An algorithm for providing all the spatial quasi-homogeneous differential systems

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Abstract

Quasi-homogeneous systems, and in particular those 3-dimensional, are currently a thriving line of research. But a method for obtaining all fields of this class is not yet available. The weight vectors of a quasi-homogeneous system are grouped into families. We found that the maximal three-dimensional quasi-homogeneous systems have the property of having only one family with minimum weight vector. This minimum vector is unique to the system, thus acting as identification code. We develop an algorithm that provides all normal forms of maximal three-dimensional quasi-homogeneous systems for a given degree. All other three-dimensional quasi-homogeneous systems can be trivially deduced from these maximal systems. We also list all the systems of this type of degree 2 using the algorithm. With this algorithm we make available to the researchers all three-dimensional quasi-homogeneous systems.

Introduction

A polynomial differential system

$$\dot{x} = P(x, y, z), \qquad \dot{y} = Q(x, y, z), \qquad \dot{z} = R(x, y, z),$$

where $P,\ Q\ R \in \mathbb{C}[x, y, z]$, is said QUASI-HOMOGENEOUS (QH) if exist positive integers s_1, s_2, s_3, d verifying that for any $\alpha \in \mathbb{R}^+$,

$$P(\alpha^{s_1}x, \alpha^{s_2}y, \alpha^{s_3}z) = \alpha^{s_1+d-1}P(x, y, z),$$
$$Q(\alpha^{s_1}x, \alpha^{s_2}y, \alpha^{s_3}z) = \alpha^{s_2+d-1}Q(x, y, z),$$
$$R(\alpha^{s_1}x, \alpha^{s_2}y, \alpha^{s_3}z) = \alpha^{s_3+d-1}R(x, y, z).$$

In this case, $\mathbf{v} = (s_1, s_2, s_3, d)$ is denominated WEIGHT VECTOR of the system.
The degree of the system is $n = \max\{deg(P), deg(Q), deg(R)\}$.

A QH system is called MAXIMAL if any new monomial added to its structure maintaining the degree of the system prevents it to be QH.

Every QH system is either maximal or contained within a maximal. Thus, knowing the maximal set we control the whole QH systems.

The goal of our algorithm is to provide all the maximal QH systems of a given degree.

Weight vectors

Definition 1 *A weight vector $\mathbf{w}_m$ is the* MINIMUM WEIGHT VECTOR *of the system if for any other weight vector $\mathbf{w}$ it is verified that $\mathbf{w}_m \le \mathbf{w}$.*

The weight vectors of a QH system form groups:

Definition 2 *The* WEIGHT VECTOR FAMILY *$F_{\mathsf{S}}(\lambda,\ \mu)$ with ratio $(\lambda,\ \mu)$ is defined as*

$$F_{\mathsf{S}}(\lambda,\ \mu) = \left\{(s_1, s_2, s_3, d) \text{ weight vector } : \frac{s_1}{s_2} = \lambda \text{ and } \frac{s_1}{s_3} = \mu\right\}$$

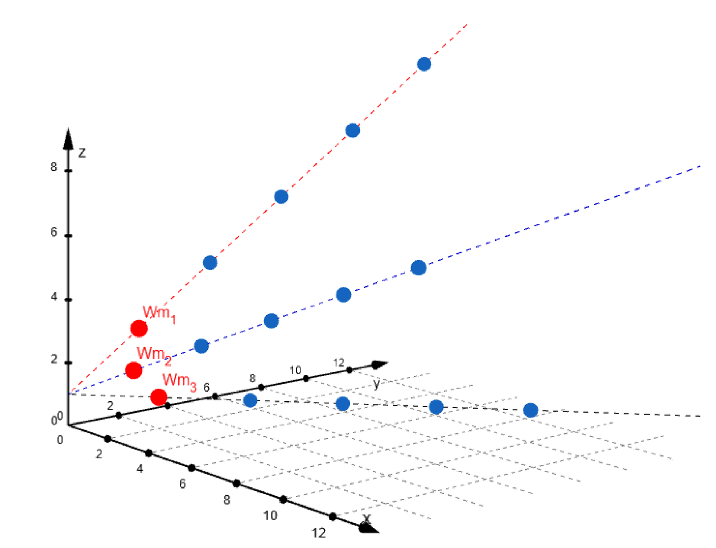

Figure 1: *Arrangement of families in an example of a plane QH system.*

Maximal systems have a single family of weight vectors, and in addition, fixed a degree, the families of two systems are disjoint. As a consequence, fixed a degree, $\mathbf{w}_m$ always exists and is a unique identifier of the maximal system.

Bricks

Given a maximal QH system of degree n, $1 \le k \le n$, $x_1, x_2 \in \{0, 1, ..., k-1\}$, $0 \le x_1 + x_2 \le k-1$, the following statements are equivalents:

1. Monomial $a_{x_1+1,x_2,k-x_1-x_2-1}x^{x_1+1}y^{x_2}z^{k-x_1-x_2-1}$ is in component P.
2. Monomial $b_{x_1,x_2+1,k-x_1-x_2-1}x^{x_1}y^{x_2+1}z^{k-x_1-x_2-1}$ is in component Q.
3. Monomial $c_{x_1,x_2,k-x_1-x_2}x^{x_1}y^{x_2}z^{k-x_1-x_2}$ is in component R.

Definition 3 *A* BRICK *is one of these sets of linked monomials of the same degree k, which are the simplest constituent elements of any maximal QH system. We denote by $[x_1, x_2; k]$ the brick associated with the previous monomials.*

As an example, the following QH system is built of five bricks with degrees running from 2 to 3:

$\dot{x} =$	$+xyz$		$+y^3$	$+x^2$	
$\dot{y} =$	$+y^2z$	$+xz^2$		$+xy$	
$\dot{z} =$	$+yz^2$			$+xz$	$+y^2$
Brick:	$[0, 1; 3]$	$[1, -1; 3]$	$[-1, 3; 3]$	$[1, 0; 2]$	$[0, 2; 2]$

Definition 4 *We denote by B_k the set of bricks of degree k,*

$$\mathsf{B}_\mathsf{k} = \left\{[x_1, x_2; k]\ :\ x_1,x_2 \in \mathbb{Z}, \le x_1, x_2, x_1 + x_2 \le k\right\}$$

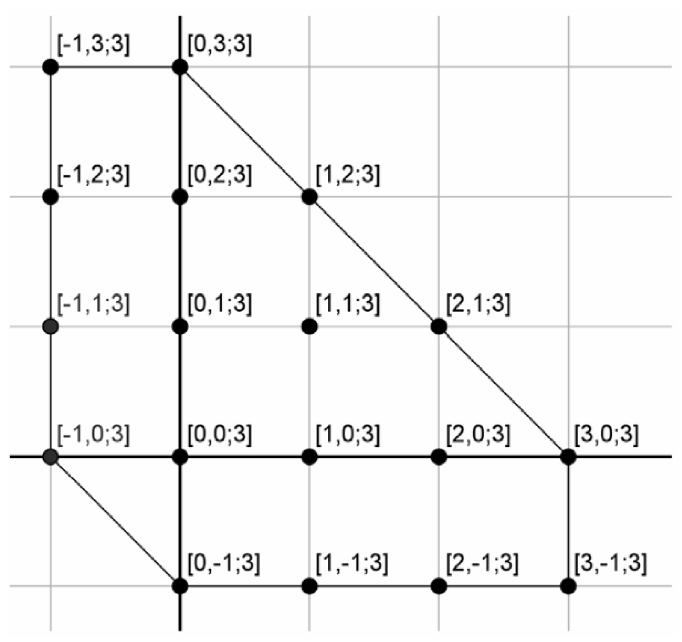

Figure 2: *B_3, the set of bricks of degree 3.*

Compatibility

Not any subset of bricks can be part of a QH system. This is determined by its compatibility.

Definition 5 COMPATIBLE *bricks are those that can coexist in a QH system, and* INCOMPATIBLE *those that cannot do so under any circumstances.*

The compatibility of two bricks is totally determined by the following results, according to the two bricks are of the same degree or not:

- The bricks $[x_1, x_2; k]$ and $[y_1, y_2; p]$ are compatible if and only if $Y_1 > 0$, or $Y_1 + Y_2 > 0$, being $Y_i = y_i - x_i$, $i = 1, 2$.
- Two different bricks $[x_1, x_2; k]$ and $[y_1, y_2; k]$ are compatible if and only if $Y_1 = 0$, or $-Y_2/Y_1 \ge 1$, being $Y_i = y_i - x_i$ for $i = 1, 2$.

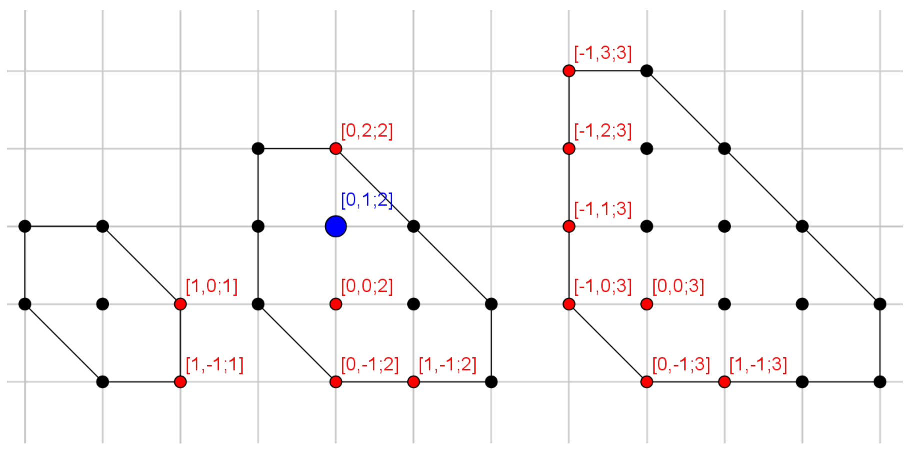

Figure 3: *The brick [0,1;2] and their compatibles in B_1, B_2 and B_3 (marked in red).*

Seeds

A maximal system always has at least a set of three bricks from which the rest of the system is totally determined. We call SEEDS to these special sets of three bricks, which encode all the algebraic information of their system by themselves.

Theorem 1 *The bricks $[x_1, x_2; n]$, $[y_1, y_2; m]$ and $[z_1, z_2; k]$ form a seed of some n-degree maximal QH system if and only if the following three conditions hold:*

- $T_1 \cdot T_2 \le 0$ *and* $|T_1| \le |T_2|$,
- $T_2 \begin{vmatrix} Y_1 & Y_2 \\ T_1 & T_2 \end{vmatrix} > 0$,
- $\dfrac{Y_1T_2 - Y_2T_1}{(n-m)\,T_2} \ge \dfrac{x_2T_1 - x_1T_2}{(n-1)\,T_2}$,

where $Y_i = y_i - x_i$, $T_i = (k-m)\,x_i + (n-k)\,y_i + (m-n)\,z_i$, for $i = 1, 2$.

Theorem 2 *If the bricks $[x_1, x_2; n]$, $[y_1, y_2; m]$ and $[z_1, z_2; k]$ form a seed of an n-degree maximal inhomogeneous QH system* S*, then*

- *The brick $[t_1, t_2; l]$ with $1 \le l \le n$, belongs to the system* S *if and only if $T_1R_2 = T_2R_1$*
- *The minimum weight vector of* S *is $\mathbf{w}_m = \left(\dfrac{\hat{s}_1}{G}, \dfrac{\hat{s}_2}{G}, \dfrac{\hat{s}_3}{G}, \dfrac{\hat{d}}{G} + 1\right)$*

where

$$Y_i = y_i - x_i, \text{ for } i = 1, 2,$$
$$T_i = (k-m)\,x_i + (n-k)\,y_i + (m-n)\,z_i, \text{ for } i = 1, 2,$$
$$R_i = (l-m)\,x_i + (n-l)\,y_i + (m-n)\,t_i, \text{ for } i = 1, 2,$$
$$\hat{s_1} = |Y_1T_2 - Y_2T_1| + (n-m)\,|T_2|,$$
$$\hat{s_2} = |Y_1T_2 - Y_2T_1| + (n-m)\,|T_1|,$$
$$\hat{s_3} = |Y_1T_2 - Y_2T_1|,$$
$$\hat{d} = (n-1)\,|Y_1T_2 - Y_2T_1| + \delta\,(n-m)\,(x_1T_2 - x_2T_1),$$
$$\delta = \operatorname{sgn}(T_2),$$
$$G = \gcd(\hat{s_1}, \hat{s_2}, \hat{s_3}).$$

The Algorithm

To find all QH systems of degree n, our algorithm follows the following sequence:

1. Find all possible seeds for systems of degree n (Th 1).
2. Build the system corresponding to each seed (Th 2).
3. Use $\mathbf{w}_m$ (unique identifier) to avoid repetition of systems that are born from two different seeds.

The algorithm is provided in pseudocode and consists of a main body plus four functions.

```
N ← (k+1)(k+6)/2
T ← n^3/6 + 2n^2 + 29/6 n
Aux ← empty matrix of 4 columns
create ordered list of bricks {B_i}_{i=1}^T, where B_i = [x_1^i, x_2^i; k^i]
for i ← 1 to N do
    for j ← N+1 to T do
        if ARECOMPAT(B_i, B_j) then
            for p ← i+1 to j−1 do
                if ARECOMPAT(B_i, B_p) and ARECOMPAT(B_p, B_j) then
                    if ARESEED(B_i, B_j, B_p) then
                        w_m ← CALCULATEWM(B_i, B_j, B_p)
                        if w_m is not a row of Aux then
                            add w_m as a new row of Aux
                            for q ← 1 to T do
                                if ISBRICKINSYSTEM(w_m, B_q) then
                                    add B_q to system S

                            output system S
```

Figure 4: *Main body of the Algorithm.*

References

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