On the number of polynomial solutions of Bernoulli and Abel polynomial differential equations

Francesc Mañosas

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Outline of the talk

- Introduction
- Bernouilli equation. Theorem A
- Fermat Theorem for polynomials and generalizations
- Abel Equation. Theorem B
- First reduction: $q(t)\dot{x} = p(t)x(x-1)(x-k)$
- Rational solutions in different levels
- Rational solutions at the same level.

Introduction

The talk is based on a joint work with Anna Cima and Armengol Gasull. (JDE, 2018)

We investigate the maximum number of polynomial solutions for the Bernouilli equation:

$$q(t)\,\dot{x} = p_n(t)\,x^n + p_1(t)\,x, \text{ with } q, p_n, p_1 \in \mathbb{C}[t] \text{ and } p_n(t) \not\equiv 0.$$

and also the same problem for the Abel equation

$$q(t)\,\dot{x} = p_3(t)\,x^3 + p_2(t)\,x^2 + p_1(t)\,x + p_0(t),$$

with coefficients in $\mathbb{R}[t]$ and $p_3(t) \not\equiv 0$.
Both equations are particular cases of the equation

$$q(t)\,\dot{x} = p_n(t)\,x^n + p_{n-1}(t)\,x^{n-1} + \cdots + p_1(t)\,x + p_0(t) \qquad (1)$$

Introduction

$$q(t)\,\dot{x} = p_n(t)\,x^n + p_{n-1}(t)\,x^{n-1} + \cdots + p_1(t)\,x + p_0(t)$$

When $n = 0, 1$ one can easily show examples having all the solutions polynomials.

There are several previous works asking for polynomial solutions of the above equation for some values of $n > 1$

When $n = 2$ (Riccati equation)

A. Gasull, J. Torregrosa and X. Zhang. *The number of polynomial solutions of polynomial Ricatti equations*. J. Differential Equations **261** (2016), 5071–5093.

About the degrees of the polynomial solutions

M. Bhargava and H. Kaufman. 1964-1966 Several papers investigating the degree of the polynomial solutions of the Ricatti equation

R. G. Huffstutler, L. D. Smith and Ya Yin Liu. 1972

Introduction

$$q(t)\,\dot{x} = p_n(t)\,x^n + p_{n-1}(t)\,x^{n-1} + \cdots + p_1(t)\,x + p_0(t)$$

About the case $q(t) \equiv 1$.
J. Gine, T. Grau and J. Llibre. *On the polynomial limit cycles of polynomial differential equations*, Israel J. Math. **106** (2013), 481–507.

Bernouilli equation: Theorem A

Theorem A *Consider Bernoulli equations*

$$q(t)\,\dot{x} = p_n(t)\,x^n + p_1(t)\,x, \tag{2}$$

with $q, p_n, p_1 \in \mathbb{C}[t]$ *and* $p_n(t) \not\equiv 0$. *Then:*

- *For* $n = 2$, *equation* (2) *has at most* $N+1$ *(resp. 2) polynomial solutions, where* $N \geq 1$ *(resp.* $N = 0$*) is the maximum degree of* q, p_2, p_1, *and these upper bounds are sharp. Moreover, when* $q, p_2, p_1 \in \mathbb{R}[t]$ *these upper bounds are reached with real polynomial solutions.*

- *For* $n = 3$, *equation* (2) *has at most seven polynomial solutions and this upper bound is sharp. Moreover, when* $q, p_3, p_1 \in \mathbb{R}[t]$ *this upper bound is reached with seven polynomial solutions belonging to* $\mathbb{R}[t]$.

Bernouilli equation: Theorem A

- *For* $n \geq 4$, *equation* (2) *has at most* $2n-1$ *polynomial solutions and this upper bound is sharp. Moreover, when* $q, p_n, p_1 \in \mathbb{R}[t]$ *it has at most three real polynomial solutions when* n *is even while it has at most five real polynomial solutions when* n *is odd, and both upper bounds are sharp.*

Sketch of the proof

$$q(t)\,\dot{x} = p_n(t)\,x^n + p_1(t)\,x. \qquad (2)$$

First observe that if $x(t)$ is a solution of our equation then $\alpha x(t)$ also is a solution for all $\alpha \in \mathbb{C}$ such that $\alpha^{n-1} = 1$. We perform the change of variable $u = x^{n-1}$ in (2). This equation is transformed into the Riccati equation

$$q(t)\,\dot{u} = (n-1)\,p_n(t)\,u^2 + (n-1)\,p_1(t)\,u. \qquad (3)$$

Now using the fact that in the above equation two non-zero solutions z_0 and z_1 determine all other solutions by

$$z_c = \frac{z_0 z_1}{c z_0 + (1-c) z_1}, \quad c \in \mathbb{C}$$

we get...

Sketch of the proof

If v^{n-1} and ω^{n-1} are different solutions of (3) and it exists another solution of type x^{n-1}, then

$$x^{n-1} = \frac{v^{n-1} \cdot \omega^{n-1}}{cv^{n-1} + (1-c)\omega^{n-1}}$$

for some number $c \in \mathbb{C}$.
This fact implies that $\left(\sqrt[n-1]{c}\, v\right)^{n-1} + \left(\sqrt[n-1]{1-c}\, \omega\right)^{n-1} = y^{n-1}$ for some polynomial y.
At this moment we use **Fermat Theorem for polynomials**.

Assume that the equation $x^k + y^k = z^k$ has non-trivial solutions in $\mathbb{C}[t]$. Then $k \leq 2$.
A trivial solution is a solution with
$y = \lambda x, \;\; z = \beta x, \;\; \beta^k = 1 + \lambda^k, \lambda, \beta \in \mathbb{C}$.

And we obtain the result for $n \geq 4$.

Sketch of the proof

For $n \geq 4$ we get that

$$q(t)\,\dot{u} = (n-1)\,p_n(t)\,u^2 + (n-1)\,p_1(t)\,u.$$

has at most two non zero solutions of the form u^{n-1} with $u \in \mathbb{C}[x]$. Therefore our original equation has at most $2n-1$ polynomial solutions.

- Equation $(t^{2n-1} - t^n)\,x' = x^n + (t^{2n-2} - 2\,t^{n-1})\,x$ has the solutions 0, $\alpha\,t$ and $\alpha\,t^2$ for each α satisfying $\alpha^{n-1} = 1$ and shows that the bound for $n \geq 4$ is sharp.

Sketch of the proof, $n = 3$

To prove the case $n = 3$, we need the following improvement of the Fermat Theorem for $k = 2$.

Theorem *Let $p, q \in \mathbb{C}[t]$. There exists at most one $c \in \mathbb{C} \setminus \{0, 1\}$ such that*

$$cp^2 + (1-c)q^2 = s^2, \quad \textit{with} \quad s \in \mathbb{C}[t]$$

Sketch of the proof, $n = 3$

A direct computation gives that equation

$$4\,t\,(t^2+1)\,(t^2-1)\,(t^2-4)\,(4\,t^2-1)\,\dot{x} = 225\,x^3 + 16\,(3\,t^8 - 17\,t^6 + 6\,t^4 - 1)\,x$$

has seven polynomial solutions. Namely $x = 0$, and

$$x_1^{\pm}(t) = \pm\,\frac{2}{5}\,t\,(t^2+1), \quad x_2^{\pm}(t) = \pm\,\frac{2}{3}\,t\,(t^2-1), \quad x_3^{\pm}(t) = \pm\,\frac{4}{15}\,(t^4-1).$$

Proof of Fermat's Theorem for polynomials

The proof of the Fermat's Theorem that we have found in the literature relies on a result, interesting by itself, called the "abc Theorem" for polynomials. It states that if a, b, c are pairwise coprime non-constant polynomials for which $a + b = c$, then the degree of each of these three polynomials cannot exceed $Z(a\,b\,c) - 1$. The "abc Theorem" for polynomials (also known as Mason's Theorem), was proved in 1981 by Stothers, and also later by Mason and Silverman. We give another proof of Fermat's Theorem based on the computation of the genus of a planar algebraic curve. The reason for introducing this proof is that the same idea will be used in several parts of the next study of the Abel equation.
Assume that there exists $p, q, r \in \mathbb{C}[t]$ be such that

$$p^k + q^k = r^k.$$

$\Rrightarrow \left(\frac{p}{r}\right)^k + \left(\frac{q}{r}\right)^k = 1 \Rrightarrow$ the curve $x^k + y^k = 1$ admits a rational parametrization$\Rrightarrow$ the curve $x^k + y^k = 1$ has genus zero $\Rrightarrow k \leq 2$.

Proof of Fermat's Theorem for polynomials

To compute the genus of $x^k + y^k = 1$ we use two basic facts

- The curve has no singular points so it is irreducible.
- If a curve F has no singular points then its genus depends only on its degree. If $deg(F) = k$ and has no singular points

$$g(F) = \frac{(k-1)(k-2)}{2}.$$

Generalizations of Fermat Theorem

Theorem (M. de Bondt (2009)) *The equation*

$$g_1^d + g_2^d + \cdots + g_n^d = 0, \quad \textit{with} \quad d \in \mathbb{N} \textit{ and } g_i \in \mathbb{C}[t]$$

can have non trivial solutions only if $d < n(n-2)$.
We will apply this theorem when $n = 4$.
Theorem (M. de Bondt (2009))*Set* $n \geq 3$ *and let* $f_1, \ldots, f_n \in \mathbb{C}[t]$ *be not all constant, such that*

$$f_1 + f_2 + \ldots + f_n = 0.$$

Assume furthermore that no proper subsum vanishes and $(f_1, \ldots, f_n) = 1$
Then for all $i \in \{1, \ldots, n\}$ *we get*

$$deg(f_i) \leq \frac{(n-1)(n-2)}{2}\big(Z(f_1 f_2 \ldots . f_n) - 1\big).$$

Abel equation, Theorem B

Theorem B *If equation*

$$q(t)\,\dot{x} = p_3(t)\,x^3 + p_2(t)\,x^2 + p_1(t)\,x + p_0(t), \tag{4}$$

with coefficients in $\mathbb{R}[t]$ *and* $p_3(t) \not\equiv 0$, *has three real polynomial solutions which are collinear then it has at most seven polynomial solutions. In this case one of the collinear solutions is the arithmetic mean of the other two and the equation reduces to a Bernoulli equation with polynomial coefficients, as the one studied in item* (ii) *of Theorem A. If this relation between the three collinear solutions does not hold then equation (4) has at most six polynomial solutions and this upper bound is sharp.*

Proof of Theorem B, first reduction

Assume that equation (4) has $x_1, x_2, x_3 \in \mathbb{R}[t]$ three different solutions which are collinear. Assume also that x_2 is between x_1 and x_3. Then the change $y = x - x_2$ transforms (4) to

$$q(t)\,\dot{y} = p_3(t)\,y^3 + \widetilde{p}_2(t)\,y^2 + \widetilde{p}_1(t)\,y, \tag{5}$$

for some $\widetilde{p}_2(t), \widetilde{p}_1(t) \in \mathbb{R}[t]$.
Notice that equation (5) has the collinear solutions
$y_1 = x_1 - x_2, y_2 = 0, y_3 = x_3 - x_2 = ky_1$, for some $k < 0$.
If $x_2 = \frac{1}{2}(x_1 + x_3)$ then a simple computation shows that $k = -1$ and $\widetilde{p}_2(t) = 0$. So in this case the result follows from Theorem A.

Proof of Theorem B, first reduction

If $x_2 \neq \frac{1}{2}(x_1 + x_3)$ then $k \neq -1$. We consider the change $z(t) := \frac{y(t)}{y_1(t)}$ that transforms equation (5) in

$$q(t)\,\dot{z} = p(t)z(z-1)(z-k) \tag{6}$$

for some $p(t) \in \mathbb{R}[t]$. Note that we can assume that $k \in (-1, 0)$.If this were not the case it suffices to consider the change $z(t) := \frac{y(t)}{y_2(t)}$ instead $z(t) := \frac{y(t)}{y_1(t)}$ and we obtain again equation (6) with $k \in (-1, 0)$.
Thus the polynomial solutions of the original equation are transformed in rational solutions of equation (6) with $k \in (-1, 0)$. So we have reduced the problem to show that equation (6) has at most six rational solutions.

Equation $q(t)\,\dot{z} = p(t)z(z-1)(z-k)$

Our equation has the following non autonomous first integral

$$\frac{(z-k)z^{k-1}}{(z-1)^k \exp\big(k(k-1)H(t)\big)},$$

where $H'(t) = \frac{p(t)}{q(t)}$.
Given a solution $z = z(t)$ we denote by $\pi(z)$ the constant value

$$\pi(z) = \frac{(z(t)-k)z(t)^{k-1}}{(z(t)-1)^k \exp\big(k(k-1)H(t)\big)}$$

Equation $q(t)\,\dot{z} = p(t)z(z-1)(z-k)$

$$\pi(z) = \frac{(z(t)-k)z(t)^{k-1}}{(z(t)-1)^k \exp\big(k(k-1)H(t)\big)}$$

Thus if z_1 and z_2 are solutions we get

$$\frac{(z_1-k)z_1^{k-1}}{(z_1-1)^k} = M\frac{(z_2-k)z_2^{k-1}}{(z_2-1)^k} \quad \text{where} \quad M = \frac{\pi(z_1)}{\pi(z_2)},$$

If $z_1(t) = \frac{y_1(t)}{x_1(t)}$ and $z_2(t) = \frac{y_2(t)}{x_2(t)}$ with $(y_1, x_1) = 1 = (y_2, x_2)$ are two non-constant rational solutions we get

$$(y_1 - kx_1)(y_1 - x_1)^{-k} y_2^{1-k} \;=\; M\,(y_2 - kx_2)(y_2 - x_2)^{-k} y_1^{1-k}$$

$q(t)\,\dot z = p(t) z(z-1)(z-k).$

$$(y_1 - kx_1)(y_1 - x_1)^{-k} y_2^{1-k} \;=\; M\,(y_2 - kx_2)(y_2 - x_2)^{-k} y_1^{1-k}$$

From this we directly deduce that

- $y_1 = y_2$. That is **all nonconstant rational solutions share the numerator**.
- $\left(\frac{y-kx_1}{y-kx_2}\right)\,\left(\frac{y-x_1}{y-x_2}\right)^{-k} \;=\; M$
- $k \in \mathbb{Q} \cap (-1, 0)$ That is **if there are more than one non constant rational solutions then** $k \in \mathbb{Q}$.

$q(t)\,\dot z = p(t)z(z-1)(z-k)$. The case $M^m \neq 1$

Proposition 1 *Assume that* $z_1(t) = \frac{y(t)}{x_1(t)}$ *and* $z_2(t) = \frac{y(t)}{x_2(t)}$, *with* $(y, x_1) = 1 = (y, x_2)$ *are two non-constant rational solutions of our equation with* $k = -\frac{n}{m}$ *and assume that* $M^m \neq 1$ *where* $M = \frac{\pi(z_2)}{\pi(z_1)}$. *Then there exist two polynomials* $P, Q \in \mathbb{R}[t]$ *with* $(P, Q) = 1$, *not simultaneously constant, such that*

$$
\begin{aligned}
y &= \frac{n}{n+m}\left(P^{n+m} - MQ^{n+m}\right),\\
x_1 &= \frac{n}{n+m}\left(P^{n+m} - MQ^{n+m}\right) - \left(P^n - MQ^n\right)P^m,\\
x_2 &= \frac{n}{n+m}\left(P^{n+m} - MQ^{n+m}\right) - \left(P^n - MQ^n\right)Q^m.
\end{aligned}
$$

$q(t)\,\dot{z} = p(t)z(z-1)(z-k)$. The case $M^m \neq 1$

The proof follows applying the previous equality

$$\left(\frac{y-kx_1}{y-kx_2}\right)\left(\frac{y-x_1}{y-x_2}\right)^{-k} = M$$

to our situation. We get

$$\left(\frac{y-kx_1}{y-kx_2}\right)^{m}\left(\frac{y-x_1}{y-x_2}\right)^{n} = M^m$$

The result follows putting

$$\frac{y-x_1}{y-x_2} = \left(\frac{P}{Q}\right)^{m} \text{ and } \frac{y-kx_1}{y-kx_2} = M\left(\frac{Q}{P}\right)^{n}$$

and using some elementary results on divisibility in $\mathbb{R}[x]$.

Rational solutions in different levels

Theorem *Assume that equation*

$$q(t)\dot{z} = p(t)z(z-1)(z-k),\; k = -\frac{n}{m}, \quad 0 < n < m$$

has two nonconstant rational solutions z_1, z_2 *with* $|\pi(z_1)| \neq |\pi(z_2)|$. *Then the equation has only five rational solutions.*

Sketch of the proof

Assume that there exists another nonconstant rational solution z_3 and assume for example that $|\pi(z_3)| \neq |\pi(z_1)|$. Put $z_i = \frac{y}{x_i}$. From the previous proposition we have that there exist P, Q, R, S with $(P, Q) = (R, S) = 1$ such that

$$\begin{aligned}
y &= \frac{n}{n+m}\left(P^{n+m} - MQ^{n+m}\right), \\
x_1 &= \frac{n}{n+m}\left(P^{n+m} - MQ^{n+m}\right) - \left(P^n - MQ^n\right)P^m, \\
x_2 &= \frac{n}{n+m}\left(P^{n+m} - MQ^{n+m}\right) - \left(P^n - MQ^n\right)Q^m.
\end{aligned}$$

Rational solutions in different levels

and

$$\begin{aligned}
y =& \frac{n}{n+m}\left(R^{n+m} - LS^{n+m}\right),\\
x_1 =& \frac{n}{n+m}\left(R^{n+m} - LS^{n+m}\right) - \left(R^{n} - LS^{n}\right) R^{m},\\
x_3 =& \frac{n}{n+m}\left(R^{n+m} - LS^{n+m}\right) - \left(P^{n} - LS^{n}\right) S^{m}.
\end{aligned}$$

In particular $(R^{n+m} - LS^{n+m}) = (P^{n+m} - MQ^{n+m})$.

Rational solutions in different levels

$$(R^{n+m} - LS^{n+m}) = (P^{n+m} - MQ^{n+m}).$$

Theorem (M. de Bondt (2009)) *The equation*

$$g_1^d + g_2^d + \cdots + g_n^d = 0, \quad \textit{with} \quad d \in \mathbb{N} \textit{ and } g_i \in \mathbb{C}[t]$$

can have non trivial solutions only if $d < n\,(n-2)$.
Our equation can have non-trivial solutions only if $n + m < 4(4-2) = 8$.
So we restrict our atention to the cases $n + m \leq 7$. We also have

$$(R^{n+m} - LS^{n+m}) - (R^n - LS^n)\,R^m = (P^{n+m} - MQ^{n+m}) - (P^n - MQ^n)\,P^m$$

Rational solutions in different levels

Now assume that $n+m \leq 7$. Calling $u = \frac{Q}{P}$ and $v = \frac{S}{R}$ from the equalities

$$(R^{n+m} - LS^{n+m}) = (P^{n+m} - MQ^{n+m}).$$

and

$$(R^{n+m} - LS^{n+m}) - (R^n - LS^n)\, R^m = (P^{n+m} - MQ^{n+m}) - (P^n - MQ^n)\, P^m$$

we deduce that

$$F(u,v) = (1 - L\, v^{n+m})\,(1 - M\, u^n) - (1 - M\, u^{n+m})\,(1 - L\, v^n) = 0. \quad (7)$$

Hence the existence of three non-constant rational solutions implies that some of the irreducible components of the above polynomial has a rational parametrization. But it is know that this happens if and only this irreducible component has genus equal to zero.

Rational solutions in different levels

$$F(u,v) = (1 - L\,v^{n+m})\,(1 - M\,u^{n}) - (1 - M\,u^{n+m})\,(1 - L\,v^{n}) = 0. \qquad (7)$$

Lemma *For $n+m \leq 7, 1 \leq n < m$, equation (7) with $M^m \neq 1 \neq L^m$ is reducible only if $M^m = L^m$, and in this case the only component of genus zero of the curve is $u - \alpha v$ for some α root of the unity. In any other case the curve is irreducible and*

$$g(F) = \frac{(2n+m-1)(2n+m-2)}{2} - \frac{3n(n-1)}{2} \neq 0$$

Computation of the genus

To see that the formula for the genus is the announced in the statement we apply the well-known formula that says that the genus of a curve G of degree k is

$$g(G) = \frac{(k-1)(k-2)}{2} - \Sigma_p \frac{m_p(G)(m_p(G)-1)}{2} \tag{8}$$

where $m_p(G)$ is the multiplicity of G at p, provided that near each multiple point p, G has $m_p(G)$ different tangents.
To get the desired result we need:

- Compute the singular points. We use resultants to solve $F(u,v) = \frac{\partial F}{\partial u}(u,v) = \frac{\partial F}{\partial v}(u,v) = 0$ to obtain that the only singular points are $(0,0)$ and the two infinite points given by the directions $u=0$ and $v=0$. Moreover one can directly see that each of these points has multiplicity n.
- In the case $M^m \neq L^m$ since $n+m \leq 7$ we can directly test that the curve is irreducible using Bezout Theorem.

Computation of the genus

- Lastly, when $L^m = M^m$ we get $L = \alpha M$ for some m-root of the unity α and it is a direct computation that $F(u, v) = (u - \alpha v)P(u, v)$.
- The same analysis shows that $P(u, v)$ is irreducible, has only three singular points and has genus different from zero.

$q(t)\,\dot{z} = p(t)z(z-1)(z-k)$ The case $M^m = 1$

Proposition 2 *Assume that* $z_1(t) = \frac{y(t)}{x_1(t)}$ *and* $z_2(t) = \frac{y(t)}{x_2(t)}$, *with* $(y, x_1) = 1 = (y, x_2)$ *are two non-constant rational solutions of our equation with* $k = -\frac{n}{m}$ *and assume that* $M^m = 1$ *where* $M = \frac{\pi(z_2)}{\pi(z_1)}$. *Then there exist two polynomials* $P, Q \in \mathbb{R}[t]$ *with* $(P, Q) = 1$, *not simultaneously constant, such that*

$$
\begin{aligned}
y =& \frac{n}{n+m}\,\frac{(P^{n+m} - Q^{n+m})}{P-Q},\\
x_1 =& \frac{n}{n+m}\,\frac{(P^{n+m} - Q^{n+m})}{P-Q} - \frac{(P^n - Q^n)\,P^m}{P-Q},\\
x_2 =& \frac{n}{n+m}\,\frac{(P^{n+m} - Q^{n+m})}{P-Q} - \frac{(P^n - Q^n)\,Q^m}{P-Q}.
\end{aligned}
$$

Rational solutions at the same level

Theorem *Assume that equation*

$$q(t)\dot{z} = p(t)z(z-1)(z-k),\ k = -\frac{n}{m}, 0 < n < m$$

has three nonconstant rational solutions z_1, z_2, z_3. *Then* $|\pi(z_1)| = |\pi(z_2)| = |\pi(z_3)|, n = 1, m = 2$ *and there are no more nonconstant rational solutions.*

From the previous proposition there exist polynomials P, Q, R, S with $(P, Q) = (R, S) = 1$ such that

$$(P^{n+m} - Q^{n+m})(R - S) = (R^{n+m} - S^{n+m})(P - Q)$$

At this moment we will need the second generalization of Fermat theorem.

Rational solutions at the same level

Theorem (M. de Bondt (2009))*Let* $f_1, \ldots, f_n \in \mathbb{C}[t]$ *be not all constant, such that*

$$f_1 + f_2 + \ldots + f_n = 0.$$

Assume furthermore that no proper subsum vanishes and $(f_1, \ldots, f_n) = 1$ *Then for all* $i \in \{1, \ldots, n\}$ *we get*

$$deg(f_i) \leq \frac{(n-1)(n-2)}{2}\big(Z(f_1 f_2 \ldots . f_n) - 1\big).$$

Corollary *If the equation*

$$(P^{n+m} - Q^{n+m})(R - S) = (R^{n+m} - S^{n+m})(P - Q)$$

with $(P, Q, R, S) = 1$ *has non trivial solutions then* $n + m \leq 83$.

Rational solutions at the same level

- $(P^{n+m} - Q^{n+m})(R - S) = (R^{n+m} - S^{n+m})(P - Q)$
- First of all we need to investigate the cases when a proper subsum vanishes. There is a lot of possible cases but all of them are easily solved using elementary results of divisibility.
- $deg(f_i) \leq \frac{(n-1)(n-2)}{2}\big(Z(f_1 f_2 \ldots . f_n) - 1\big)$.
- Let $r = \text{máx}\{deg(P), deg(Q), deg(R), deg(S)\}$. We get $Z(PQRS) \leq 4r$.Assume for example that $deg(P) = r$. we get

$$(n+m)r \leq deg(P^{n+m}R) \leq 21(4r-1) < 84r$$

Rational solutions at the same level

So we can focus our atention to the case $n+m \leq 83$. Recall that there exist $P, Q, R, S \in \mathbb{R}[t]$ such that $(P, Q, R, S) = 1$ and

- $(P^{n+m} - Q^{n+m})(R - S) = (R^{n+m} - S^{n+m})(P - Q)$
- $(P^n - Q^n)\, P^m(R - S) = (R^n - S^n)\, R^m(P - Q)$

As in the previous case putting $\frac{Q}{P} = u$ and $\frac{S}{R} = v$ we obtain from the above equations

$$G(u, v) = (1 - u^{n+m})(1 - v^n) - (1 - v^{n+m})(1 - u^n) = 0. \tag{9}$$

Therefore some of the irreducible components of the curve $G(u, v)$ must have genus zero.

Rational solutions at the same level

Proposition *Consider the algebraic curve* $G_{n,m}(u,v) = (1-u^{n+m})(1-v^n) - (1-v^{n+m})(1-u^n) = 0$ *with* $n, m > 0$. *This curve reduces in the following way*

$$G_{n,m}(u,v) = (u-v)(u-1)(v-1)P_{n,m}(u,v)$$

and when $2 < n+m \leq 83$, $n < m$ *and* $(n,m) = 1$, $P_{n,m}(u,v) = 0$ *is irreducible and has genus*

$$\frac{(2n+m-4)(2n+m-5)}{2} - 3\frac{(n-1)(n-2)}{2}.$$

- The curves $u = 1, v = 1$ and $u = v = 0$ are not compatible with our hypotheses.
- A simple computation says that the $g(P_{n,m}) = 0$ if and only if $n = 1$ and $m \in \{2,3\}$.

Rational solutions at the same level

- We need to show that for $n + m \leq 83$ the curve $P_{n,m}$ is irreducible and also has only the three singular points. The difficult situation is that we need to test this fact until $n + m = 83$. The good news are that now $P_{n,m}$ does not depend on parameters different from n, m.
- Both problems are solved using Maple packages. To find the singular points we use algcurves with the tool singularities. To see that it is irreducible we also use the algcurves package.

Rational solutions at the same level

$P_{1,3} = 1 + u + v + u^2 + uv + v^2$ that does not admit a racional parametrization with rational real functions.
$P_{1,2} = 1 + u + v$. In this case we obtain

$$y = \frac{1}{3}\frac{(P^3 - Q^3)}{P - Q}, \quad x_1 = \frac{1}{3}\frac{(P^3 - Q^3)}{P - Q} - P^2, \quad x_2 = \frac{1}{3}\frac{(P^3 - Q^3)}{P - Q} - Q^2$$

and

$$y = \frac{1}{3}\frac{(R^3 - S^3)}{R - S}, \quad x_1 = \frac{1}{3}\frac{(R^3 - S^3)}{R - S} - R^2, \quad x_3 = \frac{1}{3}\frac{(R^3 - S^3)}{R - S} - S^2,$$

which gives the solutions $R = P, S = Q$, or $R = P, S = -(P + Q)$, or $R = -P, S = P + Q$. They give rise to three different solutions with $x_1 = y - P^2$, $x_2 = y - Q^2$, $x_3 = y - (P + Q)^2$. So in this case we can obtain six rational solutions.

Rational solutions at the same level

To get un example with six solutions in the case $k = -\frac{1}{2}$ we simply choose $P(t) = t$ and $Q(t) = 1$ in the corresponding set of equations. Then the equation is

$$3t(t+1)(t^2+t+1)\,\dot{z} = -2(2t+1)(t-1)(t+2)\,z(z-1)(z+\frac{1}{2}).$$

This equation has the solutions $0, 1, -\frac{1}{2}$ and

$$z_1(t) = -\frac{t^2+t+1}{(2t+1)(t-1)}, \quad z_2(t) = \frac{t^2+t+1}{(t+2)(t-1)}$$

and

$$z_3(t) = -\frac{t^2+t+1}{(t+2)(2t+1)}.$$

THANK YOU FOR YOUR ATTENTION!!