Cyclicity of canard cycles with hyperbolic saddles located away from the critical curve

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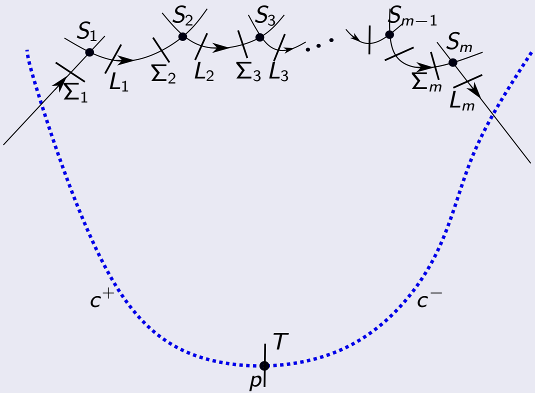
Motivation

- Our goal is to study the cyclicity of canard cycle Γ.



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- Predator-prey systems with a Holling type II or IV response function P:

$$\begin{cases} \dot{x} &= rx\left(1-\frac{x}{K}\right)-yP(x) \\ \dot{y} &= y\left(-\epsilon+cP(x)\right), \end{cases}$$

where $x \geq 0$ is the population density of prey, $y \geq 0$ is the population density of predator, $\epsilon \geq 0$ is the death rate of the predator kept small and the parameters c, K and r are strictly positive.

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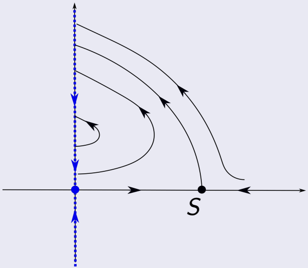
- The function $P(x)=\frac{mx}{b+x}$, with $m>0$ and $b>0$, is a response function of Holling type II
- When $P(x)=\frac{mx}{1+bx+ax^2}$ with $m>0$, $a>0$ and $b>-2\sqrt{a}$, we have the Holling type IV response function. Its simplified version is given by $P(x)=\frac{mx}{a+x^2}$ where $m>0$ and $a>0$.

Motivation



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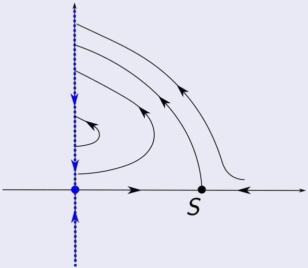

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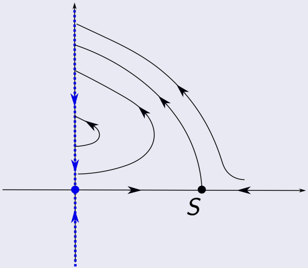

- At most one limit cycle!
- See [Ting-Hao Hsu, Number and Stability of Relaxation Oscillations for Predator-Prey Systems with Small Death Rates, 2019]
- See also H. Zhu, X. Zhang, C. Li, B. W. Kooi, J. C. Poggiale...

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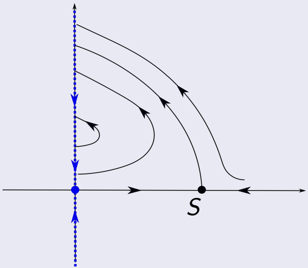

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Motivation

- For the study of the cyclicity of degenerate limit periodic sets (thus, containing curves of singularities) inside polynomial systems see [Artés, Dumortier, Llibre,2009], [Bobieński, Mardesic, Novikov,2013], [Bobieński, Gavrilov,2016], [De Maesschalck, Dumortier, 2010,2011], [De Maesschalck, Huzak, 2014], [Dumortier, Panazzolo, Roussarie, 2007], [Dumortier, Roussarie, 1996,2009], [Dumortier, Rousseau, 2009], [H. Zhu, C. Li, 2013], etc.

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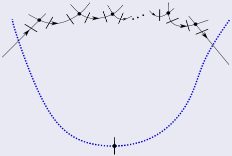
- Our goal is to study the cyclicity of degenerate limit periodic sets with singularities outside the slow curve.

Definitions

- We deal with a slow-fast Hopf point at $p \in M$ for $(\epsilon, \mu) = (0, 0)$, i.e. $X_{\epsilon,\mu}$ is locally smoothly equivalent to

$$\begin{cases} \dot{x} &= y \\ \dot{y} &= -xy + \epsilon\big(b(\mu) - x + x^2 g(x, \epsilon, \mu)\big) + \epsilon y^2 H(x, y, \epsilon, \mu). \end{cases}$$



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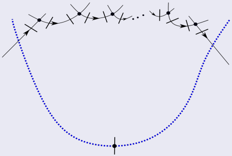


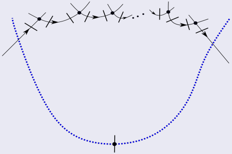
- See [Dumortier, Roussarie, 1996,2009], [Krupa, Szmolyan, 2001], [De Maesschalck, Dumortier, 2005,2008], [Huzak, 2016,2017,2018]

Definitions

- Along the slow curve, away from $p \in M$, $X_{\epsilon,\mu}$ can be studied by using the slow dynamics

$$x' = f(x, \mu), \ \ \mu \sim 0,$$

with a smooth function f.



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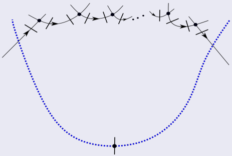

- We consider a smooth (ϵ,μ)-family of planar slow-fast vector fields locally given by $\{\dot{x} = f(x,y,\epsilon,\mu), \dot{y} = \epsilon g(x,y,\epsilon,\mu)\}$.

Definitions

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- If we divide $\{\dot{x} = f(x, y, \epsilon, \mu), \dot{y} = \epsilon g(x, y, \epsilon, \mu)\}$ by $\epsilon > 0$, we get $\{\epsilon x' = f(x, y, \epsilon, \mu), y' = g(x, y, \epsilon, \mu)\}$.

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- When $\epsilon = 0$, we get a slow subsystem: $\{0 = f(x, y, 0, \mu), y' = g(x, y, 0, \mu)\}$.

Definitions

- If the slow dynamics is regular and points towards p on c^- and away from p on c^+ (i.e. $f < 0$), then the slow divergence integral $\mathcal{I}_-(u_m, \mu)$ along c^- (resp. $\mathcal{I}_+(s_1, \mu)$ along c^+) is well defined:

$$\mathcal{I}_-(u_m, \mu) := \int_{\omega(u_m,\mu)}^{0} \frac{\operatorname{div} X_{0,\mu} dx}{f(x,\mu)} < 0, \ \ u_m \in L_m,$$

$$\left(\text{resp. } \mathcal{I}_+(s_1, \mu) := \int_{\alpha(s_1,\mu)}^{0} \frac{\operatorname{div} X_{0,\mu} dx}{f(x,\mu)} < 0, \ \ s_1 \in \Sigma_1\right).$$

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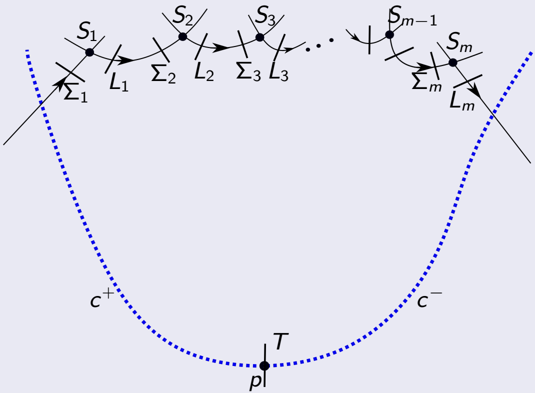
Definitions

- The slow divergence integral:



Definitions

- Sometimes, the slow dynamics $f(x, 0)$ can have isolated (nonzero) singularities, i.e. the slow dynamics is negative, except at a finite number of singularities, near which the passage from the right to the left can be possible for $(\epsilon, \mu) \sim (0, 0)$ and $\epsilon > 0$.

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- A typical example is a singularity of saddle-node type in the slow dynamics
- Besides the "regular" hyperbolic saddles $S_1,\ldots,S_m$, we can have "singular" hyperbolic saddles (they don't exist when $\epsilon=0$), generated by hyperbolic singularities in the slow dynamics.
- For example, $f(\omega(\bar{u}_m,0),0)=0$ and $\frac{\partial f}{\partial x}(\omega(\bar{u}_m,0),0)>0$.

Definitions

Definition (Hyperbolicity ratio)

Let $\lambda_- < 0$ and $\lambda_+ > 0$ be the eigenvalues of a hyperbolic saddle. The hyperbolicity ratio of the hyperbolic saddle is the quantity $r = -\frac{\lambda_-}{\lambda_+} > 0$. The hyperbolic saddle is attracting (resp. repelling) if $r > 1$ (resp. $r < 1$). The hyperbolic saddle is neutral if $r = 1$.

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- We denote by $r_i = r_i(\epsilon, \mu)$ the hyperbolicity ratio of S_i, for $i = 1, \ldots, m$.
- For a singular hyperbolic saddle, we denote ratio of the eigenvalue attached to the slow direction to the eigenvalue attached to the normal direction by $-\epsilon\rho_-$ where $\rho_- = \rho_-(\epsilon, \mu) > 0$ for all $\epsilon \geq 0$

Definitions

Near hyperbolic saddles we use Mourtada's normal form ([Mourtada,1990], [Ilyashenko, Yakovenko,1991]):

Theorem

The Dulac map $\mathcal{D}_i$, near the hyperbolic saddle S_i, has the following structure:

$$\mathcal{D}_i(s_i) = \mathcal{D}_i(s_i, \delta, b_0, \mu) = s_i^{r_i}\big(1 + \Psi(s_i, \delta, b_0, \mu)\big), \qquad (1)$$

where $s_i \geq 0$, $(\delta, b_0, \mu) \sim (0,0,0)$, $\delta \geq 0$, $r_i = r_i(\delta, b_0, \mu)$ is the hyperbolicity ratio of S_i, Ψ is a smooth function for $s_i > 0$ and for all $n \in \mathbb{N}$ we have $\lim_{s_i \to 0} s_i^n \frac{\partial^n \Psi}{\partial s_i^n}(s_i, \delta, b_0, \mu) = 0$, uniformly in (δ, b_0, μ). If $r_i(0,0,0) \notin \mathbb{Q}$, then $\Psi \equiv 0$. If $r_i(0,0,0) \in \mathbb{Q}$, then

$$s_i^n \frac{\partial^n \Psi}{\partial s_i^n} = o(s_i^\nu), \; s_i \to 0, \; \forall \nu < 1. \qquad (2)$$

Results

- We write $\mathcal{I} = \mathcal{I}_-(\bar{u}_m, 0) - \mathcal{I}_+(\bar{s}_1, 0)$.

Theorem

Suppose that the slow dynamics is regular on I (i.e. $f(x,0) < 0$ for all $x \in I$). If S_i is attracting (resp. repelling) for all $i = 1, \dots m$ and $(\epsilon, \mu) = (0,0)$ and $\mathcal{I} < 0$ (resp. > 0), then $\mathrm{Cycl}(X_{\epsilon,\mu}, \Gamma) \leq 1$. *If a limit cycle exists, it is hyperbolic and attracting (resp. repelling).*

Results

- Proof: We study the number of zeros w.r.t. $s_1 \sim 0$ of

$$\mathcal{I}_-(u_m,\mu) - \mathcal{I}_+(s_1,\mu) + \delta^2 \sum_{i=1}^{m} (r_i - 1) \ln s_i + o(1) = 0,$$

where $u_m = \mathcal{T}_m(s_1) \sim 0$ and the $o(1)$-term tends to 0 as $\delta \to 0$, uniformly in $(s_1, b_0, \mu) \sim (0,0,0)$ and $s_1 > 0$.

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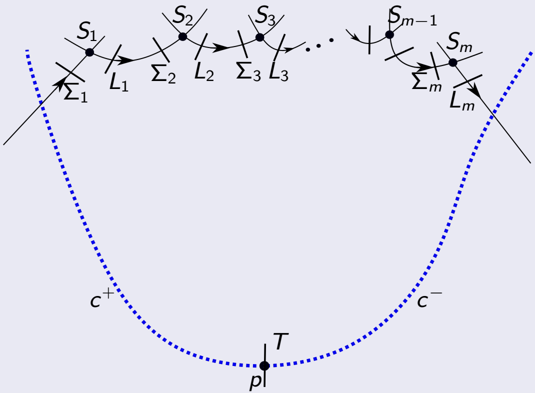
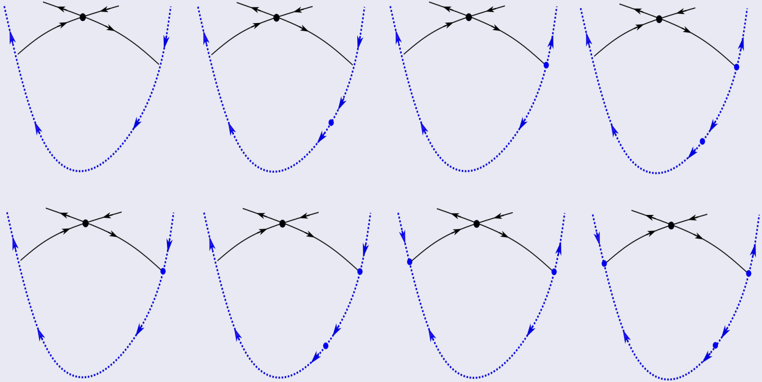
Results

- The difference map:



Results

- When Γ is neither attracting nor repelling, the study of the cyclicity of Γ is more difficult, and we will therefore suppose that $m = 1$.



Results

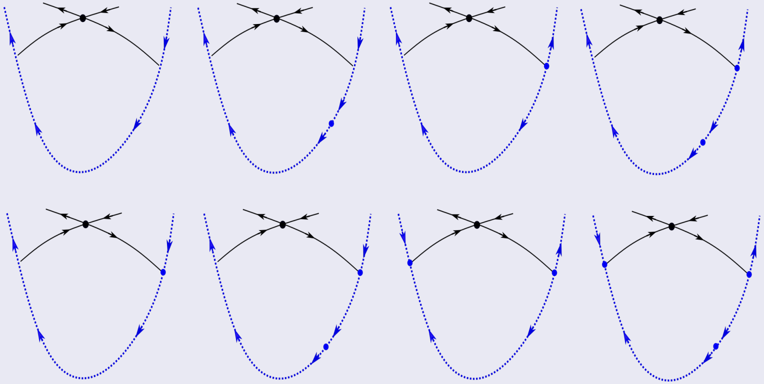
Theorem

Suppose that S is not neutral (i.e. $r(0,0) \neq 1$). The following statements are true:

1. *If the slow dynamics is regular on I, then* $\mathrm{Cycl}(X_{\epsilon,\mu}, \Gamma) \leq 3$.
2. *When the slow dynamics has a finite number of singularities in the interior of I, then* $\mathrm{Cycl}(X_{\epsilon,\mu}, \Gamma) \leq 3$.

Results

- We assume now that the slow dynamics has a singularity at precisely one corner point. If the singularity is hyperbolic, we have:



Results

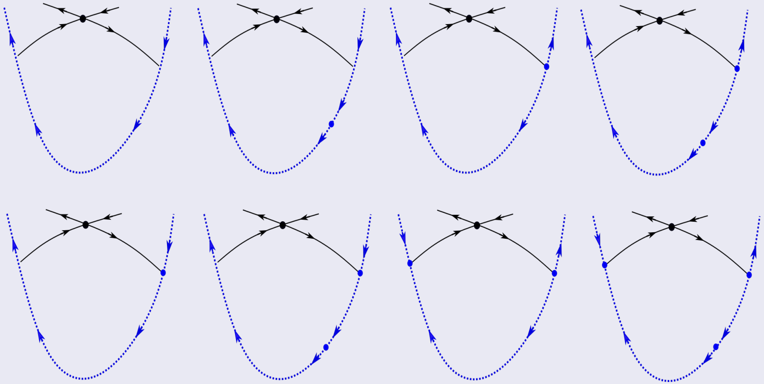
Theorem

Suppose that the slow dynamics has a hyperbolic saddle at precisely one corner point, for example at S_-. Then:

1. *If the slow dynamics is regular on $I \setminus \{\omega(\bar{u}_m, 0)\}$ and the connection between S and S_- cannot be broken for all $(\epsilon, \mu) \sim (0, 0)$ and $\epsilon > 0$, then* $\mathrm{Cycl}(X_{\epsilon,\mu}, \Gamma) \leq 1$.
2. *If the slow dynamics is regular on $I \setminus \{\omega(\bar{u}_m, 0)\}$ and the hyperbolic saddle S is not neutral, then* $\mathrm{Cycl}(X_{\epsilon,\mu}, \Gamma) \leq 3$.
3. *When the slow dynamics has (a finite number of) singularities in the interior of I and the connection between S and S_- cannot be broken, then* $\mathrm{Cycl}(X_{\epsilon,\mu}, \Gamma) \leq 2$.
4. *If S is not neutral and the slow dynamics has extra singularities on c^-, then we have* $\mathrm{Cycl}(X_{\epsilon,\mu}, \Gamma) \leq 3$.

Results

- In the following theorem we suppose that the singularity is not hyperbolic.



Results

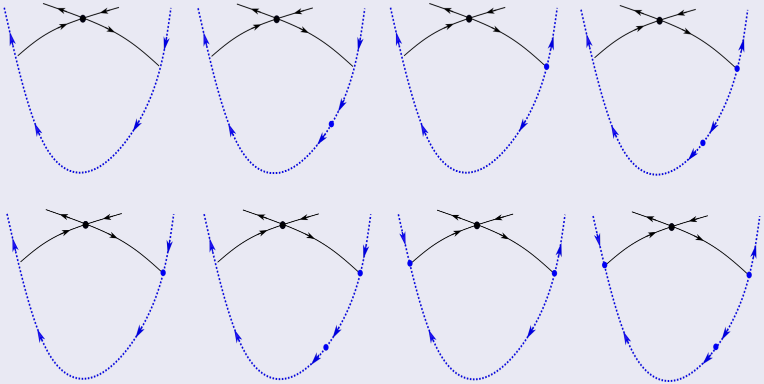
Theorem

Assume that the slow dynamics has a singularity at precisely one corner point. Then the following statements are true:

1. *If the slow dynamics is regular in the interior of I and the hyperbolic saddle S is not neutral, then* $\mathrm{Cycl}(X_{\epsilon,\mu}, \Gamma) \leq 3$.
2. *If S is not neutral and the slow dynamics has extra singularities on the same part of the slow curve where the corner singularity is located, then* $\mathrm{Cycl}(X_{\epsilon,\mu}, \Gamma) \leq 3$.

Results

- The following theorem deals with a hyperbolic singularity at both corner points.



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Results

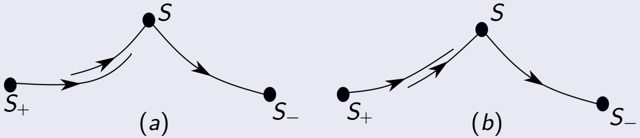

Figure: The connection between S and S_- remains unbroken. (a) $\kappa \leq 0$. (b) $\kappa > 0$.

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Theorem

Suppose that the slow dynamics has a hyperbolic saddle at both corner points and that $\rho_+ r\frac{1}{\rho_-} \neq 1$ *for* $(\epsilon, \mu) = (0, 0)$. *Then:*

1. *If the slow dynamics is regular in the interior of* I *and both connections cannot be broken, then* $\mathrm{Cycl}(X_{\epsilon,\mu}, \Gamma) \leq 1$.
2. *If the slow dynamics has singularities in the interior of* I *and both connections cannot be broken, then* $\mathrm{Cycl}(X_{\epsilon,\mu}, \Gamma) \leq 2$.
3. *If the slow dynamics is regular in the interior of* I *and precisely one of the two connections cannot be broken, then* $\mathrm{Cycl}(X_{\epsilon,\mu}, \Gamma) \leq 2$.
4. *If the slow dynamics has singularities in the interior of* I *and precisely one of the two connections cannot be broken, then* $\mathrm{Cycl}(X_{\epsilon,\mu}, \Gamma) \leq 3$.

Results

- When both connections are broken then we get the following result:

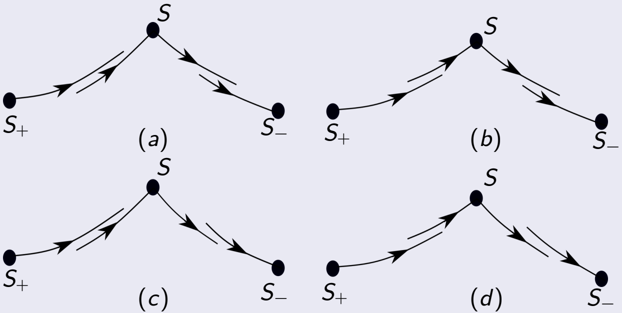

Figure: Both connections can be broken. (a) $\kappa_+ \geq 0$, $\kappa_- \leq 0$. (b) $\kappa_+ < 0$, $\kappa_- \leq 0$. (c) $\kappa_+ \geq 0$, $\kappa_- > 0$. (d) $\kappa_+ < 0$, $\kappa_- > 0$.

Canard cycles with hyperbolic saddles

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Results

Theorem

Suppose that the slow dynamics has a hyperbolic saddle at both corner points and that the slow dynamics is regular in the interior of I. Moreover, suppose that $\rho_+ r^{\frac{1}{\rho_-}} \neq 1$, $\rho_+ \neq \rho_-$ and $r \neq 1$ for $(\epsilon, \mu) = (0, 0)$. If $\rho_+ r^{\frac{1}{\rho_-}} > 1$ and $r < 1$ (resp. $\rho_+ r^{\frac{1}{\rho_-}} < 1$ and $r > 1$), then $\mathrm{Cycl}(X_{\epsilon,\mu}, \Gamma) \leq 4$.

Proof

- Proof: Let the slow dynamics be regular in the interior of I. Then the cyclicity of Γ is bounded by $1+$the number of zeros (counting multiplicity) w.r.t. $s>0$ of

$$\left(\frac{1}{\rho_-}-\delta^2\right)\ln\beta_-(\mathcal{D}_1(s))-\left(\frac{1}{\rho_+}-\delta^2\right)\ln\beta_+(s)+\delta^2(r-1)\ln s+*$$

where $*$ is δ-regularly C^k in $(s,\mathcal{D}_1(s),\Psi_1(s),b_0,\mu)$.

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where $*$ is δ-regularly C^k in $(s, \mathcal{D}_1(s), \Psi_1(s), b_0, \mu)$.

- $\mathcal{D}_1(s) = s^r \left(1 + \Psi(s, \delta, b_0, \mu)\right)$

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  where $*$ is δ-regularly C^k in $(s, \mathcal{D}_1(s), \Psi_1(s), b_0, \mu)$.
- $\mathcal{D}_1(s) = s^r\left(1 + \Psi(s, \delta, b_0, \mu)\right)$
- $\beta_-(u) = \kappa_- + u\tilde{\beta}_-(u)$ and $\beta_+(s) = \kappa_+ + s\tilde{\beta}_+(s)$

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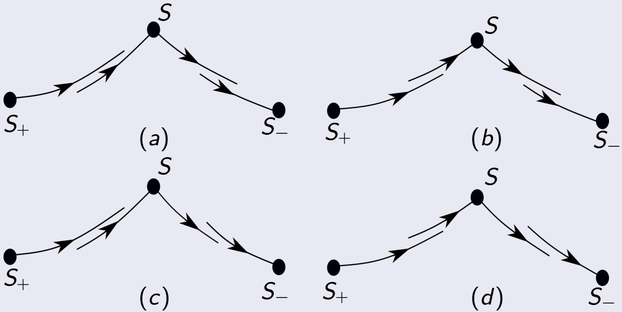

Figure: Both connections can be broken. (a) $\kappa_+ \geq 0$, $\kappa_- \leq 0$. (b) $\kappa_+ < 0$, $\kappa_- \leq 0$. (c) $\kappa_+ \geq 0$, $\kappa_- > 0$. (d) $\kappa_+ < 0$, $\kappa_- > 0$.

Proof

- *(a) "*$\kappa_+ \geq 0, \kappa_- \leq 0$*".* In this case, S is located in front of S_+ and S_- lies in front of S.

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Proof

- *(a)* "$\kappa_+ \geq 0, \kappa_- \leq 0$". In this case, S is located in front of S_+ and S_- lies in front of S.

$$
\begin{aligned}
P(s) \leq & \left(\frac{1}{\rho_-} - \delta^2\right) \ln \mathcal{D}_1(s) \tilde{\beta}_-(\mathcal{D}_1(s)) - \left(\frac{1}{\rho_+} - \delta^2\right) \ln s \tilde{\beta}_+(s) \\
& + \delta^2 (r-1) \ln s + O(1) \\
= & \left(\frac{r}{\rho_-} - \delta^2 r\right) \ln s - \left(\frac{1}{\rho_+} - \delta^2\right) \ln s + \delta^2 (r-1) \ln s + O(1) \\
= & \left(\frac{r}{\rho_-} - \frac{1}{\rho_+}\right) \ln s + O(1).
\end{aligned}
$$

Proof

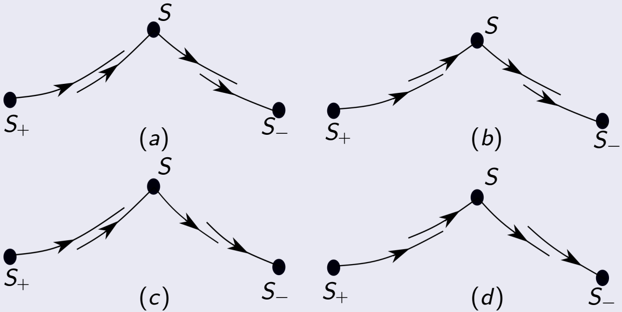

Figure: Both connections can be broken. (a) $\kappa_+ \geq 0$, $\kappa_- \leq 0$. (b) $\kappa_+ < 0$, $\kappa_- \leq 0$. (c) $\kappa_+ \geq 0$, $\kappa_- > 0$. (d) $\kappa_+ < 0$, $\kappa_- > 0$.

Proof

- *(b) "$\kappa_+ < 0$, $\kappa_- \leq 0$".* We use a blow-up in the (κ_+, s)-space and a blow-up in the (κ_-, s^r)-space.

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- Our goal is to prove that P has at most 2 zeros (counting multiplicity) w.r.t. $s > 0$ for each fixed $(\delta, b_0, \mu) \sim (0, 0, 0)$ such that $\kappa_+(\delta, b_0, \mu) < 0$ and $\kappa_-(\delta, b_0, \mu) \leq 0$.

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- We write $\kappa_+ = -\bar{\kappa}_+ s$, with $s > 0$, $s \sim 0$ and $\bar{\kappa}_+ \in]0, \tilde{\beta}_+(0)[$. Note that $\beta_+(s) = s(-\bar{\kappa}_+ + \tilde{\beta}_+(s))$ has to be positive.

Proof

- We have

$$
\begin{aligned}
P(s) \leq & \left(\frac{1}{\rho_-} - \delta^2\right) \ln \mathcal{D}_1(s)\tilde{\beta}_-(\mathcal{D}_1(s)) - \left(\frac{1}{\rho_+} - \delta^2\right) \ln \beta_+(s) \\
& + \delta^2 (r-1) \ln s + O(1) \\
= & \left(\frac{r}{\rho_-} - \delta^2 r\right) \ln s - \left(\frac{1}{\rho_+} - \delta^2\right) \ln s(-\bar{\kappa}_+ + \tilde{\beta}_+(s)) \\
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\end{aligned}
$$

- When $\bar{\kappa}_+ \sim \tilde{\beta}_+(0)$, the above expression is of type $-\infty + \infty$ and we have to blow up the origin in the (κ_-, s^r)-space, i.e. we write $\kappa_- = -\bar{\kappa}_- s^r$ where $s > 0$, $s \sim 0$ and $\bar{\kappa}_- \in [0, \tilde{\beta}_-(0)[$.

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Proof

- The derivative of P, multiplied by $s\beta_-(\mathcal{D}_1(s))\beta_+(s) > 0$, is given by

$$\begin{aligned}&(\frac{1}{\rho_-} - \delta^2)\beta_-'(\mathcal{D}_1(s))\beta_+(s)rs^r(1+\Psi_1(s))\\ &\quad - (\frac{1}{\rho_+} - \delta^2)s\beta_-(\mathcal{D}_1(s))\beta_+'(s)\\ &\quad + \beta_-(\mathcal{D}_1(s))\beta_+(s)\Big(\delta^2(r-1) + * \cdot s + * \cdot s^r + * \cdot s\Psi_1'(s)\Big)\end{aligned}$$

where $*$-functions are δ-regularly C^k in $(s, \mathcal{D}_1(s), \Psi_1(s), b_0, \mu)$.

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- We have

$$
s^{r+1}\Big(-\frac{1}{\rho_+}\tilde{\beta}_+(0)(-\bar{\kappa}_- + \tilde{\beta}_-(0)) + o(1)\Big).
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- If in the O-term we put the expressions for $\beta_\pm$, we get

$$s^r\left(r(\frac{1}{\rho_-} - \frac{1}{\rho_+})\beta_-'(0)\beta_+'(0) + o(1)\right)$$

where the $o(1)$-term tends to 0 as $s \to 0$ and $\bar{\kappa}_\pm \to \tilde{\beta}_\pm(0)$. We use $s\Psi_1', s^2\Psi_1'' \to 0$ as $s \to 0$.

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Proof

- Using Rolle's theorem, we find at most 1 zero of the derivative of P when $\kappa_+ = -\bar{\kappa}_+ s$ and $\kappa_- = -\bar{\kappa}_- s^r$ with $\bar{\kappa}_+ \sim \tilde{\beta}_+(0)$ and $\bar{\kappa}_- \in [0, \tilde{\beta}_-(0)[$.

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- When $\bar{\kappa}_+$ is kept in an interval uniformly away from $\tilde{\beta}_+(0)$, s is kept in an interval on which P is negative.

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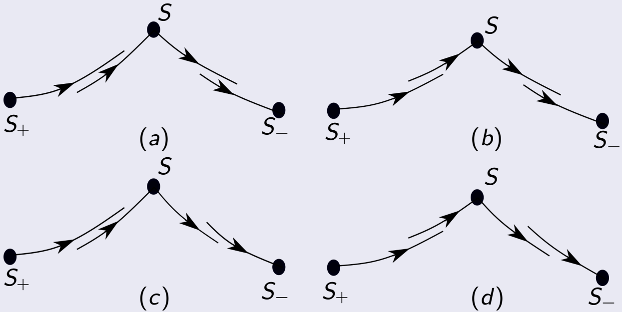
- Indeed, we fix $(\delta, b_0, \mu) \sim (0, 0, 0)$ with the property that $\kappa_+(\delta, b_0, \mu) < 0$ and $\kappa_-(\delta, b_0, \mu) \leq 0$. Thus, $\kappa_\pm$ are fixed. As $\bar{\kappa}_+$ increases, s decreases because $\kappa_+ = -\bar{\kappa}_+ s$.
- When $\bar{\kappa}_+$ is kept in an interval uniformly away from $\tilde{\beta}_+(0)$, s is kept in an interval on which P is negative.
- When $\bar{\kappa}_+$ is kept close to $\tilde{\beta}_+(0)$, s decreases further and P can have at most 2 zeros using Rolle's theorem.

Proof



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- *(c) "$\kappa_+ \geq 0$, $\kappa_- > 0$".* We will prove that P has at most 3 zeros (counting multiplicity) in $s > 0$ and $s \sim 0$ under condition that $r < 1$.

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- We write $\kappa_- = \bar{\kappa}_- s^r$ where $\bar{\kappa}_- \in]0, \bar{\kappa}_-^0]$ with $\bar{\kappa}_-^0 > 0$ arbitrary and fixed. Then P is negative. Indeed,

$$P(s) \leq \left(\frac{r}{\rho_-} - \frac{1}{\rho_+}\right) \ln s + \left(\frac{1}{\rho_-} - \delta^2\right) \ln\left(\bar{\kappa}_- + (1+\Psi)\tilde{\beta}_-(\mathcal{D}_1)\right) + O(1).$$

  To cover the limiting case "$\bar{\kappa}_-^0 \to \infty$" we write $s^r = \bar{s}_- \kappa_-$ where $\bar{s}_- \sim 0$ and $\bar{s}_- > 0$. Then $\beta_-(\mathcal{D}_1(s)) = \kappa_-(1 + O(\bar{s}_-))$.
- Here we have to consider two charts in the (κ_+, s)-space.

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Proof

- First, we write $\kappa_+ = \bar{\kappa}_+ s$ where $\bar{\kappa}_+ \in [0, \bar{\kappa}_+^0]$ with $\bar{\kappa}_+^0 > 0$ large and fixed. Then $\beta_+(s) = s(\bar{\kappa}_+ + \tilde{\beta}_+(s))$.

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- The derivative of P can be written as

$$
\begin{aligned}
&(\frac{r}{\rho_-} - \delta^2 r)\beta_-'(\mathcal{D}_1(s))s(\bar{\kappa}_+ + \tilde{\beta}_+)\bar{s}_-\kappa_-(1+\Psi_1) \\
&-(\frac{1}{\rho_+} - \delta^2)s\kappa_-(1 + O(\bar{s}_-))\beta_+'(s) \\
&+ s\kappa_-(1 + O(\bar{s}_-))(\bar{\kappa}_+ + \tilde{\beta}_+)\Big(\delta^2(r-1) + * \cdot s + * \cdot s^r + * \cdot s\Psi_1'(s)\Big) \\
&= s\kappa_-\big(-\frac{1}{\rho_+}\beta_+'(0) + o(1)\big).
\end{aligned}
$$

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Proof

- In the limiting case "$\bar{\kappa}_+^0 \to \infty$" we have $s = \bar{s}_+ \kappa_+$, with $\bar{s}_+ \sim 0$ and $\bar{s}_+ > 0$, and $\beta_+(s) = \kappa_+(1 + O(\bar{s}_+))$. This is the most difficult chart. P''' is given by

$$
\begin{aligned}
&\Big(\frac{2r}{\rho_-} - \frac{1+r}{\rho_+} + o(1)\Big)\beta_-'(\mathcal{D}_1)\beta_+'(s)rs \\
&+ \Big(\frac{r(r-1)}{\rho_-} + o(1)\Big)\beta_-'(\mathcal{D}_1)\beta_+(s)r \\
&+ O(\beta_- s s^{1-r}, \beta_+ s s^{2-r}\Psi_1'', \beta_- s s^{1-r}\Psi_1', \beta_- s s^{2-r}\Psi_1'', \beta_-\beta_+ s^{2-r}(\Psi_1')^2, \\
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\end{aligned}
$$

$$
\kappa_+\Big(\big(\frac{r(r-1)}{\rho_-} + o(1)\big)\beta_-'(\mathcal{D}_1)(1 + O(\bar{s}_+))r + o(1)\Big).
$$

Canard cycles with hyperbolic saddles

Proof

- Using Rolle's theorem twice we conclude that P' has at most 2 zeros counting multiplicity in the chart "$\bar{\kappa}_-^0 \to \infty$" and $\kappa_+ \geq 0$.

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- If we apply Rolle's theorem once more, we find that P has at most 3 zeros for $\kappa_+ \geq 0$ and $\kappa_- > 0$ (note that P is nonzero when $\bar{\kappa}_-^0$ is finite and $\bar{\kappa}_- \in]0, \bar{\kappa}_-^0]$).

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- Thank you!