# On the complete integrability of the $N$–dimensional differential systems

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It is well known the classical result that if a 2–dimensional differential system

$$\frac{dx_1}{dt} = \dot{x}_1 = X_1(x_1, x_2) = X_1, \qquad \frac{dx_2}{dt} = \dot{x}_2 = X_2(x_1, x_2) = X_2,$$

has an integrating factor $J = J(x_1, x_2)$, then doing the change in the independent variable $t \to \tau$ given by

$$d\tau = Jdt,$$

this differential can be written as

$$\frac{dx_1}{d\tau} = x_1' = -\frac{1}{J}\frac{\partial H}{\partial x_2} = X_1, \qquad \frac{dx_2}{d\tau} = x_2' = \frac{1}{J}\frac{\partial H}{\partial x_1} = X_2,$$

for a convenient function $H = H(x_1, x_2)$, being $H$ a first integral of the system.

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1) The extension of the previous classical result for 2–dimensional differential systems to complete integrable $N$–dimensional differential systems in different directions, using as the main tool the Nambu bracket.

2) These new results on the complete integrable systems in dimension $N$ using the Nambu bracket allows to do some interesting applications.

In the rest of this talk we will work with the $N$–dimensional differential system

$$\dot{x}_j = X_j(x_1, \ldots, x_N), \quad \text{for} \quad j = 1, \ldots, N,$$

or with its associated vector field

$$\mathcal{X} = X_1 \frac{\partial}{\partial x_1} + X_2 \frac{\partial}{\partial x_2} + \ldots + X_N \frac{\partial}{\partial x_N}.$$

Let $V \subset U$ be an open subset. Here a first integral of a vector field $\mathcal{X}$ defined in $U$ is a $\mathcal{C}^1$ non–locally constant function $H : V \to \mathbb{R}$ such that it is constant on the solutions $(x_1(t), \ldots, x_N(t))$ of the vector field $\mathcal{X}$ contained in $V$, Let $V \subset U$ be an open subset. Here a first integral of a vector field $\mathcal{X}$ defined in $U$ is a $\mathcal{C}^1$ non–locally constant function $H : V \to \mathbb{R}$ such that it is constant on the solutions $(x_1(t), \dots, x_N(t))$ of the vector field $\mathcal{X}$ contained in $V$,

i.e.

$$\dot{H} = \frac{\partial H}{\partial x_1} X_1 + \frac{\partial H}{\partial x_2} X_2 + \dots + \frac{\partial H}{\partial x_N} X_N \equiv 0 \quad \text{in } V.$$

For an $N$–dimensional differential systems the existence of $N-1$ independent first integrals $H_1, \ldots, H_{N-1}$ means that the system is completely integrable, For an $N$–dimensional differential systems the existence of $N-1$ independent first integrals $H_1,\ldots,H_{N-1}$ means that the system is completely integrable,

i.e. the orbits of the vector field $\mathcal{X}$ are contained in the curves

$$\left\{H_1 = h_1\right\} \cap \left\{H_2 = h_2\right\} \cap \ldots \cap \left\{H_{N-1} = h_{N-1}\right\}$$

where $h_1,\ h_2\ldots,h_{N-1}$ vary in $\mathbb{R}$.

If $H_r : V_r \longrightarrow \mathbb{R}$ for $r = 1, \ldots, K$ are $K$ first integrals of the vector field $\mathcal{X}$ , we say that they are independent in $\tilde{V} := V_1 \cap V_2 \cap \ldots \cap V_K$, if their gradients are independent in all the points of $\tilde{V}$ except perhaps in a zero Lebesgue measure set.

Let $J = J(x_1, \ldots, x_N)$ be a non–negative function non–identically zero on an open subset $V$ of $U$, being $U$ the domain of definition of the vector field $\mathcal{X}$.

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Then $J$ is a Jacobi multiplier of the vector field $\mathcal{X}$ if

$$\int_{\Omega} J(x_1, \ldots, x_N) dx_1 \ldots dx_N = \int_{\varphi_t(\Omega)} J(x_1, \ldots, x_N) dx_1 \ldots dx_N,$$

being $\Omega$ any open subset of $V$, $\varphi_t$ is the flow defined by $\mathcal{X}$, and $\varphi_t(\Omega)$ is the image of the domain $\Omega$ under the flow $\varphi_t$.

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$$\operatorname{div}(J\mathcal{X}) := \frac{\partial(JX_1)}{\partial x_1} + \ldots + \frac{\partial(JX_N)}{\partial x_N} = 0.$$

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Note that if $N = 2$ then the definition of Jacobi multiplier coincides with the definition of integrating factor.

JACOBI'S THEOREM. Assume that the $N$–dimensional vector field $\mathcal{X}$ has a Jacobi multiplier $J$ and $N-2$ independent first integrals $H_1, H_2, \ldots, H_{N-2}$.

JACOBI'S THEOREM. Assume that the $N$–dimensional vector field $\mathcal{X}$ has a Jacobi multiplier $J$ and $N-2$ independent first integrals $H_1, H_2, \ldots, H_{N-2}$. Then $\mathcal{X}$ admits an additional first integral independent of the previous ones given by

$$H_{N-1} = \int \frac{\tilde{J}}{\tilde{\Delta}} \left( \tilde{X}_2 dx_1 - \tilde{X}_1 dx_2 \right),$$

where $\tilde{}$ denotes quantities expressed in the variables $(x_1, x_2, h_1, \ldots, h_{N-2})$ with $H_j = h_j$ for $j = 1, \ldots, N-2$ and

$$\Delta = \begin{vmatrix} \dfrac{\partial H_1}{\partial x_3} & \dfrac{\partial H_1}{\partial x_4} & \cdots & \dfrac{\partial H_1}{\partial x_N} \\ \dfrac{\partial H_2}{\partial x_3} & \dfrac{\partial H_2}{\partial x_4} & \cdots & \dfrac{\partial H_2}{\partial x_N} \\ \vdots & \vdots & \cdots & \vdots \\ \dfrac{\partial H_{N-2}}{\partial x_3} & \dfrac{\partial H_{N-2}}{\partial x_4} & \cdots & \dfrac{\partial H_{N-2}}{\partial x_N} \end{vmatrix}$$

Then $\mathcal{X}$ is completely integrable.

The previous theorem goes back to Jacobi, for a proof see for instance Theorem 2.7 of the book:

A. Goriely, Integrability and nonintegrability of dynamical systems, Advances Series in Nonlinear Dynamics **19**, World Scientific Publishing Co., Inc., River Edge, NJ, 2001.

For the $\mathcal{C}^1$ real functions $H_j$ for $j = 1, \ldots, N-1$ defined in some open set $U$ of $\mathbb{R}^n$ the Nambu vector field is the $N$–dimensional vector field

$$\left\{H_1, H_2, \ldots, H_{N-1}, *\right\} = \begin{vmatrix} \dfrac{\partial H_1}{\partial x_1} & \dfrac{\partial H_1}{\partial x_2} & \cdots & \dfrac{\partial H_1}{\partial x_N} \\ \dfrac{\partial H_2}{\partial x_1} & \dfrac{\partial H_2}{\partial x_2} & \cdots & \dfrac{\partial H_2}{\partial x_N} \\ \vdots & \vdots & \cdots & \vdots \\ \dfrac{\partial H_{N-1}}{\partial x_1} & \dfrac{\partial H_{N-1}}{\partial x_2} & \cdots & \dfrac{\partial H_{N-1}}{\partial x_N} \\ \dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial x_2} & \cdots & \dfrac{\partial}{\partial x_N} \end{vmatrix} .$$

For the $\mathcal{C}^1$ real functions $H_j$ for $j = 1, \ldots, N-1$ and $F$ defined in some open set $U$ of $\mathbb{R}^n$ the Nambu bracket is the function

$$\{H_1, H_2, \ldots, H_{N-1}, F\}.$$

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Many properties and applications of the Nambu vector field can be found in the book:

J. Llibre and R. Ramírez, Inverse problems in ordinary differential equations and applications, Progress in Math. **313**, Birkhäuser, 2016.

THEOREM 1. For $j = 1, \ldots, N-1$ let $H_j$ be $N-1$ independent $C^2$ functions.

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$$\dot{x}_j = \frac{1}{J}\left\{H_1, H_2, \ldots, H_{N-1}, x_j\right\} = X_j, \quad \text{for} \quad j = 1, \ldots, N,$$

where $J = J(x_1, x_2, \ldots, x_N)$ is a Jacobi multiplier and $\left\{H_1, H_2, \ldots, H_{N-1}, *\right\}$ is the Nambu vector field.

THEOREM 1 generalizes the classical result that if a 2–dimensional differential system

$$\dot{x}_1 = X_1(x_1, x_2), \qquad \dot{x}_2 = X_2(x_1, x_2),$$

has an integrating factor $J$, then it can be written as

$$\dot{x}_1 = -\frac{1}{J}\frac{\partial H_1}{\partial x_2} = X_1, \qquad \dot{x}_2 = \frac{1}{J}\frac{\partial H_1}{\partial x_1} = X_2,$$

to $N$–dimensional differential systems, because

$$-\frac{\partial H_1}{\partial x_2} = \{H_1, x_1\}, \text{ and } \frac{\partial H_1}{\partial x_1} = \{H_1, x_2\},$$

where $\{H_1, x_1\}$ and $\{H_1, x_2\}$, are Nambu brackets.

COROLLARY. Assume that an $N$–dimensional vector field $\mathcal{X}$ has $N-2$ independent first integrals $H_1, H_2, \ldots, H_{N-2}$ and a Jacobi multiplier $J$, COROLLARY. Assume that an $N$–dimensional vector field $\mathcal{X}$ has $N-2$ independent first integrals $H_1,\ H_2,\ldots,H_{N-2}$ and a Jacobi multiplier $J$, then another independent first integral $H_{N-1}$ can be obtained as a solution of the first order partial differential system

$$\left\{H_1, H_2, \ldots, H_{N-2},\ H_{N-1}, x_j\right\} = JX_j, \qquad \text{for} \quad j = 1, \ldots, N,$$

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The COROLLARY provides a different way to compute the additional independent first integral $H_{N-1}$ found by Jacobi.

THEOREM 2. $\mathcal{X}$ is completely integrable if and only if $\mathcal{X}$ has $N-r$ first integrals $H_1, \ldots, H_{N-r}$, and $r-1$ Jacobi multipliers $J_{N-r+1}, \ldots, J_{N-1}$ such that $H_n = J_n/J_{N-1}$ for $n = N-r+1, \ldots, N-2$ are non locally constant, and the functions $H_1, \ldots, H_{N-2}$ are independent.

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$$\dot{x}_j = \frac{1}{J}\left\{H_1, H_2, \ldots, H_{N-1}, x_j\right\} = X_j, \quad \text{for} \quad j = 1, \ldots, N,$$

where $J = J(x_1, x_2, \ldots, x_N)$ is a Jacobi multiplier and $\left\{H_1, H_2, \ldots, H_{N-1}, *\right\}$ is the Nambu vector field.

We recall the definition of the Nambu vector field:

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Since the $N$–dimensional vector field $\mathcal{X}$ is completely integrable, there exist $N-1$ independent first integrals $H_1, \ldots, H_{N-1}$ such that

$$\mathcal{X}(H_j) = X_1 \frac{\partial H_j}{\partial x_1} + \ldots + X_{N-1} \frac{\partial H_j}{\partial x_{N-1}} + X_N \frac{\partial H_j}{\partial x_N} = 0,$$

for $j = 1, 2, \ldots, N-1$, or equivalently

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$$X_1 \frac{\partial H_j}{\partial x_1} + \ldots + X_{N-1} \frac{\partial H_j}{\partial x_{N-1}} = -X_N \frac{\partial H_j}{\partial x_N},$$

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for $j = 1, 2, \ldots, N-1$. We solve this linear system in the variables $X_1, \ldots, X_{N-1}$, and we obtain

$$X_i = \frac{X_N \begin{vmatrix} \frac{\partial H_1}{\partial x_1} & \cdots & \frac{\partial H_1}{\partial x_{i-1}} & \frac{\partial H_1}{\partial x_N} & \frac{\partial H_1}{\partial x_{i+1}} & \cdots & \frac{\partial H_1}{\partial x_{N-1}} \\ \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ \frac{\partial H_{N-1}}{\partial x_1} & \cdots & \frac{\partial H_{N-1}}{\partial x_{i-1}} & \frac{\partial H_{N-1}}{\partial x_N} & \frac{\partial H_{N-1}}{\partial x_{i+1}} & \cdots & \frac{\partial H_{N-1}}{\partial x_{N-1}} \end{vmatrix}}{\begin{vmatrix} \frac{\partial H_1}{\partial x_1} & \cdots & \frac{\partial H_1}{\partial x_{N-1}} \\ \vdots & \cdots & \vdots \\ \frac{\partial H_{N-1}}{\partial x_1} & \cdots & \frac{\partial H_{N-1}}{\partial x_{N-1}} \end{vmatrix}}$$

for $i = 1, \ldots, N-1$.

Consequently

$$\mathcal{X} = X_1 \frac{\partial}{\partial x_1} + X_2 \frac{\partial}{\partial x_2} + \ldots + X_N \frac{\partial}{\partial x_N} = \lambda \left\{ H_1, H_2, \ldots, H_{N-1}, * \right\}.$$

where

$$\lambda = \frac{X_N}{\begin{vmatrix} \frac{\partial H_1}{\partial x_1} & \cdots & \frac{\partial H_1}{\partial x_{N-1}} \\ \vdots & \vdots & \vdots \\ \frac{\partial H_{N-1}}{\partial x_1} & \cdots & \frac{\partial H_{N-1}}{\partial x_{N-1}} \end{vmatrix}}.$$

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Now we prove that $J = \dfrac{1}{\lambda}$ is a Jacobi multiplier.

Indeed from the properties of the Nambu vector field (see for instance Proposition 1.2.1 of

J. Llibre and R. Ramírez, Inverse problems in ordinary differential equations and applications, Progress in Math. **313**, Birkhäuser, 2016.

we get that the divergence of the Nambu vector field is zero, i.e.

$$\mathrm{div}\left(\{H_1, H_2, \ldots, H_{N-1}, *\}\right) = 0.$$

On the other hand, $\frac{1}{\lambda}\mathcal{X} = \{H_1, H_2, \ldots, H_{N-1}, *\}$. Therefore in view of the previous equality we get that

$$0 = \text{div}\{H_1, \ldots, H_{N-1}, *\} = \text{div}(\frac{\mathcal{X}}{\lambda}) = \frac{\partial(\frac{X_1}{\lambda})}{\partial x_1} + \frac{\partial(\frac{X_2}{\lambda})}{\partial x_2} + \ldots + \frac{\partial(\frac{X_N}{\lambda})}{\partial x_N}.$$

So $\frac{1}{\lambda}$ is a Jacobi multiplier by Whittaker's Theorem.

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Since $\mathcal{X} = \dfrac{1}{J}\Big\{H_1, H_2, \ldots, H_{N-1}, *\Big\}$, we have

$$\mathcal{X}(H_j) = \frac{1}{J}\{H_1, H_2, \ldots, H_{N-1}, H_j\Big\} = 0,$$

for $j = 1, 2, \ldots, N-1$.

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for $j = 1, 2, \ldots, N-1$.

In short THEOREM 1 is proved.

We recall the statement of the COROLLARY:

COROLLARY. Assume that an $N$–dimensional vector field $\mathcal{X}$ has $N-2$ independent first integrals $H_1, H_2, \ldots, H_{N-2}$ and a Jacobi multiplier $J$, then another independent first integral $H_{N-1}$ can be obtained as a solution of the first order partial differential system

$$\left\{H_1, H_2, \ldots, H_{N-2}, H_{N-1}, x_j\right\} = JX_j, \qquad \text{for} \quad j = 1, \ldots, N,$$

where $\left\{H_1, H_2, \ldots, H_{N-2}, H_{N-1}, *\right\}$ is the Nambu vector field.

We shall illustrated the applications of the COROLLARY in the determination of the second first integral of the Jacobi Theorem in the following three particular cases of 3–dimensional Lotka–Volterra differential systems. We study the existence of a second first integral $H_2$

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(i) of the integrable asymmetric May–Leonard model,

(ii) of the integrable symmetric May–Leonard model.

(iii) of some integrable cases for special Lotka–Volterra systems studied by Aziz and Christhoper.

W. Aziz and C. Christopher, Local integrability and linearizability of three–dimensional Lotka-Volterra systems, Appl. Math. and Comput. **219** (2012), 4067–4081.

The class of asymmetric May–Leonard models

$$\begin{aligned}
\dot{x} &= x\,(1 - x - (2 - b_2)y - b_1 z) = X_1(x, y, z),\\
\dot{y} &= y\,(1 - y - (2 - b_3)z - b_2 x) = X_2(x, y, z),\\
\dot{z} &= z\,(1 - z - (2 - b_1)x - b_3 y) = X_3(x, y, z),
\end{aligned}$$

with $b_1 + b_2 + b_3 = 3$ is completely integrable.

The class of asymmetric May–Leonard models

$$\begin{aligned} \dot{x} = &\quad x\,(1 - x - (2 - b_2)y - b_1 z) = X_1(x, y, z), \\ \dot{y} = &\quad y\,(1 - y - (2 - b_3)z - b_2 x) = X_2(x, y, z), \\ \dot{z} = &\quad z\,(1 - z - (2 - b_1)x - b_3 y) = X_3(x, y, z), \end{aligned}$$

with $b_1 + b_2 + b_3 = 3$ is completely integrable.

More precisely, this system has the first integral $H_1$ and the Jacobi multiplier $J$:

$$H_1 = \left|\frac{z}{y}\right|^{b_2 - 1} \left|\frac{x}{y}\right|^{b_3 - 1}, \qquad J = \frac{1}{|xyz(x + y + z - 1)|},$$

and a second independent first integral $H_2$, which is determined by using the COROLLARY, i.e. $H_2$ is determined solving for $H_2$ the partial differential equation of the COROLLARY for $N = 3$:

$$\begin{aligned}
\frac{\partial H_1}{\partial y}\frac{\partial H_2}{\partial z} - \frac{\partial H_1}{\partial z}\frac{\partial H_2}{\partial y} &= JX_1,\\
\frac{\partial H_1}{\partial z}\frac{\partial H_2}{\partial x} - \frac{\partial H_1}{\partial x}\frac{\partial H_2}{\partial z} &= JX_2,\\
\frac{\partial H_1}{\partial x}\frac{\partial H_2}{\partial y} - \frac{\partial H_1}{\partial y}\frac{\partial H_2}{\partial x} &= JX_3,
\end{aligned}$$

we get that $H_2$ is equal to

$$\frac{1}{H_1}\log\left(\left(\frac{z}{y|x+y+z-1|^{b_3-1}}\right)^{1-b_2}\left(\frac{x}{y|x+y+z-1|^{1-b_2}}\right)^{b_3-1}\right).$$

Consider the symmetric May–Leonard model

$$\begin{aligned} \dot{x} &= x\,(1 - x - ay - bz)\,, \\ \dot{y} &= y\,(1 - y - bx - az)\,, \\ \dot{z} &= z\,(1 - z - ax - by)\,, \end{aligned}$$

If $a + b = 2$ this system has the first integral $H_1$ and the Jacobi multiplier $J$, where

$$H_1 = \frac{xyz}{(x+y+z)^3}, \quad \text{and} \quad J = \frac{1}{|xyz(x+y+z-1)|}.$$

Now using the COROLLARY a second independent first integral is

$$H_2 = \frac{1}{H_1}\left((b-1)\log|x+y+z-1| + \Lambda\left(\frac{y}{x},\frac{z}{x}\right)\right),$$

where $\Lambda = \Lambda\left(\frac{y}{x},\frac{z}{x}\right)$ is a solution of the partial differential equation

$$-(2\eta-\xi-1)\xi\frac{\partial\tilde{\Lambda}}{\partial\xi} + (2\xi-\eta-1)\eta\frac{\partial\tilde{\Lambda}}{\partial\eta} = -(1+\xi+\eta),$$

where $\tilde{\Lambda} = \Lambda(\xi,\eta)$, $\xi = y/x$, and $\eta = z/x$.

The Lotka–Volterra differential system

$$\begin{aligned} \dot{x} &= x(2+ax), \\ \dot{y} &= y(-1+dx+hy+kz), \\ \dot{z} &= z(1+gx+hy+kz), \end{aligned}$$

has the first integral $H_1 = z(2+ax)^{\frac{d+a-g}{a}}/(xy)$ and the Jacobi multiplier $J = x^{-\frac{5}{2}} y^{-3} (2+ax)^{-\frac{2(g-2d)-a}{2a}}$.

Then it has the second first integral

$$H_2 = \frac{(2+ax)^{\frac{a+2d}{2a}}}{\sqrt{x}} \left( \frac{1}{y} + T(\frac{z}{y}, x) \right).$$

where the function $T = T(\frac{z}{y}, x)$ is a solution of the partial differential equation

$$(2+(g-d)x)\eta \frac{\partial T}{\partial \eta} + x(2+ax)\frac{\partial T}{\partial x} + (x\,d-1)T(\eta, x) = h + k\eta,$$

with $\eta = \frac{z}{y}$.

# The end

THANK YOU VERY MUCH FOR YOUR ATTENTION