Non-integrability criteria for polynomial differential systems in $\mathbb{C}^2$

Jaume Giné[1], Jaume Llibre[2]

Universitat de Lleida, Spain
Universitat Autònoma de Barcelona, Spain

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Introduction

How to “solve” the differential equations that appear in many phenomena?
What is “solve” a differential equation?

- For physicist and applied mathematician means: to derive a closed-form solution.
- For mathematician: existence and uniqueness of the solutions.

The first attempt to solve differential equations either explicitly or by series expansions goes back to Euler, Newton and Leibniz.

Introduction

The theory of integration of equations was subsequently expanded by analysts and mechanicians as Lagrange, Poisson, Hamilton, Liouville in 18th and 19th centuries.

The solution can always be represented by the combination of known functions or by perturbations expansions.

Integrability

The property of equations for which all the local and global information can be obtained either explicitly from the solutions or implicitly from the constant of motion (first integrals)

Introduction

Two different works have radically changed the program of classical mechanics of 19th century.

- Kovalevskaya's study of the Euler equations. Technique based in the behavior of the solution near the singularities in the complex plane.
- Poincaré's geometric theory of solutions. He study asymptotic solutions as geometric sets which define the global qualitative behavior of solutions in the long time limit.

The two approaches share a common feature: The local analysis of the differential equation, close to its complex time singularities for Kovalevskaya and its space singularities for Poincaré, allows to find global properties of the system.

Introduction

As a consequences of these works, mathematicians and physicists shifted their interest away from the integrability theory.

It is introduced the notion of **Dynamical system** by Birkhoff.

The success of dynamical systems theory was so overwhelming that exact methods for integration were considered for years useless and non-generic. This way of thinking continues nowadays.

- The important discovery Zabusky and Kruskal of solitons of the Korteweg-de Vries equation. Solitons, pattern formation and ordered structures are the key feactures of systems with infinite degrees of freedom.

Introduction

This seems shocking with the chaos, strange attractors and ergodicity of dynamical systems with few degrees of freedom.

However in order to analyze a family of dynamical systems, is usual to begin detecting the elements of the family that satisfy some non-generic property (including integrability).

Next, some systems in the family are described as a perturbations of the non-generic systems studied, and the dynamical behavior of the perturbed systems can be analyzed.

This shows how crucial is the understanding of the phenomena of **integrability** in dynamical systems.

Introduction

Are the solutions of system $\dot{x} = P(x, y)$ and $\dot{y} = Q(x, y)$ in $\mathbb{C}^2$ expressable in terms of elementary functions? And the first integrals?

First Integrals: functions that are constants on solution curves (to deduce properties of the solutions).

Poincaré (1888) begun the qualitative theory of differential equations.

INTEGRABILITY PROBLEM: When does a system of differential equations have a first integral that can be expressed in terms of "known functions" and how does one find such an integral?

Objective

Determine when a differential system in $\mathbb{C}^2$ has or has not a first integral is one of the main problems in the qualitative theory of differential systems.

The Liouville integrability is based on the existence of invariant algebraic curves and their multiplicity through the exponential factors.

Recently generalizations on the Liouville integrability theory have been done.

Objective

There exist differential systems which are integrable that are non–Liouville integrable. Some of them are Weierstrass integrable.

How to detect these non–Liouville and also non-Weierstrass integrable systems?

In this talk we give a new criterium that detects weak formal Weierstrass and strong formal Weierstrass non–integrability.

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Liouville integrability

Consider the complex polynomial differential system

$$\dot{x} = P(x,y), \qquad \dot{y} = Q(x,y), \tag{1}$$

where P, $Q \in \mathbb{C}[x,y]$. The *degree* of system (1) is $m = \max\{\deg P, \deg Q\}$.

Obviously system (1) has the associated differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{Q(x,y)}{P(x,y)}, \tag{2}$$

and the associated vector field $\mathcal{X} = P(x,y)\partial/\partial x + Q(x,y)\partial/\partial y$.

Invariant curve

$f(x,y)=0$ is an invariant curve of system (1) if the orbital derivative $\dot{f}=\mathcal{X}f=P\partial f/\partial x+Q\partial f/\partial y$ vanishes on $f=0$.

$f(x,y)=0$ with $f\in\mathbb{C}[x,y]$, is an invariant algebraic curve of system (1) if

$$\mathcal{X}f=P\frac{\partial f}{\partial x}+Q\frac{\partial f}{\partial y}=Kf. \tag{3}$$

where $K(x,y)\in\mathbb{C}[x,y]$ of degree less than or equal to $m-1$, called the cofactor associated to the curve $f(x,y)=0$.

First integral

A non-constant function $H : U \subset \mathbb{C}^2 \to \mathbb{C}$ is a first integral of system (1) in the open set U if this function is constant in each solution $(x(t), y(t))$ of system (1) contained in U. Clearly $H \in C^1(U)$ is a first integral of system (1) on U if and only if $\mathcal{X}H = P\partial H/\partial x + Q\partial H/\partial y \equiv 0$ on U.

A function R is an integrating factor associated to a first integral H of system (1) if

$$RP = -\frac{\partial H}{\partial y}, \quad \text{and} \quad RV = \frac{\partial H}{\partial x},$$

or equivalently

$$P\frac{\partial R}{\partial x} + Q\frac{\partial R}{\partial y} = -\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) R = -\mathrm{div}(\mathcal{X})R. \tag{4}$$

Liouvillian integrability

A polynomial differential system (1) has a Liouvillian first integral H if its associated integrating factor is of the form

$$R = \exp\left(\frac{D}{E}\right)\prod_i C_i^{\alpha_i}, \tag{5}$$

where D, E and the C_i are polynomials in $\mathbb{C}[x, y]$ and $\alpha_i \in \mathbb{C}$.

The functions of the form (5) are called Darboux functions. Note that the curves $C_i = 0$ and $E = 0$ are invariant algebraic curves of the polynomial differential system (1), and the exponential $\exp(D/E)$ is a product of some exponential factors associated to the invariant algebraic curves of system (1) or to the invariant straight line at infinity when such invariant curves have multiplicity greater than one.

Liouville integrability theory

There was the belief that a Liouvillian integrable system has always an invariant algebraic curve in the finite phase portrait. Moreover this statement was proved under certain hypothesis by Zernov & Scárdua (2001).
However there exist Liouvillian integrable system without any finite invariant algebraic curve as the folowing example shows.

Proposition (Giné & Llibre (2011))

Consider the differential system

$$\dot{x} = -1 - 2x^2 - xy, \qquad \dot{y} = 2x(2x + y). \tag{6}$$

System (6) is a Liouvillian integrable system without any finite invariant algebraic curve.

Liouville integrability theory

The proof previous theorem is based in the two following propositions:

Proposition

System (6) is Liouville integrable because it has the integrating factor $R = e^{-(2x+y)^2/4}$. Moreover it has the first integral

$$H(x,y) = 2e^{-\frac{(2x+y)^2}{4}} x - \sqrt{\pi}\,\mathrm{Erf}\left(\frac{1}{2}(2x+y)\right).$$

Proposition

System (6) has not finite invariant algebraic curves.

Liouville integrability theory

The proof of the second proposition is based in mentioned method to detect invariant algebraic curves of arbitrary degree. We impose that $F(x,y)=0$, where F is a polynomial of degree N be an invariant algebraic curve of system (6) i.e.,

$$(-1-2x^2-xy)F_x(x,y)+2x(2x+y)F_y(x,y)=K(x,y)F(x,y),$$

where $K(x,y)$ is a polynomial of degree 1 and is the cofactor of the curve $F(x,y)$. In fact for the computations we take $K(x,y)=a_{00}+a_{10}x+a_{01}y$. Now we develop the invariant curve $F(x,y)$ in homogeneous terms into the form

$$F(x,y)=f_N(x,y)+f_{N-1}(x,y)+\cdots\,, \tag{7}$$

where f_i are homogeneous trigonometric polynomials of degree i.

Liouville integrability theory

We substitute the expression of $F(x,y)$ into the equation that must satisfy and we consider the homogeneous term of largest degree which is $N+1$ given by

$$(-2x^2 - xy) f_{N_x} + 2x(2x+y) f_{N_y} = (a_{10}x + a_{01}y) f_N. \qquad (8)$$

We made the change $(x,y) \to (v, uv)$. We have that $f_N(x,y) = v^N \tilde{f}_N(1,u)$ and from here we assume that $\tilde{f}_N(1,u)$ is not identically a constant in u, i.e., $f_N \neq x^N$. Equation (8) takes the form

$$(4 + 4u + u^2) \tilde{f}_{N_u} = (a_{10} + 2N + (a_{01} + N)u)\, \tilde{f}_N,$$

Liouville integrability theory

and its solution is given by

$$\tilde{f_N} = C_N \exp\left(\frac{2a_{01} - a_{10}}{2+u}\right)(2+u)^{a_{01}+N}$$

which implies $a_{10} = 2a_{01}$. Moreover taking into account that $\tilde{f_N}$ must be of degree at most N we have that $a_{01} \in \{-(N-1), -(N-2), \ldots, -1, 0\}$. Undoing the change we have that $f_N = C_N\,(2x+y)^{a_{01}+N} x^{-a_{01}}$.

The next homogeneous terms of higher degree given by

$$(-2x^2-xy)f_{N-1_x}+2x(2x+y)f_{N-1_y} = a_{00}C_N(2x+y)^{a_{01}+N}x^{-a_{01}}.$$

Liouville integrability theory

The change $(x, y) \to (v, uv)$ transforms this equation into

$$
\begin{aligned}
&(4 + 4u + u^2)\tilde{f_{N-1}}_u \\
&= a_{00} C_N \, (2+u)^{a_{01}+N} + (2+u)(a_{01} + N - 1)\tilde{f}_{N-1}.
\end{aligned} \tag{9}
$$

The solution equation (9) is

$$
\tilde{f}_{N-1} = (2+u)^{a_{01}+N-1}(C_{N-1} + C_N a_{00} \ln|2+u|),
$$

which has logarithmic terms that disappear when we take the condition $a_{00} = 0$ and in this case we have that $\tilde{f}_{N-1}$ has the form

$$
\tilde{f}_{N-1} = C_{N-1}(2+u)^{a_{01}+N-1}.
$$

Liouville integrability theory

The same process for the next homogeneous term give us that $\tilde{f}_{N-2}$ reads for

$$\tilde{f}_{N-2} = (2+u)^{a_{01}+N-2}\Big(C_{N-2} - C_N a_{01}(2+u) \\ +2\, C_N (a_{01}+N) \ln|2+u|\Big). \tag{10}$$

Finally, in order to vanish the logarithmic term we obtain the condition $a_{01} = -N$. Since $\tilde{f}_{N-2}$ must a polynomial we obtain $C_N \neq 0$ which is contradiction. The case $f_N = C_N x^N$ also gives a contradiction.

Liouville integrability theory

Are there polynomial systems with a non-Liouvillian first integral ?

- Cairó, Giacomini & Llibre (2002): They give some example in the Lotka-Volterra quadratic systems that have a non-Liouvillian first integral.
- García & Giné (2002): They give some examples in Liénard systems that have a non-Liouvillian first integral.
- Chavarriga, Giacomini & Grau (2003): They give the proof of the non-Liouvillian integrability of the algebraic limit cycles of quadratic systems.

Example of non-Liouvillian integrable system

Consider the polynomial Liénard system

$$\dot{x} = P(x,y) = -y + x^4 \ , \quad \dot{y} = Q(x,y) = x \ . \tag{11}$$

From Odani's works, it follows that system (11) does not have any invariant algebraic curve.
The only possible Darboux integrating factor, if it exists, is an exponential factor of the form $\exp(h)$ with $h \in \mathbb{C}[x,y]$. From the definition of inverse integrating factor we have

$$(-y + x^4)\partial h/\partial x + x\partial h/\partial y = 4x^3 \ . \tag{12}$$

Let $h(x,y) = \sum_{i=0}^{N} h_i(y)x^i$, where $h_i(y) \in \mathbb{C}[y]$ with $h_N(y) \not\equiv 0$. Equating the highest degree terms in both members of (12) gives $Nh_N(y)x^{N+3} = 0$. Therefore $N = 0$ which gives a contradiction with equation (12). Therefore system (11) does not have any Liouvillian first integral.

Example of non-Liouvillian integrable system

The change of the dependent and independent variables $dw/dy = -2x^2 w$ and $z = 4^{1/3} y$ transforms the equation $dy/dx = x/(-y + x^4)$ to an Airy differential equation $w''(z) = zw(z)$.
Its general solution is given by $w(z) = c_1 \mathrm{Ai}(z) + c_2 \mathrm{Bi}(z)$ where c_i are arbitrary constants and $\mathrm{Ai}(z)$ and $\mathrm{Bi}(z)$ is a pair of linearly independent solutions of the Airy equation.
$H(x, y) = f_1 f_2^{-1}$ is a non-Liouvillian first integral of system (11), where

$$
\begin{aligned}
f_1(x, y) &= 2^{1/3} x^2 \mathrm{Ai}(4^{1/3} y) + \mathrm{Ai}'(4^{1/3} y), \\
f_2(x, y) &= 2^{1/3} x^2 \mathrm{Bi}(4^{1/3} y) + \mathrm{Bi}'(4^{1/3} y).
\end{aligned}
$$

are invariant curves with associated generalized cofactors $K_1 = K_2 = 2x^3$. A non-Liouvillian inverse integrating factor is $V(x, y) = f_2^2(x, y)$.

Generalizations of Liouvillian integrability

This type of examples are included in the first generalization of the Liouville theory of integrability where the cofactors for non–algebraic invariant curves are defined.

I.A. García, J. Giné, *Generalized cofactors and nonlinear superposition principles*, Appl. Math. Lett. 16 (2003), no. 7, 1137–1141.

The next question is whether a non-algebraic invariant curve has always a polynomial cofactor. The answer to this question in general is negative. The following example is given in:

J. Giné, M. Grau, J. Llibre, *On the extensions of the Darboux theory of integrability*, Nonlinearity 26 (2013), no. 8, 2221–2229.

Generalizations of Liouvillian integrability

Consider the polynomial differential system

$$\dot{x} = 2y, \quad \dot{y} = x - y^4. \tag{13}$$

System (13) has the non-algebraic invariant curve $f = 0$ where $f = y + \sqrt{\mathrm{Ai}(x)/\mathrm{Ai}'(x)}$, with the non-polynomial cofactor

$$K(x,y) = -y^3 + y^2\sqrt{\mathrm{Ai}(x)/\mathrm{Ai}'(x)} - y\mathrm{Ai}(x)/\mathrm{Ai}'(x) + x\sqrt{\mathrm{Ai}'(x)/\mathrm{Ai}(x)}.$$

By Seidenberg a formal invariant curve $f(x,y) = 0$ of a planar autonomous differential system given by a formal power expansion $f \in \mathbb{C}[[x,y]]$ must satisfy an equation $\mathcal{X}f = Lf$ where $L \in \mathbb{C}[[x,y]]$ is also a formal power expansion.

Cofactor of a formal invariant curve

Moreover the form of the non-polynomial cofactor suggests to define the so-called quasipolynomial cofactor.

Proposition

Let $g(x) \in \mathbb{C}[[x]]$. A formal invariant curve of the form $y - g(x) = 0$ of a polynomial differential system (1) of degree m has a quasipolynomial cofactor of the form

$$K(x, y) = K_{m-1}(x)y^{m-1} + \cdots + K_1(x)y + K_0(x). \tag{14}$$

I.A. García, H. Giacomini, J. Giné, *Generalized nonlinear superposition principles for polynomial planar vector fields*, J. Lie Theory 15 (2005), no. 1, 89–104.

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Formal Weierstrass polynomial

Let $\mathbb{C}[[x]]$ be the set of the formal power series in the variable x with coefficients in $\mathbb{C}$, and $\mathbb{C}[y]$ the set of the polynomials in the variable y with coefficients in $\mathbb{C}$. A function of the form

$$\sum_{i=0}^{\ell} a_i(x) y^i \in \mathbb{C}[[x]][y] \tag{15}$$

is called a formal Weierstrass polynomial in y of degree ℓ. Here we have privileged the variable y but of course we can privileged the variable x instead of y. A formal Weierstrass polynomial whose coefficients are convergent is called a Weierstrass polynomial.

Weierstrass integrability

A planar autonomous differential system is Weierstrass integrable if admits an integrating factor of the form

$$R = \exp\left(\frac{D}{E}\right)\prod_i C_i^{\alpha_i}, \tag{16}$$

where D, E and the C_i's are Weierstrass polynomials. In several works some Liénard differential systems and Abel differential equations that are Weierstrass integrable are studied.

J. Giné, M. Grau, *Weierstrass integrability of differential equations*, Appl. Math. Lett. 23 (2010), no. 5, 523–526.

Weierstrass integrability

A first question is if any Liouville integrable system is formal Weierstrass integrable. The answer in negative.

Given a polynomial, for instance, $x^3y^2 + x^2y - x - 2 = 0$ the y-roots of this polynomial are not formal, in fact are of the form

$$y_{1,2} = \frac{-x^2 \pm x^{3/2}\sqrt{8+5x}}{2x^3}$$

whose expansions are

$$y_1 = \frac{\sqrt{2}}{x^{3/2}} - \frac{1}{2x} + \frac{5}{8\sqrt{2x}} - \frac{25\sqrt{x}}{256\sqrt{2}} + \mathcal{O}(x^{3/2})$$

$$y_2 = -\frac{\sqrt{2}}{x^{3/2}} - \frac{1}{2x} - \frac{5}{8\sqrt{2x}} + \frac{25\sqrt{x}}{256\sqrt{2}} + \mathcal{O}(x^{3/2})$$

Puiseux Weierstrass integrability

Let $\mathbb{C}((x)))$ be the set of series in fractionary powers in the variable x with coefficients in $\mathbb{C}$ (these series are called Puiseux series), and $\mathbb{C}[y]$ the set of the polynomials in the variable y with coefficients in $\mathbb{C}$. We call a function of the form

$$\sum_{i=0}^{\ell} a_i(x) y^i \in \mathbb{C}((x))[y] \tag{17}$$

a Puiseux Weierstrass polynomial in y of degree ℓ. This definition is a generalization of the *formal Weierstrass polynomial* presented before.

Puiseux Weierstrass integrability

A planar autonomous differential system is Puiseux Weierstrass integrable if it admits an integrating factor of the form

$$R = \exp\left(\frac{D}{E}\right) \prod_i C_i^{\alpha_i}, \tag{18}$$

where D, E and the C_i's are Puiseux Weierstrass polynomials.

By definition the Puiseux Weierstrass integrable systems include the Liouville integrable systems as the following result shows.

Puiseux Weierstrass integrability

Let $\mathbb{C}[[x,y]]$ be the set of all formal power series in the variables x and y with coefficients in $\mathbb{C}$.

Theorem (Puiseux)

If $f \in \mathbb{C}[[x,y]]$ then it has a unique decomposition of the form

$$f = ux^r \prod_{j=1}^{\ell} (y - g_j(x)), \tag{19}$$

where $g_j(x)$ are Puiseux series and $r \in \mathbb{Z}$, $r \geq 0$ and $u \in \mathbb{C}[[x,y]]$ is invertible inside the ring $\mathbb{C}[[x,y]]$.

We note that a Darboux integrating factor (5) is analytic function where it is defined consequently it can be written into the form (19).

Formal weakly Weierstrass integrability

Does not exist any criterium for detecting non–Liouville integrability or non–Weierstrass integrability. In general to detect the non–integrability of a differential system is a very difficult problem.

We say that a polynomial differential system (1) is formal weakly Weierstrass integrable if it has an integrating factor of the form

$$R = \prod_{i=1}^{\ell} (y - f_i(x))^{\alpha_i}, \tag{20}$$

but where the functions $f_i(x) \in \mathbb{C}[[x]] \subset \mathbb{C}((x))$.

Now we give a criterium for detecting when a polynomial differential system (1) is not formal weakly Weierstrass integrable.

Formal weakly Weierstrass integrable

The criterion is based in the following result which is an extension of the results of Darboux integrability.

Theorem

Assume that the polynomial differential system (1) *is formal weakly Weierstrass integrable, that is, it has an integrating factor of the form* (20)*. If K_i is the cofactor of the invariant curve $y - f_i(x) = 0$, then*

$$\sum_{i=1}^{\ell} \alpha_i K_i = -\mathrm{div}(\mathcal{X}).$$

Formal weakly Weierstrass integrable

The proof of the result is based on the given proposition.

Proposition

Let $g(x) \in \mathbb{C}[[x]]$. A formal invariant curve of the form $y - g(x) = 0$ of a polynomial differential system (1) of degree m in y has a quasipolynomial cofactor of the form

$$K(x, y) = K_{m-1}(x)y^{m-1} + \cdots + K_1(x)y + K_0(x). \tag{21}$$

Then any formal invariant curve $y - f_i(x) = 0$ has a formal Weierstrass polynomial, that is, $K \in C[[x]][y]$.

Criterium for detect formal weakly Weierstrass non–integrability

We compute the solutions of the form $y = f_i(x) = g_i(x) + \mathcal{O}(x^{r+1}) = \sum_{j=0}^{r} a_j x^j + \mathcal{O}(x^{r+1}) \in \mathbb{C}[[x]]$ of system (1) through the equation $Eq := \dot{x} dy/dx - \dot{y} = 0$ up to order r in the variable x. After we compute the corresponding cofactor $L_i \in \mathbb{C}[[x]][y]$ of the invariant curve $y - f_i(x) = 0$ up to order r, using the equality

$$\mathcal{X}(y - g_i(x)) = L_i(y - g_i(x)) + \mathcal{O}(x^{r+1}).$$

From Theorem 6 if the cofactors L_i's do not verify the equality

$$\sum_{i=1}^{\ell} \alpha_i L_i = -\mathrm{div}_r(\mathcal{X}) + \mathcal{O}(x^{r+1}), \tag{22}$$

where $\mathrm{div}_r(\mathcal{X})$ is the divergence $\mathrm{div}(\mathcal{X})$ up to order r in the variables x and y, then system (1) cannot be formal weakly Weierstrass integrable.

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Example 1

Consider the polynomial differential system

$$\dot{x} = -y + x^2, \qquad \dot{y} = x + ax^2. \tag{23}$$

System (23) has a focus at the origin if $a \neq 0$ because the first Poincaré-Liapunov constant is $V_4 = a$. Now we study if system (23) can be formal weakly Weierstrass integrable. We find four solutions curves up to order 5 for system (23), two of them not passing through the origin and two passing through the origin that we call y_1, y_2, y_3 and y_4.

Next we compute their Weierstrass polynomial cofactors up to order 5 that we call L_1, L_2, L_3 and L_4 respectively.

Example 1

Now we try to see if there is a linear combination of them equal to minus the divergence, that is

$$c_1 L_1 + c_2 L_2 + c_3 L_3 + c_4 L_4 = -\mathrm{div}\mathcal{X} + \mathcal{O}(x^5),$$

and the system does not have any solution. Hence system (23) has not an integrating factor of the form

$$R = \prod_{i=1}^{\ell} (y - f_i(x))^{\alpha_i}, \tag{24}$$

Consequently system (23) is not formal weakly Weierstrass integrable privileging the variable y. The same happens if we privilege the variable x.

Example 1

When $a = 0$ equation

$$c_1 L_1 + c_2 L_2 + c_3 L_3 + c_4 L_4 = -\mathrm{div}\mathcal{X} + \mathcal{O}(x^5),$$

has the solution

$$c_1 = c_2 = -\frac{5}{2}(1 + c_4) \quad c_3 = 1 + c_4.$$

In fact for $a = 0$ the differential system (23) is a time-reversible system because is invariant by the the symmetry $(x, y, t) \to (-x, y, -t)$. Hence it has a center at the origin. Moreover it has a Darboux integrating factor of the form $R = (1 + 2y - 2x^2)^{-1}$. So for $a = 0$ system (23) is Liouvillian integrable.

Example 2

Consider the differential system

$$\dot{x} = -y + x^4, \qquad \dot{y} = x. \tag{25}$$

System (25) is a time-reversible system because is invariant under the symmetry $(x, y, t) \to (-x, y, -t)$. Hence it has a center at the origin of coordinates. Moreover, as we have seen, system (25) has not a Liouville first integral, but it has a formal weakly Weierstrass integrating factor because as we have seen before it has the integrating factor

$$V(x, y) = \left(2^{1/3} x^2 \mathrm{Bi}(4^{1/3} y) + \mathrm{Bi}'(4^{1/3} y)\right)^2 .$$

Now we are going to apply the criterium to detect if system (25) is formal weakly Weierstrass integrable, as it is the case.

Example 2

We propose a solution curve of the form $y = \sum_{j=0}^{4} a_j x^j + \mathcal{O}(x^5)$ and substituting this solution into the differential equation $Eq := \dot{x} dy/dx - \dot{y} = 0$ up to order 4, and solving it we find five solutions curves, two passing through the origin and three not passing through it. We call them y_1, y_2, y_3, y_4 and y_5.

We compute the Weierstrass polynomial cofactors of these solution curves up to order 4. Since system (25) is of degree 1 in y the Weierstrass polynomial cofactors are of the form $k_0(x)$. Using the equation

$$\mathcal{X}(y - y_i(x)) = L_i(y - y_i(x)) + \mathcal{O}(x^5).$$

Example 2

for each solution curves we compute the Weierstrass polynomial cofactor up to order 4 and we call them L_1, L_2, L_3, L_4 and L_5. Now we study if system (25) satisfies the condition

$$c_1L_1 + c_2L_2 + c_3L_3 + c_4L_4 + c_5L_5 = -\mathrm{div}\mathcal{X} + \mathcal{O}(x^5),$$

and this equation has the solution $c_3 = c_4$ and

$$c_1 = -\frac{15i-5\sqrt{3}+6ic_4-2\sqrt{3}c_4+10ic_5}{5(-i+\sqrt{3})}, \quad c_2 = \frac{10\sqrt{3}+4\sqrt{3}c_4+5ic_5+5\sqrt{3}c_5}{5i-5\sqrt{3}}.$$

Consequently system (25) could have a formal weakly Weierstrass integrating factor of the form (24), as we know that it has.

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Formal strongly Weierstrass integrability

A polynomial differential system (1) is formal strongly Weierstrass integrable if it has an integrating factor of the form

$$M(x,y) = \alpha(x) \prod_{k=1}^{\ell} (y - g_k(x))^{\alpha_k}, \tag{26}$$

where the functions $\alpha(x), g_k(x) \in C[[x]]$ for $i = 1, \ldots, k$.

Note that this integrating factor is a generalization of the previous one.

Main result

Theorem

Assume that a polynomial differential system (1) is formal strongly Weierstrass integrable, and let $H(x,y)$ be a first integral.

(a) *Let $h(x) \in \mathbb{C}[[x]]$ and $y = h(x)$ be an invariant curve of the system such that $H(x,y)$ is defined on the curve $y = h(x)$. Then there exists an integrating factor $M(x,y)$ of the form (26) such that $M(x,h(x)) = 0$.*

(b) *Assume that the origin of system (1) is a singular point, and the first integral $H(x,y)$ and $M(x,y)$ of statement (a) are well-defined at the origin. Then a linear combination of the formal Weierstrass cofactors up to order r of the solutions of the form $y = f(x)$ satisfying $Eq := \dot{x}dy/dx - \dot{y} = 0$ must be equal to minus the divergence of system (1) up to order r.*

Example

Consider the differential system

$$\dot{x} = y, \quad \dot{y} = -(\zeta x^2 + \alpha)y - (\varepsilon x^3 + \sigma x). \qquad (27)$$

System (27) contains the famous force-free Duffing oscillator ($\zeta = 0, \ \varepsilon \neq 0$) and the Duffing-Van der Pol ($\zeta \neq 0, \ \varepsilon \neq 0$).

Theorem

System (27) can be formal strongly Weierstrass integrable if, and only if, one of the following cases holds:

(a) $\sigma = 2\alpha^2/9$,

(b) $\sigma \neq 2\alpha^2/9$, $\sigma \neq 0$ *and* $3\alpha\varepsilon - 4\zeta\sigma = 0$,

(c) $\sigma \neq 2\alpha^2/9$, $\sigma \neq 0$ *and* $-21\alpha\varepsilon^2 + 6\alpha^2\varepsilon\zeta + 24\varepsilon\zeta\sigma - 7\alpha\zeta^2\sigma = 0$,

(d) $\sigma \neq 2\alpha^2/9$, $\sigma = 0$ *and* $-6\varepsilon(7\varepsilon - 2\alpha\zeta) = 0$.

Example

The Liouville integrability of system (27) was studied in previously an the following results were established.

Theorem

System (27) with $\zeta = 0$ and $\varepsilon \neq 0$ is Liouvillian integrable if and only if either $\alpha = 0$ or $\sigma = 2\alpha^2/9$.

In the case $\zeta \neq 0$ by a suitable rescaling of the variables for the Duffing-Van der Pol system we can take $\zeta = 3$ without loss of generality.

Theorem

System (27) with $\zeta = 3$ and $\varepsilon \neq 0$ is Liouvillian integrable if and only if $\alpha = 4\varepsilon/3$ and $\sigma = \varepsilon^2/3$.

Proof of the main theorem

We assume that system (1) is formal strongly Weierstrass integrable, that is, it has an integrating factor of the form

$$M(x,y) = \alpha(x) \prod_{k=1}^{\ell} (y - g_k(x))^{\alpha_k}, \tag{28}$$

where the functions $\alpha(x), g_k(x) \in C[[x]]$ for $i = 1, \ldots, k$.

Moreover we know that a first integral H and an integrating factor M of the form given in statement (a) can be found.

Proof of the main theorem

We compute the solutions $y = f_i(x)$ where $f_i(x) = \sum_{j=0}^{\infty} a_j x^j$ with a_i arbitrary coefficients that must satisfy the equation $Eq := \dot{x} dy/dx - \dot{y} = 0$ up to certain order r. Note that these solutions satisfy that

$$\text{either} \quad M(x, f_i(x)) = \mathcal{O}(x^r), \quad \text{or} \quad M(x, f_i(x)) = c_2 + \mathcal{O}(x^r),$$

with $c_2 \neq 0$. The first ones correspond to the $f_i(x)$ that approximate the invariant curves $y = g_k(x)$ that appear in the integrating factor (28).

Proof of the main theorem

For such $f_i(x)$ we compute the cofactor K_i up to certain order r though the equation

$$\mathcal{X}(y - f_i(x)) = \bar{K}_i(y - f_i(x)) + \mathcal{O}(x^r). \tag{29}$$

Hence these cofactors $\bar{K}_i$ of the solutions $y - f_i(x)$ are the approximations up to order r of the cofactors K_k of the invariant curves $y - g_k(x)$ of the integrating factor (28).

Proof of the main theorem

The second ones satisfy

$$M(x, f_i(x)) = \alpha(x) \prod_{k=1}^{\ell} (f_i(x) - g_k(x))^{\alpha_k} = c_2 + \mathcal{O}(x^r). \tag{30}$$

Hence, since $c_2 \neq 0$, $M(x, f_i(x)) = c_2 + \mathcal{O}(x^r)$, and from (26) we have that $\alpha(0) \neq 0$. Then up order r we have

$$\prod_{k=1}^{\ell} (f_i(x) - g_k(x))^{\alpha_k} = \left[\frac{c_2}{\alpha(x)} \right]_r + \mathcal{O}(x^r), \tag{31}$$

where here $[\,]_r$ means up to order r.

Proof of the main theorem

Consequently $y = f_i(x)$ is an approximation up to order r of the equation

$$\prod_{k=1}^{\ell} (y - g_k(x))^{\alpha_k} = \frac{c_2}{\alpha(x)}. \tag{32}$$

We apply the vector field operator to (32) and we obtain

$$\mathcal{X}\Big(\prod_{k=1}^{\ell} (y - g_k(x))^{\alpha_k} \Big) = \mathcal{X}\left(\frac{c_2}{\alpha(x)} \right) = -\frac{c_2 \alpha'(x)}{\alpha(x)^2} \dot{x} = -K_\alpha \frac{c_2}{\alpha(x)}, \tag{33}$$

because $\mathcal{X}(\alpha(x)) = K_\alpha(x,y)\alpha(x)$ where K_α is a formal Weierstrass polynomial cofactor.

Proof of the main theorem

This happens because $\alpha(x) = 0$ is an invariant algebraic curve of the vector field $\mathcal{X}$. Indeed, $\alpha(x)$ is a factor of the integrating factor $M(x, y)$ given in (28).
Moreover we have that $\mathcal{X}(\alpha(x)) = \alpha'(x)\dot{x} = \alpha'(x)P(x, y)$ and then $K_\alpha = \alpha'(x)P(x, y)/\alpha(x)$.

In summary from equations (32) and (33) we have

$$\mathcal{X}\Big(\prod_{k=1}^{\ell}(y - g_k(x))^{\alpha_k}\Big) = -K_\alpha \prod_{k=1}^{\ell}(y - g_k(x))^{\alpha_k} \qquad (34)$$

Proof of the main theorem

Now we apply the vector field operator to (31) and we obtain

$$\mathcal{X}\left(\prod_{k=1}^{\ell}(f_i(x)-g_k(x))^{\alpha_k}\right)=\mathcal{X}\left(\left[\frac{c_2}{\alpha(x)}\right]_r\right)+\mathcal{O}(x^r), \qquad (35)$$

where $\mathcal{X}(\mathcal{O}(x^r))=\mathcal{O}(x^{r-1})\,P(x,f_i(x))=\mathcal{O}(x^r)$. Taking into account equation (33) we define the new cofactor $\tilde{K}_\alpha$ through the equation

$$\mathcal{X}\left(\left[\frac{c_2}{\alpha(x)}\right]_r\right)=-\tilde{K}_\alpha\left(\left[\frac{c_2}{\alpha(x)}\right]_r\right) \qquad (36)$$

which is equation (33) taking the lower terms up to r and where $\tilde{K}_\alpha$ is an approximation up to r of the cofactor K_α.

Proof of the main theorem

Therefore from (31), (35) and (36) we obtain an approximation of the cofactor of $\alpha(x)$ up to order r computing

$$\frac{\mathcal{X}\left(\prod_{k=1}^{\ell}(f_i(x)-g_k(x))^{\alpha_k}\right)}{\prod_{k=1}^{\ell}(f_i(x)-g_k(x))^{\alpha_k}} = -\tilde{K}_\alpha + \mathcal{O}(x^r). \tag{37}$$

By the definition of integrating factor (28) we have that

$$\mathcal{X}(M) = -\mathrm{div}(\mathcal{X})M. \tag{38}$$

Proof of the main theorem

In short the other solutions $y - f_i(x)$ not passing through the origin with cofactor $\tilde{K}_i$ given by equation (37) an approximation up to order r of the cofactor $\tilde{K}_\alpha$ of $\alpha(x)$, i.e.

$$\sum_{i=1}^{s} \mu_i \tilde{K}_i = -\tilde{K}_\alpha. \tag{39}$$

Therefore, from (30), (38) and (39) we obtain that

$$\sum_{i=1}^{\ell} \lambda_i \bar{K}_i + \sum_{i=1}^{s} \mu_i \tilde{K}_i = -\mathrm{div}_r(\mathcal{X}) + \mathcal{O}(x^r). \tag{40}$$

This proves statement (b) of the theorem.

Thank you!

Jaume Giné
Universitat de Lleida, Spain
gine@matematica.udl.cat