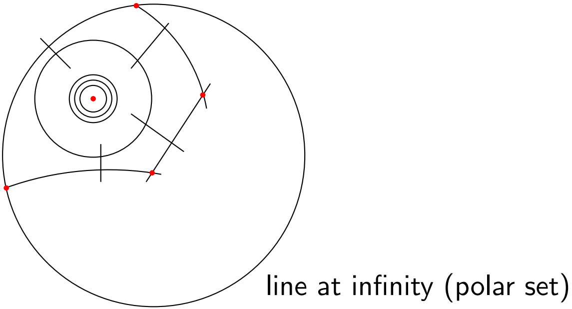
Motivation: Dulac map and time as building block

Qualitative behavior (bifurcation) of the period function of a center at the outer boundary of its period annulus (polycycle).



Difficulty: The regularity of the period function drops out at the polycycle.

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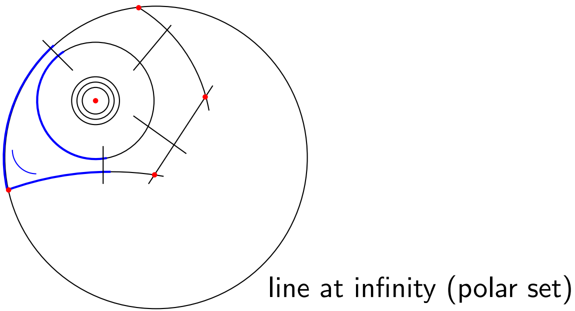

Difficulty: The regularity of the period function drops out at the polycycle.

Tool: Asymptotic expansion of the period function at the polycycle, uniform with respect to parameters.

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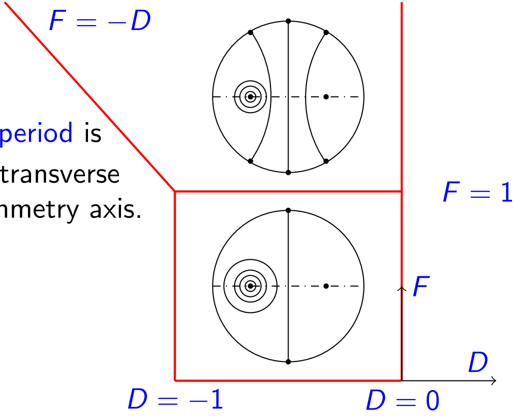
Test Example: Loud family $(-y+xy)\partial_x + (x + Dx^2 + Fy^2)\partial_y$, symmetric system with Darboux first integral $(1-x)^{\alpha}(y^2 - P_2(x))$ for $F(F-1)(F-1/2) \neq 0$ (Liouville first integral in general).

Symmetry implies half period is the Dulac time between transverse sections located at the symmetry axis.



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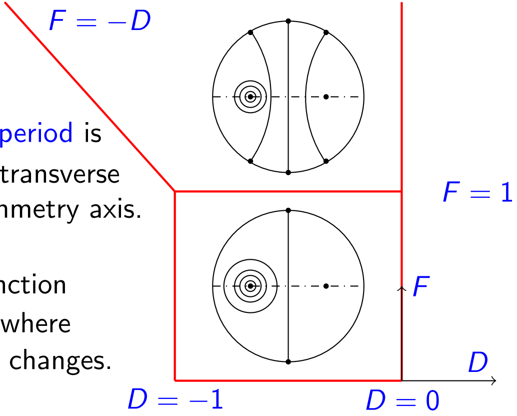

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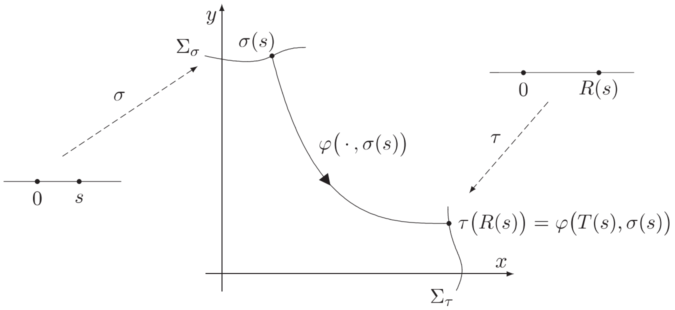
Increasing period function outside the red line, where the polycycle's topology changes.

Dulac map and time of families of hyperbolic saddles

Building block in hyperbolic monodromic polycycles:

$$X_\mu = \frac{1}{x^m y^n}\left(P_\mu(x,y)x\frac{\partial}{\partial x} + Q_\mu(x,y)y\frac{\partial}{\partial y}\right), \ \lambda = -\frac{Q_\mu(0,0)}{P_\mu(0,0)} > 0,$$

where P, Q are $\mathscr{C}^\infty$ functions on $\Omega \times U \subset \mathbb{R}^2 \times \mathbb{R}^N$ amb $m, n \in \mathbb{Z}_+$.



FLP: X_μ is locally orbitally linearizable ($\Leftarrow$ Darboux integrable).

(L, K)-Flatness condition

Definition: If $W \subset \mathbb{R}^{N+1}$ is an open neighborhood of $\{0\} \times U$ and $f : W \cap ((0, +\infty) \times U) \to \mathbb{R}$ is of class $\mathscr{C}^K$ we say that $f(s; \mu) \in \mathcal{F}_L^K(\mu_0)$ if $\forall \nu = (\nu_0, \nu_1, \ldots, \nu_N) \in \mathbb{Z}_+^{N+1}$, $|\nu| \leq K$, $\exists V \ni \mu_0$, $\exists C, s_0 > 0$ such that $\forall \mu \in V$ and $\forall s \in (0, s_0)$

$$|\partial^\nu f(s; \mu)| \leq C s^{L-\nu_0},$$

where $\partial^\nu = \partial_s^{\nu_0} \partial_{\mu_1}^{\nu_1} \cdots \partial_{\mu_N}^{\nu_N}$ and $|\nu| = \nu_0 + \nu_1 \cdots + \nu_N$.

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Lemma: If $L > K$ every $f \in \mathcal{F}_L^K(\mu_0)$ extends to a $\mathscr{C}^K$ function $\tilde{f}$ in a neighborhood of $(0, \mu_0)$ such that $\partial^\nu \tilde{f}(0;\mu) = 0$ for $|\nu| \leq K$.

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Lemma: If $L' > L$ and $f \in \mathcal{F}_{L'}^K(\mu_0)$ then $f \in s^L \mathcal{I}_K(\mu_0)$, i.e. for every $n \leq K$ there is a neighborhood $V \ni \mu_0$ such that $\mathscr{D}^n(f(s; \mu)/s^L) \to 0$ as $s \to 0^+$ uniformly on $\mu \in V$, where $\mathscr{D} = s\partial_s$ is the Euler operator. ($\mathcal{I}_K$ are the Mourtada's classes.)

Unifom asymptotic expansion (in the FLP case)

Theorem: For every $\mu_0 \in U$ there exist a neighborhood $V \ni \mu_0$ and polynomials $D_{ij}, T_{ij} \in \mathscr{C}^\infty(V)[w]$ such that $\forall L \in \mathbb{R}$

$$D(s;\mu) = s^\lambda \sum_{0 \leq i+\lambda_0 j \leq L} s^{i+\lambda j} D_{ij}(\omega;\mu) + \mathcal{F}_L^\infty(\mu_0),$$

$$T(s;\mu) = \tau_0(\mu)\log s + \sum_{0 \leq i+\lambda_0 j \leq L} s^{i+\lambda j} T_{ij}(\omega;\mu) + \mathcal{F}_L^\infty(\mu_0),$$

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where $\lambda_0 = \lambda(\mu_0)$. If $\lambda_0 = p/q$ then ω is the Ecalle-Roussarie compensator (deformation of the logarithm $-\ln s = \omega(s;0)$):

$$\omega(s;\alpha(\mu)) := \int_s^1 x^{-\alpha(\mu)} \frac{dx}{x} = \frac{s^{-\alpha(\mu)} - 1}{\alpha(\mu)}, \quad \alpha(\mu) = p - \lambda(\mu) q.$$

Moreover $\deg D_{ij} = \deg T_{ij} = 0$ if $\lambda_0 \notin \Delta_{ij} \subset \mathbb{Q}_{>0}$ discrete subset and $\tau_0 \equiv 0$ except for $(m,n) = (0,0)$.

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Work in progress: elimination of the FLP hypothesis.

Formulae for the first coefficients of the Dulac time (2018)

Assume $m=0$, $n>0$ and define $\sigma_{ij}=\sigma_i^{(j)}(0)$, $\tau_{ij}=\tau_i^{(j)}(0)$,

$$T_{00}=\int_{\sigma_{20}}^{0}\frac{x^{n-1}}{Q(x,0)}dx,\quad \lambda=-\frac{Q(0,0)}{P(0,0)},$$

$$L(u)=\exp\int_0^u\left(\frac{P(0,y)}{Q(0,y)}+\frac{1}{\lambda}\right)\frac{dy}{y},$$

$$M(u)=\exp\int_0^u\left(\frac{Q(x,0)}{P(x,0)}+\lambda\right)\frac{dx}{x}.$$

- If $\lambda>1/n$ then $T(s)=T_{00}+T_{10}s+s\mathcal{I}_1$ with

$$T_{10}=-\frac{\sigma_{21}\sigma_{20}^{n-1}}{Q(0,\sigma_{20})}+\frac{\sigma_{11}\sigma_{20}^{1/\lambda}}{L(\sigma_{20})}\int_0^{\sigma_{20}}\frac{\partial_1 Q(0,y)L(y)}{Q(0,y)^2}\frac{dy}{y^{1/\lambda-n+1}}$$

- If $\lambda<1/n$ then $T(s)=T_{00}+T_{0n}s^{\lambda n}+s^{\lambda n}\mathcal{I}_1$ with

$$\frac{L(\sigma_{20})^{\lambda n}\,T_{0n}}{\sigma_{11}^{\lambda n}\sigma_{20}^{n}}=\frac{\tau_{10}^{-\lambda n}}{nQ(0,0)}+\int_0^{\tau_{10}}\left(\frac{M(x)^n}{P(x,0)}-\frac{M(0)^n}{P(0,0)}\right)\frac{dx}{x^{\lambda n+1}}$$

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Theorem (Mardešić-M.-Villadelprat, 2003)

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- If $\lambda \approx \frac{1}{n}$ then $T(s) = T_{00} + s[T_{100} + T_{101}\omega(s;1-\lambda n)] + s\mathcal{I}_1$ with $T_{101} = (1-\lambda n)T_{0n}$ and $T_{100} = T_{10} + T_{0n}$ extending to $\lambda = \frac{1}{n}$.

Modifying Mellin transform

Mellin transform: $f(x) \mapsto \{\mathscr{M} f\}(\alpha) = \int_0^\infty x^\alpha f(x) \frac{dx}{x}$.

Example 0: The Gamma function

$$\Gamma(\alpha) = \begin{cases} \{\mathscr{M}(e^{-x})\}(\alpha) & \text{if} \quad \alpha > 0 \\ \{\mathscr{M}(e^{-x} - 1)\}(\alpha) & \text{if} \quad \alpha \in (-1, 0) \\ \{\mathscr{M}(e^{-x} - (1 - x))\}(\alpha) & \text{if} \quad \alpha \in (-2, -1) \\ \quad \vdots & \quad \vdots \end{cases}$$

Definition-Proposition: If $T_0^r f(x) = \sum_{i=0}^{r} \frac{f^{(i)}(0)}{i!} x^i$ and $\alpha \notin \mathbb{Z}_-$ then

$$\hat{f}_\alpha(u) := \sum_{i=0}^{k-1} \frac{f^{(i)}(0)}{i!(i+\alpha)} u^i + u^{-\alpha} \int_0^u [f(x) - T_0^{k-1} f(x)] x^{\alpha-1} dx$$

does not depend on $k > -\alpha$.

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$$\hat{f}_\alpha(u) = u^{-\alpha} \int_0^u f(x) x^{\alpha-1} dx \quad \text{for} \quad \alpha > 0.$$

Remark: $\lim_{\alpha \to -i} (i + \alpha) \hat{f}_\alpha(u) = \frac{f^{(i)}(0)}{i!} u^i$ residue at pole $\alpha = -i \in \mathbb{Z}_-$.

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Example 1: If $f(x; b, c) = (1 + cx^2)^b$ and $b < -\frac{\alpha}{2}$ then

$$\lim_{u \to +\infty} u^\alpha \hat{f}_\alpha(u; b, c) = \frac{c^{-\frac{\alpha}{2}}}{2} B\left(\frac{\alpha}{2}, -b - \frac{\alpha}{2}\right),$$

where $B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ is the Euler Beta function.

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does not depend on $k > -\alpha$.

Example 2: If $f(x; a, c; d) = (1 - dx)^{-a}(1 - x)^{c-1}$, $c > 0$ and $d < 1$ then

$$\lim_{u \to 1^-} \hat{f}_\alpha(u; a, c; d) = B(\alpha, c)\,{}_2F_1(a, \alpha; c + \alpha; d),$$

where ${}_2F_1(a, b; c; d)$ is the hypergeometric Gauss function.

Formulae for the first coefficients of the Dulac time (2003)

Define $A(u) = L(u)\partial_1 Q^{-1}(0,u)$, $B(u) = L(u)\partial_1\left(\frac{P}{Q}(0,u)\right)$,
$C(u) = L^2(u)\partial_1^2 Q^{-1}(0,u) + 2A(u)\widehat{B}_{-1/\lambda}(u)$, $D(u) = \frac{M^n(u)}{P(u,0)}$,
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Then

$$T_{10} = -\frac{\sigma_{21}\sigma_{20}^{n-1}}{Q(0,\sigma_{20})} - \frac{\sigma_{11}\sigma_{20}^{n}}{L(\sigma_{20})}\widehat{A}_{n-1/\lambda}(\sigma_{20}) \text{ for } \lambda \notin \Delta_{10} = \left\{\frac{1}{n+i}\right\}_{i=0}^{\infty}$$

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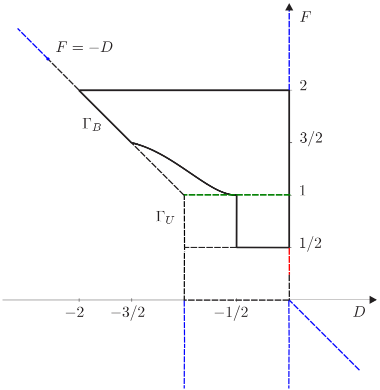
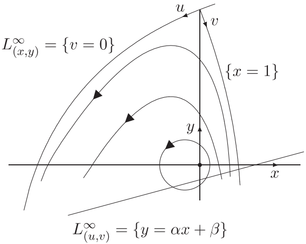
for $\lambda \notin \Delta_{0,n+1} = \left\{\frac{i}{n+1}\right\}_{i=1}^{\infty}$. There is a longer explicit expression for T_{20} involving $\widehat{A}_{n-\frac{1}{\lambda}}$, $\widehat{B}_{-\frac{1}{\lambda}}$ and $\widehat{C}_{n-\frac{2}{\lambda}}$ valid for $\lambda \notin \Delta_{20} = \{\frac{2}{n+i}\}_{i=0}^{\infty}$.

Period function criticality of quadratic Loud centers

$$\dot{x} = -y + xy, \quad \dot{y} = x + Dx^2 + Fy^2$$

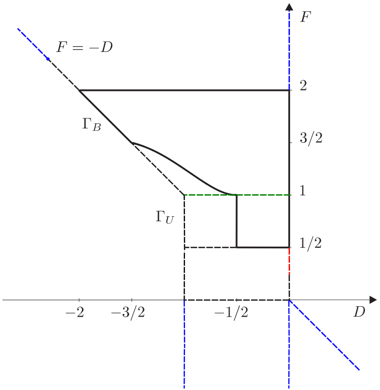


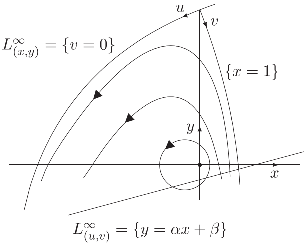


In $\Gamma_B \setminus \{D(F+D)(F-\frac{4}{3})(F-\frac{1}{2}) = 0\} \cup \{(-\frac{1}{2}, \frac{1}{2}), (-\frac{1}{2}, 2)\}$ we have criticality 1 and criticality 2 in $\Gamma_B \cap \{F = \frac{4}{3}\}$

Period function criticality of quadratic Loud centers

$$\dot{x} = -y + xy, \quad \dot{y} = x + Dx^2 + Fy^2$$





In $\Gamma_B \setminus \{D(F+D)(F-\frac{4}{3})(F-\frac{1}{2}) = 0\} \cup \{(-\frac{1}{2}, \frac{1}{2}), (-\frac{1}{2}, 2)\}$ we have criticality 1 and criticality 2 in $\Gamma_B \cap \{F = \frac{4}{3}\} \Leftarrow$ Explicit expressions of T_{ij} in terms of Gamma and hypergeometric functions.

Explicit expressions of the coefficients for Loud centers

For $\mu = (D, F) \in (-1, 0) \times [(0, 1) \setminus \{1/2\}]$, $\lambda = \frac{F}{1-F}$ and

$$T_{00}(\mu) = \frac{\pi}{2\sqrt{F(D+1)}}, \quad T_{01}(\mu) = \rho_1(\mu)\frac{\Gamma(-\frac{\lambda}{2})}{\Gamma(\frac{1-\lambda}{2})},$$

$$T_{10}(\mu) = \rho_2(\mu)(2D+1)\frac{\Gamma(1-\frac{1}{2\lambda})}{\Gamma(\frac{3}{2}-\frac{1}{2\lambda})},$$

$$T_{20}(\mu) = \rho_3(\mu)\frac{\Gamma(\frac{1}{2}-\frac{1}{\lambda})}{\Gamma(1-\frac{1}{\lambda})} + \rho_4(\mu)(2D+1).$$

where $\rho_i(\mu)$ are analytic functions which are positive por $i = 1, 2, 3$.

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For $\mu = (D, F) \in \{F + D > 0, D < 0, F > 1\}$, $\lambda = \frac{1}{2(F-1)}$ and the outer boundary of the period annulus is contained in the line at infinity and an invariant hyperbola $\frac{y^2}{2} = (a(\mu)x^2 + b(\mu)x + c(\mu))$ meeting the axis $\{y = 0\}$ at the points $(p_1, 0), (p_2, 0)$ with $p_1 < p_2$.

Explicit expressions of the coefficients for Loud centers

For $\mu = (D, F) \in (-1, 0) \times [(0, 1) \setminus \{1/2\}]$, $\lambda = \frac{F}{1-F}$ and

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For $\mu = (D, F) \in \{F + D > 0, D < 0, F > 1\}$, $\lambda = \frac{1}{2(F-1)}$ and

$$T_{00}(\mu) = \frac{\sqrt{2}}{\sqrt{a}(1-p_1)}\, {}_2F_1\left(1, -\frac{3}{2}; -\frac{1}{2}; \frac{1-p_2}{1-p_1}\right),$$

$$T_{01}(\mu) = \rho_1(\mu) B\left(-\lambda, \frac{1}{2}\right),$$

$$T_{10}(\mu) = \rho_2(\mu) B\left(1-\frac{1}{\lambda}, -\frac{1}{2}\right) {}_2F_1\left(-1-\frac{1}{\lambda}, -\frac{1}{2}; \frac{1}{2}-\frac{1}{\lambda}; \frac{1-p_2}{1-p_1}\right)$$

$$T_{20}(\mu) = \rho_3(\mu) B\left(1-\frac{2}{\lambda}, -\frac{3}{2}\right) {}_2F_1\left(-\frac{2}{\lambda}-3, -\frac{3}{2}; -\frac{1}{2}-\frac{2}{\lambda}; \frac{1-p_2}{1-p_1}\right) + \rho_4(\mu) T_{10}(\mu).$$

Thanks for your attention!